PART 2

SOLAR CONVECTION AND DIFFERENTIAL ROTATION

THEORY OF CONVECTION IN A DEEP ROTATING SPHERICAL SHELL, AND ITS APPLICATION TO THE SUN

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Abstract. The theory of convection in a rotating spherical shell, when applied to the Sun, should ultimately satisfy at least three broad constraints. The radial heat flux by the convection must not vary significantly with latitude; an equatorial acceleration must be produced; and the convection must give the right dynamo action to produce the gross features of a solar cycle.

Important quantities to look for in the observations with which a convection theory can be compared include evidence of global velocities of giant convection cells, differences in motion features between low latitudes and high, persistence of motion features over successive rotations, evidence of excess brightness and variations in total solar luminosity, correlations of velocities between northern and southern hemispheres, and between north-south and east-west motions, and time periodic changes in motion fields.

The general theory of convective motions in a rotating spherical shell such as the convection zone of the Sun has developed rapidly over the last several years, but is still a long way from providing quantitative models which agree in a satisfactory way with the main solar observations. Most work has (for mathematical reasons) employed the Boussinesq approximation, and most has been either linear or nearly so. Early work demonstrated the basic ability of global convection to transport momentum toward the equator, but whether an equatorial acceleration results depends on the effects of competing angular momentum transport processes, namely transport in the radial direction by convection, and transport by axisymmetric meridional circulations. The end result can be assessed only by nonlinear calculations.

Some recent calculations directed at this question by the author indicate that, at least at Prandtl number of unity, convection growing from an initial state of solid rotation produces equatorial acceleration only when equatorial modes dominate and the rotational constraint is sufficiently strong, which results in a convective heat flux which is strongly dependent on latitude. When the rotational constraint is broken enough to give convection at all latitudes and therefore more nearly uniform heat flux, equatorial deceleration and angular momentum mixing in latitude results. A few calculations for Prandtl number substantially smaller than one give results which suggest it is possible to produce equatorial acceleration and nearly uniform heat flux, but these solutions appear not to be unique. In particular, the amount of differential rotation present in the initial conditions appears to be important in determining the final state.

1. Introduction

The theory of thermal convection for a deep rotating spherical shell such as the convection zone of the Sun is still in its early stages of development. On the one hand, it is clearly still impractical (and unwise) to include in a computer model all the physics that could be said to be important for this problem. On the other hand, there is much room for progress in developing simple, but still meaningful models. Working with such simpler models, I and others have obtained a number of promising results, but at the same time have revealed some perplexing and difficult problems.

Before going any further, let me indicate I am talking about modeling those scales of motion which are large enough in physical scale, and peristent enough in time, to be significantly influenced by rotation. Thus, for the Sun we are talking about the differential rotation, together with its fluctuations in time as well as global scale

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eddies or giant cells imbedded in it. In this context, granules and even supergranules are too small in physical extent and too short in lifetime to be influenced much by rotation. Because of the central importance of differential rotation in maintaining solar activity, this paper will give particular emphasis to the angular momentum transport effects of global scale convection.

2. Broad Constraints Imposed by Observations

Even when developing models with highly simplified physics compared to the real Sun, certain broad observational constraints should be kept in mind for testing model progress. First, the heat flux coming out of the convection zone does not vary substantially with latitude – certainly no more than a few parts in 10^3, and perhaps substantially less. Thus, to be interesting for the Sun, the convection which is produced in a spherical shell model should also not show large differentials in heat flux. As we will see later, this imposes quite severe constraints on convective theory.

Second, even simple models should produce rotation rates which are higher in equatorial regions than in higher latitudes, as occurs on the real Sun. This is a substantially less severe constraint than on the heat flux, but still we find that the convection has a way of *de*celerating the equator when we least want it to.

Third, even if the convective model passes both of the above tests, it must still be shown to give hydromagnetic dynamo action which either produces the gross features of a solar cycle, or which does not interfere with some other mechanism which does. To test how well a convective spherical shell model satisfies this third constraint will clearly require considerable effort.

In addition to the broad constraints given above, there is also other observational evidence we should pay attention to in building and testing a model. For example, from Howard's observations (reviewed earlier at this meeting) it is clear that the kinetic energy at the surface in any giant cells on the sun is not large compared to the kinetic energy present in the shears of the mean differential rotation itself and may be substantially smaller. Furthermore, the fluctuations in the differential rotation (which may be hard to separate from giant cells) are of roughly the same magnitude or less than this mean shear. In addition, any axisymmetric meridional circulation (i.e., north-south motion) is at least one order of magnitude smaller than the differential rotation, and perhaps smaller still. Also, sunspots appear to rotate systematically faster at the surface than does the solar plasma, but some structures of very large latitudinal extent, such as coronal holes, show rather little differential rotation and rotate at nearly the equatorial plasma rate.

3. Important Parameters which are Poorly Known

Despite the fact there are many observational facts (alluded to above) with which we can compare a model, there are also a number of important parameters which we simply do not know well. For example, while we have some knowledge of the energy spectrum of motions at the surface (granules are somewhat more energetic than supergranules, which in turn are more energetic than giant cells or the differential rotation), we have really very little idea how this spectrum varies with depth. We suspect on theoretical grounds the dominant eddy size increases in scale with increasing depth and the velocities decrease in magnitude, but we have little in the way of observational evidence to support this. Consequently, parameterizing the effects of small scale motion, as we must in order to make progress, is a very uncertain matter.

In addition, while the depth of the convection zone can probably be reasonably estimated from solar structure calculations, we have little idea how much the convection may overshoot into the interior. Consequently, proper boundary conditions are difficult to define.

4. What Should be Looked for in Observations

The observations of the global circulation of the Sun have already been reviewed for this symposium by Howard and Yoshimura. I would like to list certain effects which, if found, could provide very useful guidance to theoreticians such as myself in attempting circulation and dynamo models.

(1) Differences in the form of large scale motion when comparing low latitudes with high, on scales substantially larger than supergranules. In this regard, Howard and Yoshimura's evidence of velocity anomalies in low and middle latitudes in each hemisphere is most encouraging.

(2) Evidence for actual giant convective cells – which Howard and Yoshimura's discovery may be. One particularly important parameter to know is the wavelength in longitude.

(3) Persistence of motion features (in addition to the mean differential rotation itself) for several solar rotations.

(4) Any large scale persistent features of excess brightness, no matter how small in amplitude, which cannot be attributed to plages or other obvious active region features. Long-lasting variations in both longitudinal and latitudinal directions are important, because they could be evidence of the thermal properties of giant convective cells. The important recent observational work of Hill and colleagues at the University of Arizona on solar oblateness, excess brightness and oscillations encourages me to believe it may be possible to look for such variations.

(5) Fluctuations in the solar constant of periods of weeks, months or longer, if any are found with new measurements, could also be related to thermal structure in giant convective cells. I note that Schwarzchild (1975) has recently invoked time dependent giant convective cells for red giants and supergiants as a possible explanation of their irregular fluctuations in energy output. Perhaps giant cells on the Sun, of which presumably only a few would be on the observable disk at any time, could produce small fluctuations in the solar flux visible from the Earth as a function of time, both from their own possible time dependence, and from rotation of new convective cells onto the disk. In turn, small modulations of the giant cells by the changing magnetic fields of the solar cycle could give longer period variations in heat flux.

(6) Any tendency in velocity anomalies to show symmetry between the two hemispheres, or correlation in the fluctuations in time of motions in the two

hemispheres. Theoretical models of spherical shell convection with rotation do indicate certain symmetries. With regard to time correlations, I recently did a simple analysis of the published parameters b and c in Howard and Harvey's (1970) differential rotation Ω as a function of latitude ϕ

$$\Omega = a + b \sin^2 \phi + c \sin^4 \phi$$

Howard and Harvey reported separate values for northern and southern hemispheres for a, b and c for each day observations were taken. I correlated $b(n)$ with $b(s)$, and $c(n)$ with $c(s)$, obtaining significantly positive values in both cases:

$$r(b(n), b(s)) = 0.27 \pm 0.11$$

$$r(c(n), c(s)) = 0.34 \pm 0.11$$

(with attached 95% confidence limits determined according to Hoel (1947), p. 122.) Thus, it would appear that mid and high latitudes in the two hemispheres tend to speed up and slow down together. It has already been established by Yoshimura (1971) and later by Wolff (1975) that within a single hemisphere, mid-latitudes tend to speed up while higher latitudes are slowing down, and vice versa. Since all these results are found from the result of least squares fitting done by Howard and Harvey (1970), it is possible that some of these effects, if real, represent not actual changes in the differential rotation but rather correlations (which are linked between the two hemispheres) in the global scale convective cell structure. The effect needs to be looked for in new, larger data samples.

(7) Any evidence from the large scale velocity anomalies which allows us to distinguish between the longitudinal and latitudinal motions in them. In cases where the velocity anomaly persists for a few days on both sides of the central meridian, and its rotation rate can be determined, then by successively adding and subtracting matched data from the eastern and western hemispheres the two components can be separated (provided the radial motion is small) since the Doppler shift of the longitudinal component changes sign with central meridian passage, while that of the latitudinal component remains the same.

(8) If such a separation of longitudinal and latitudinal velocity anomalies can be made (call them u' and v', respectively) then the latitudinal angular momentum transport

$$r = \cos \phi \, u' v'$$

can be computed. Theory indicates this ought to indicate transport toward the equator, in order to help maintain the equatorial acceleration there. Sunspot motion statistics computed by Ward (1965) indicate equatorward transport.

(9) Some knowledge of u' and v' separately might also allow us to estimate the radial component of vorticity, as well as the velocity divergence in horizontal surfaces. These quantities could provide comparison points with convective structures generated by the models.

(10) Knowledge of eddy latitudinal motions should also allow the estimation of transport in latitude of radial magnetic flux. This is an important quantity to know for understanding the solar dynamo.

(11) Periods of any long-lived fluctuations in either the differential rotation or cell velocities could indicate the presence of inertial oscillations, as well as helping in determining the nature of nonlinear interactions between the cells and the differential rotation.

5. Recent Developments in the Theory of Convection in a Rotating Spherical Shell

Having commented on various limitations to our knowledge of the dynamical properties of the Sun, let me turn to the theory, and describe the present state of the subject as I see it. Where it is possible to do so, I will try to relate theory back to the observational questions raised earlier.

There have been quite a variety of theories proposed for the solar differential rotation, only some of which involve explicit calculation of the effects of global scale convection. Much of the more general work will be discussed in the invited papers by Durney and Weiss. Here I will concentrate only on the theory of non-axisymmetric convection, and what it has to say about differential rotation. Most of the important contributions to this subject have been made in the last five to seven years, by Busse (1970a, 1973), Durney (1970, 1971), Gilman (1972, 1975), Yoshimura (1971, 1972, 1974), and Yoshimura and Kato (1971). The work of Roberts (1968), Busse (1970b), and Gierasch (1975) is also related and relevant. Simon and Weiss (1968) have considered simple models of global solar convection which include compressibility but leave out rotation.

5.1. Model assumptions

There are certain assumptions common to most of the spherical shell convection models referenced above which should be mentioned. Some of these are made to render the problem mathematically more tractable and are not particularly well justified in physical terms. In particular, all except Gierasch (1975) consider a so-called Boussinesq system, that is, one in which all density variations are ignored except where coupled with gravity. This clearly cannot be justified for the solar convection zone, but does provide a relevant analogue, with which to concentrate on rotational effects. In addition, small scale diffusion of heat and momentum is parameterized in linear constant eddy viscosity and thermal diffusivities, clearly a considerable oversimplification. Finally, except for Gilman (1972) and, to a lesser degree Busse (1973), the models presented are at best only slightly nonlinear. Consequently, not much can be said from most of the models about amplitudes of convection and differential rotation. Gilman (1972) demonstrated a number of nonlinear effects. Recent results obtained by me with a much more nonlinear model than presented previously, on which I will report shortly, indicate the nonlinearities are even more important.

Some of the early models (particularly Busse, 1970, 1973) considered only thin shells (depth $\ll$ radius) but other results (Gilman, 1972, 1975) indicate deep shell effects are important, particularly Coriolis forces in the radial direction affecting the radial momentum flux. Similarly, Yoshimura and Kato (1971) and Yoshimura (1971,

1974) assumed the convective motions were hydrostatic, which also excludes deep shell Coriolis force effects.

The boundary conditions at the top and bottom of the convection zone for the real Sun are clearly quite complex, but not well known. Models to date have restricted themselves to quite simple conditions, usually assuming stress-free fixed temperature top and bottom. In a few cases, a non slip bottom has been considered.

5.2. Fluid dynamical parameters

There are several dimensionless parameters which need to be defined in order to facilitate description of spherical shell convection theory. These are the Rayleigh number R, Taylor number T, Prandtl number P, and a depth parameter β. They are summarized below

$$R = \frac{g_0 \alpha \Delta\Theta d^3}{\kappa\nu}$$

in which g_0 is gravity at the outer edge of the shell, α is the coefficient of volume expansion (= temperature^{-1} in a perfect gas), $\Delta\Theta$ is the imposed radial temperature difference, d is the depth of the convecting layer, κ is the thermal diffusivity, and ν the kinematic viscosity. The Rayleigh number measures the strength of the buoyancy forces. The Taylor number

$$T = 4\Omega^2 d^4/\nu^2,$$

in which Ω is the rotation rate of the coordinate system in which the relative angular momentum is zero. The Taylor number measures the relative importance of Coriolis and viscous forces. The Prandtl number

$$P = \nu/\kappa$$

is simply a measure of the relative importance of viscous and thermal diffusion processes. Finally β = radius of inner spherical shell/depth of convecting layer. The various results which will be referred to are for a variety of values of these parameters. For linear results and perturbation nonlinear results, R must be near R_c, the critical Rayleigh number for onset of convection. Nonlinear calculations have been done for the spherical shell up to $R \approx 10^5$. Mixing length arguments can provide only extremely crude estimates of the effective Rayleigh number for the Sun. One indirect approach is to compare motion amplitudes to what observations are available, but this, too, is very approximate, since compressibility is not included.

The Taylor number T for the Sun, if we use $d = 1.4 \times 10^{10}$ cm for the total depth of the convection zone, a mean rotation rate $\Omega \approx 2.6 \times 10^{-6}\ s^{-1}$, and an eddy viscosity of $10^{12}\ cm^2\ s^{-1} \leq \nu \leq 10^{14}\ cm^2\ s^{-1}$, falls in the range $10^2 \leq T \leq 10^6$. At $T = 10^2$, rotation is a minor perturbation on the nonrotating case; for $T = 10^6$, the flow is very strongly rotationally influenced. With κ, ν representing eddy diffusivities, we should expect to take the Prandtl number $P \sim 1$. This assumption will be re-examined later. Busse and Yoshimura have concentrated on thin shell theory, i.e., $\beta \gg 1$, while Durney and Gilman have taken a finite depth, typically 25% of the inner radius, or $\beta = 4$, which is a reasonable depth for the actual solar convection zone.

5.3. Linear analyses: the basic mechanism for producing equatorial acceleration

The early work of Busse (1970a), Durney (1970, 1971), and Yoshimura and Kato (1971) principally illustrated the fundamental mechanism by which global convection can generate and maintain an equatorial acceleration. Without rotation, convection in a spherical shell is degenerate in that many different modes first become unstable at the same Rayleigh number. They showed that when a small amount of rotation perturbs these modes, i.e., Taylor number small, then one mode is selected as the initial unstable mode. This mode is highly elongated in latitude and has the desirable property of transporting angular momentum towards the equator from higher latitudes. The mechanism of this transport is perhaps best illustrated in Figure 1, taken from my own linear numerical work, which shows the horizontal velocity vectors associated with the most unstable mode for $T = 10^3$ compared to $T = 0$. The Coriolis forces acting in the horizontal plane have turned the horizontal velocity vectors to the right, so that flow in longitude in the direction of rotation has a component toward the equator, flow against the rotation has a component toward the pole. Thus the Reynolds stress $u'v'$ gives momentum flux toward the equator. The effect remains even as T is increased to the point where the original perturbation analysis employed for small T by several authors is no longer valid. This results in horizontal motions with the form of closed swirls as seen, for example, in the bottom figures for $T = 3 \times 10^4$. The profile of momentum flux is shown in Figure 2 indicating it typically peaks for these modes somewhere in the neighborhood of 20° north latitude, and extends in depth largely through the convecting layer.

In the actual nonlinear system, of course, this equatorward transport of momentum by global convection must compete with other mechanisms of transport. In particular, all these modes also transport momentum in the radial direction through the Reynolds stress $u'w'$, or correlation between longitudinal motion u and radial motion w. This is particularly important if the depth of the convecting layer is large enough that Coriolis forces in the radial direction are active. Some profiles of radial momentum flux I have calculated for the most unstable modes at various Taylor number are shown in Figure 3 for increasing Taylor number. We see that at low T, the transport is negative (dashed contours) or radially inward. Only at and above $T = 10^5$ is the flux outward. Thus at low T, even though momentum is transported toward the equator near the surface, it is also carried below. Whether an equatorial acceleration would still result at the surface is not clear. On the other hand, at large T, an equatorial acceleration seems much more likely, since momentum is being brought to the equatorial outer surface both from high latitudes and greater depths. The reason for the switch from inward to outward momentum flux is illustrated in Figure 4. At $T = 0$, rising and sinking fluid particles have no rotational momentum to carry. At low but nonzero T, e.g., $T = 10^3$, fluid particles essentially conserve total angular momentum (except for frictional effects) and thus transport momentum relative to the rotating frame inward. Rising particles acquire a component of longitudinal motion opposite to rotation, sinking particles a component in the direction of rotation. At high $T = 10^5$ organized longitudinal pressure torques develop which accelerate rising particles in the direction of rotation in longitude, sinking particles in

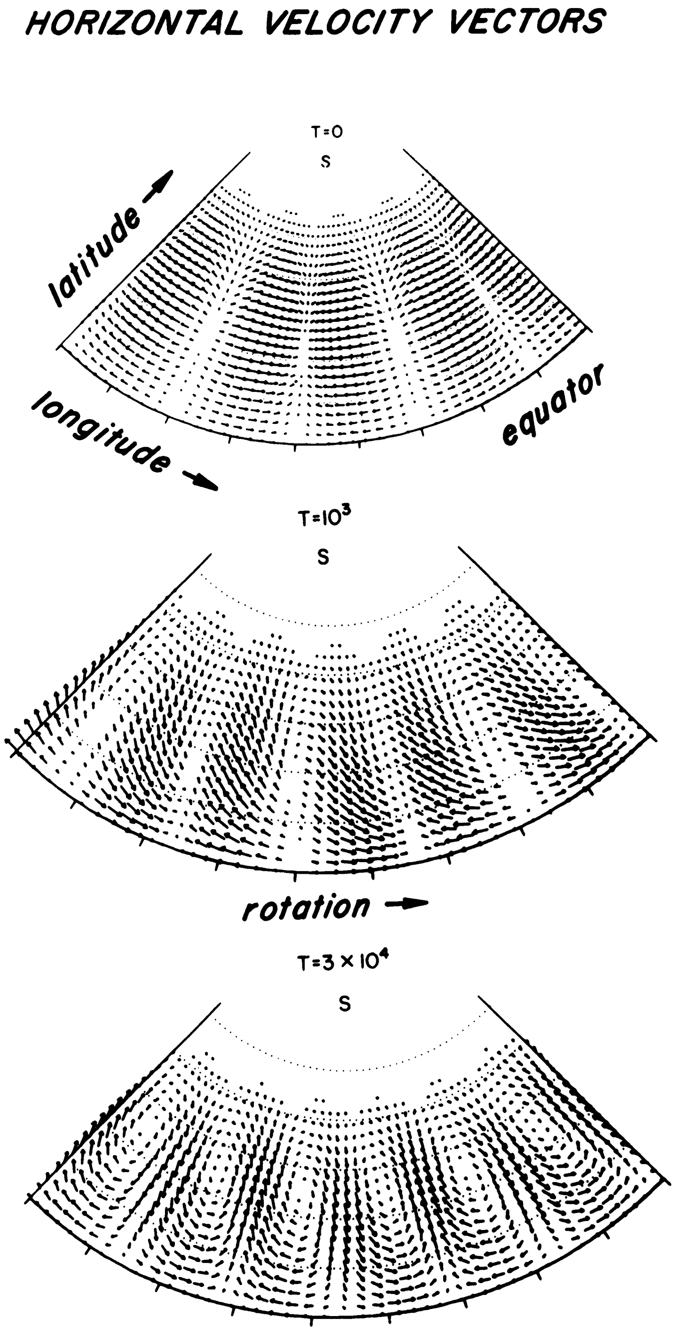

Fig. 1. Horizontal velocity vectors near outer surface of spherical shell in northern hemisphere for most unstable longitudinal wave number m, as function of Taylor number T (defined in text) (for $T=0$, $m=9$, $T=10^3$, $m=10$, $T=3\times10^4$, $m=12$). Light dotted arcs are latitude belts spaced by 10°. Tick marks are spaced 10° in longitude.

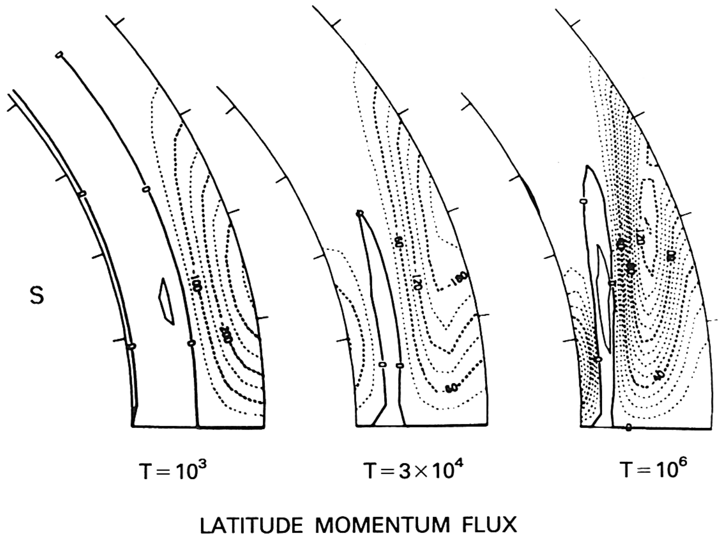

Fig. 2. Latitude-radius sections showing latitudinal angular momentum flux associated with most unstable mode m as function of Taylor number T. Dashed contours denote negative, or equatorward, momentum flux. Equatorial plane is at lower edge of each figure.

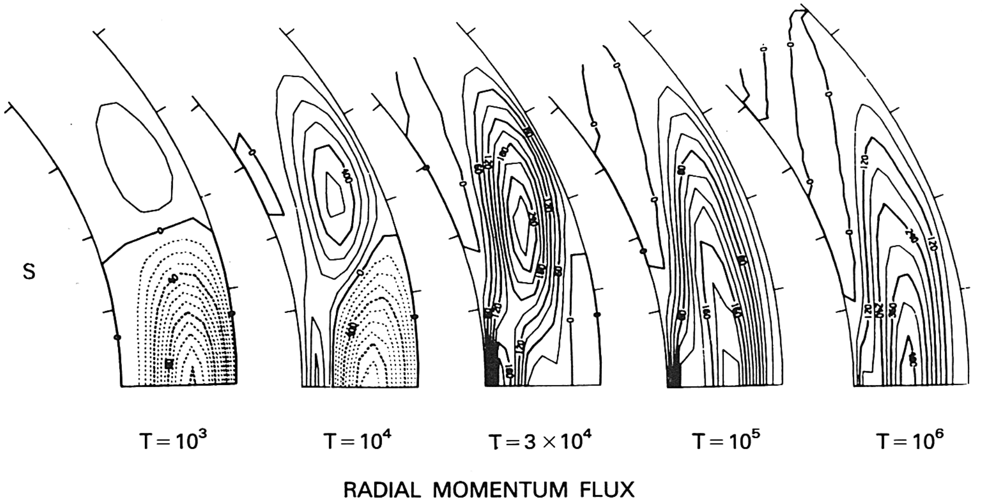

Fig. 3. Latitude-radius sections showing radial angular momentum flux associated with most unstable modes as function of Taylor number T. Solid contours denote positive or outward momentum flux; dashed contours negative or inward flux.

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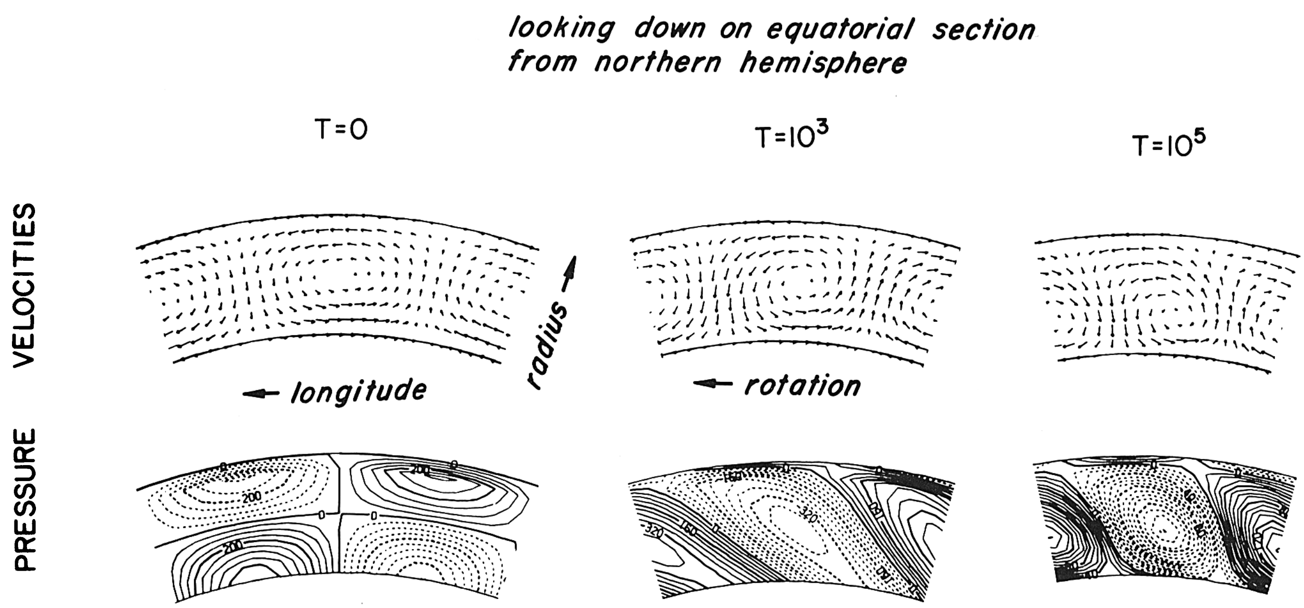

Fig. 4. Longitude-radius section in equatorial plane showing velocity vectors and perturbation pressure fields associated with modes with same T whose radial momentum flux is shown in Figure 3.

the opposite direction, resulting in a net outward flux of momentum. These pressure forces are slightly larger than needed to keep the flow in heliostrophic balance (pressure forces balancing Coriolis forces). This effect is also described in Gilman (1975).

A third mechanism of angular momentum transport arises from the fact that other Reynolds stresses, namely v^2, w^2, and vw, as well as thermal stresses, force an axisymmetric meridional circulation. As Busse (1975) has pointed out, this circulation occurs even in the limit of small rotation, provided some mode selectivity is present. Consequently, the lower the Taylor number the more important is this meridional circulation relative to the other momentum transport mechanisms. In general, it tends to produce equatorial deceleration because the rings of fluid being advected around in the circulation tend to conserve their angular momentum.

The problem of competing and perhaps conflicting mechanisms of momentum transport becomes even more acute when one realizes that at small Taylor number, the modes with the most desirable properties are only slightly favored over other modes with different properties. The behavior of the system can (and does) change markedly when nonlinear effects come in. We will illustrate this below.

From Gilman (1972, 1975) it is clear that as Taylor number is increased beyond the point where small T effects are dominant, the favoring at instability onset of modes which have the desirable momentum transport properties becomes much stronger, and equatorial acceleration is more nearly assured. These modes also produce a differential rotation which is symmetric about the equator, which the Sun's rotation is, predominantly. Gilman (1975) showed that for Prandtl numbers near unity the most unstable mode structures in latitude for each longitudinal wave number m break at high T into just two classes: a large number of modes above a certain m ($m \leq 4$ for $T \geq 10^3$) all of which peak at or near the equator, and which transport momentum towards it from higher latitudes; and a small number of very differently structured low m modes which peak near the pole. (This result is one reason why it would be very interesting to see if any differences between equatorial and polar regions occur in large scale convection on the real Sun.)

The equatorial modes for Prandtl number $P \sim 1$ are excited at much lower Rayleigh numbers than are the polar modes. This leads to the prospect of the convective heat flux being a strong function of latitude in the nonlinear case, clearly not observed on the Sun. The key question to be answered in nonlinear calculations then is, does a regime of solutions exist in which the heat flux has been more or less equalized in latitude, which at the same time still gives equatorial acceleration? As will be shown below, so far we have not found one for $P \approx 1$, but solutions for $P \ll 1$ look promising.

5.4. Nonlinear analyses at Prandtl number $P = 1$: testing the momentum transport mechanisms

In the Spring of 1975 I began running a nonlinear spherical shell convection model to try to find out, among other things, what kind of differential rotation is produced. I am able to give some preliminary results. In the model calculations, the nonlinear Boussinesq convection equations are solved numerically as an initial value problem, using a grid of points in the meridian plane (latitude-radius section) with all variables represented by Fourier series in longitude. Typically, up to 7 different longitudinal modes were retained, most often longitudinal wave numbers $m = 0, 4, 8, 12, 16, 20, 24$, in order to span the most unstable convective modes (denser arrays of wave numbers are much more time consuming on the computer, and can be done only for a few cases).

The calculations were done for a fluid layer 20% of the outer radius, and at first we focused on calculations with the Prandtl number $P = 1$, as well as stress free, fixed temperature upper and lower boundary conditions.

The basic regimes of convection which occur are described in Figure 5. The stability boundary below which no convection occurs is given by the lower solid curve. Above this line, but below the next one, a spectrum of convection occurs (that

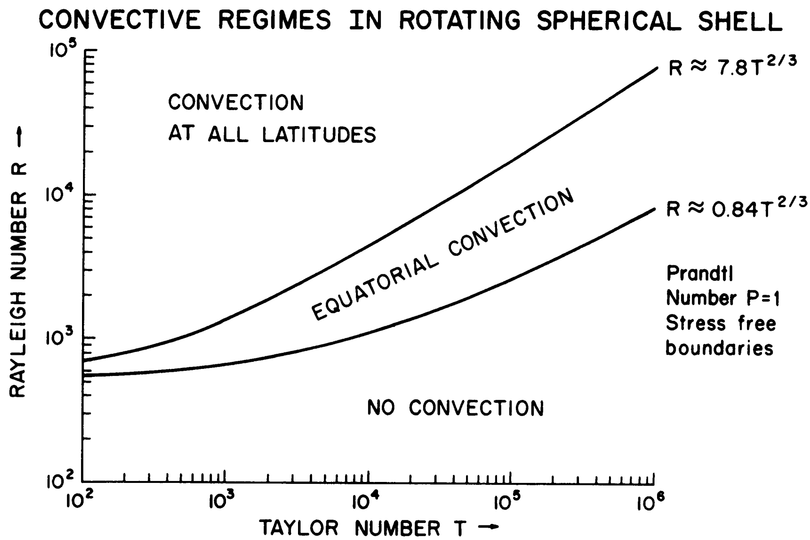

Fig. 5. Simplified regime diagram showing convection occurrence as function of Rayleigh number R and Taylor number T for Prandtl number $P = 1$.

is, many modes, each with different m, are needed to adequately describe it) but is confined to equatorial latitudes. Above the second curve, the polar modes come in, and convection occurs at all latitudes, though at high Taylor number, it is definitely concentrated near the poles for Rayleigh numbers near the line. Substantially above the second line, the heat flux is more nearly equal at low and high latitudes, but some variations remain.

What about the differential rotation that is produced? Figure 6 summarizes this. We have found that in general, equatorial regions rotate faster in the nonlinear case only when the convection itself is confined to equatorial regions. Even here, as we increase R for a given T, a deceleration near the equator starts to appear. For R high enough that convection is occurring at all latitudes, there is a strong tendency for high

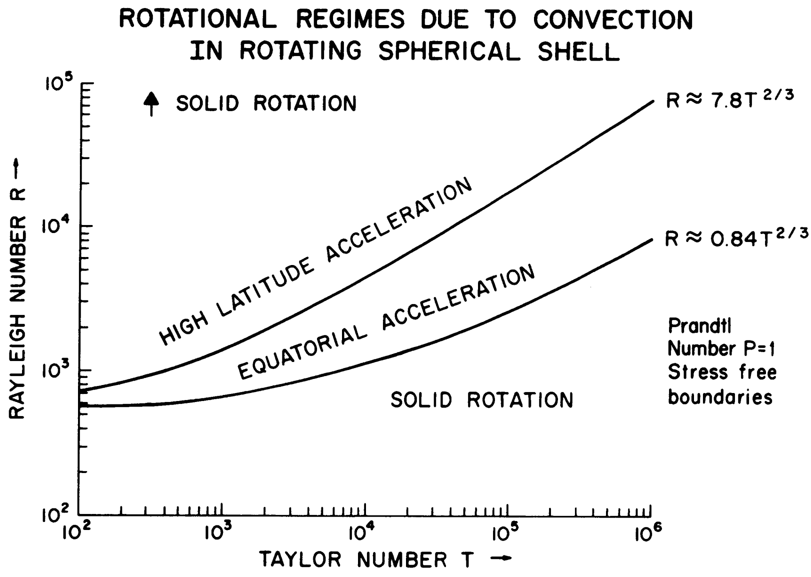

Fig. 6. Rotational regimes corresponding to convection regimes in Figure 5, as deduced from integrations of the nonlinear spherical shell convection equations.

latitudes to rotate faster than low, exactly the opposite of the result we want for the Sun. So at least at Prandtl number $P = 1$, the price paid for equalizing the heat flux at low and high latitudes is the destruction of equatorial acceleration.

Let me give a few examples of the more detailed results, and explain what causes the effect.

Let us consider a set of integrations at $P = 1$, $T = 10^5$, for R between 4000 and 80 000, for stress free top and bottom, which spans the regime diagram. At the beginning, the fluid is in solid rotation and the temperature field is given by conduction alone. Then essentially arbitrary temperature perturbations are added which grow if the fluid is convectively unstable. At $R = 4000$ the resulting convection is confined to equatorial regions, has a spectrum which peaks rather sharply near the most unstable mode. It drives a very simple differential rotation, essentially constant on cylinders, shown in Figure 7. So at this R, the rotation falls off nicely with latitude and decreases inward. A weak meridional circulation with the profile shown in Figure 8 is also produced, including poleward flow near the surface, equatorward flow underneath. The differential rotation profile is maintained by equatorward and radially outward angular momentum flux in the convection cells, as expected from the linear results quoted earlier. Momentum transport by meridional circulation is too weak to alter this. The total heat flux peaks at the equator, since the convection is confined to low latitudes.

Now, as the Rayleigh number R is increased, so the convection becomes stronger, the differential rotation produced begins to evolve substantially. Figures 9–12 show the differential rotation for $R = 10^4$, 2×10^4, 4×10^4, 8×10^4, respectively. At $R = 10^4$ (Figure 9) we see that the maximum velocity has moved away from the

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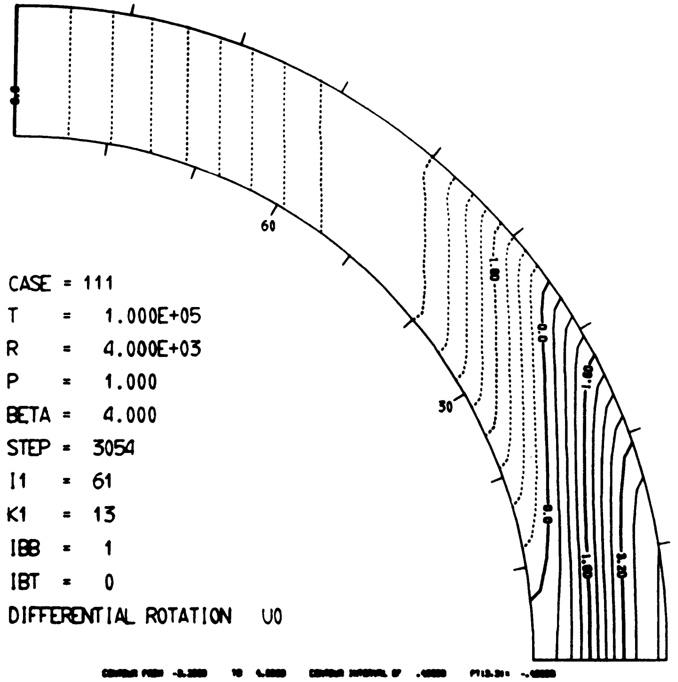

Fig. 7. Computer produced meridional cross section of linear rotation velocity u_0 relative to uniformly rotating frame, for Taylor number $T=10^5$, Rayleigh number $R=4\times10^3$, Prandtl number $P=1$. Units dimensionless, with velocity scaled by κ/d, in which κ is thermal diffusivity, d is convection zone depth. Positive u_0 indicated by solid contours; negative u_0 by dashed contours.

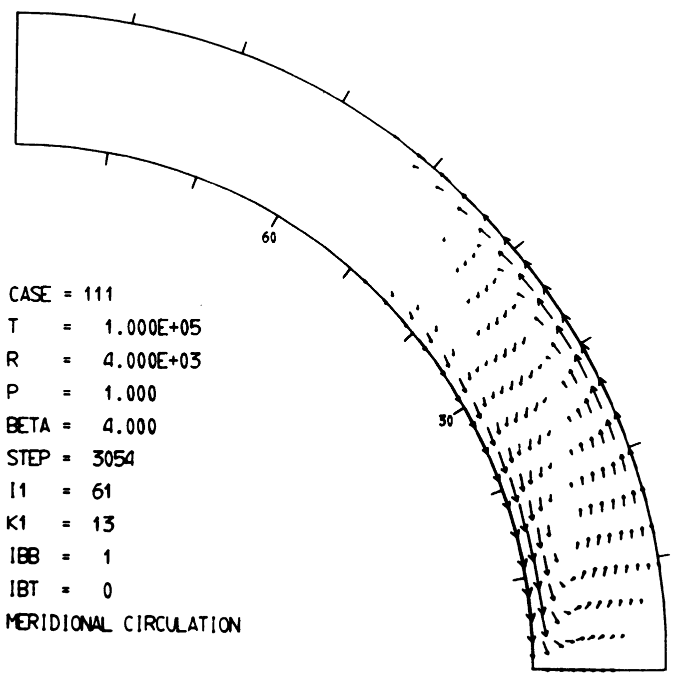

Fig. 8. Meridional circulation vectors for same case as Figure 7.

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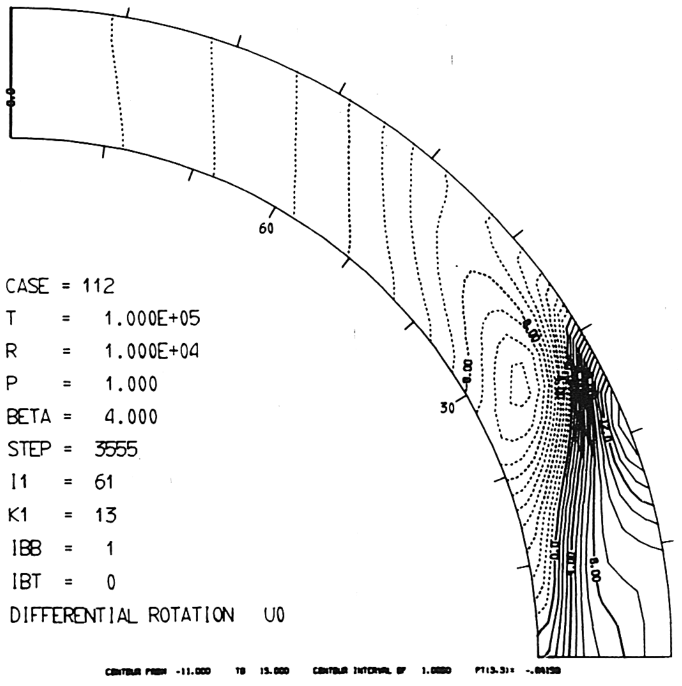

Fig. 9. Typical differential rotation profile for Rayleigh number R increased to 10^4.

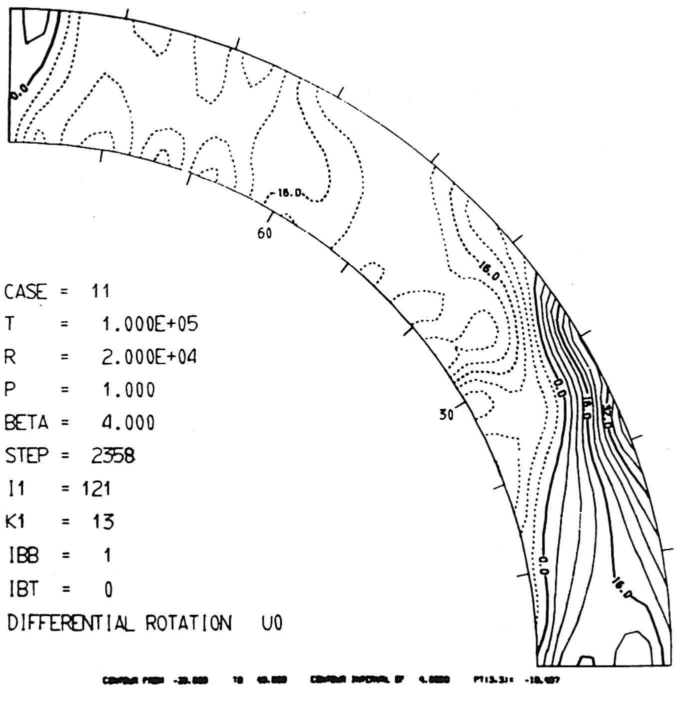

Fig. 10. Typical differential rotation for Rayleigh number R raised to 2×10^4.

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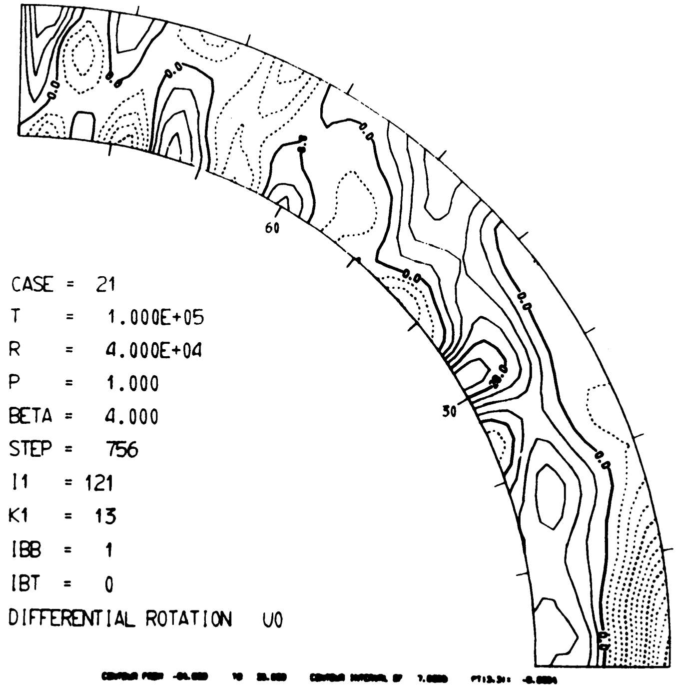

Fig. 11. Typical differential rotation for Rayleigh number R raised to 4×10^4.

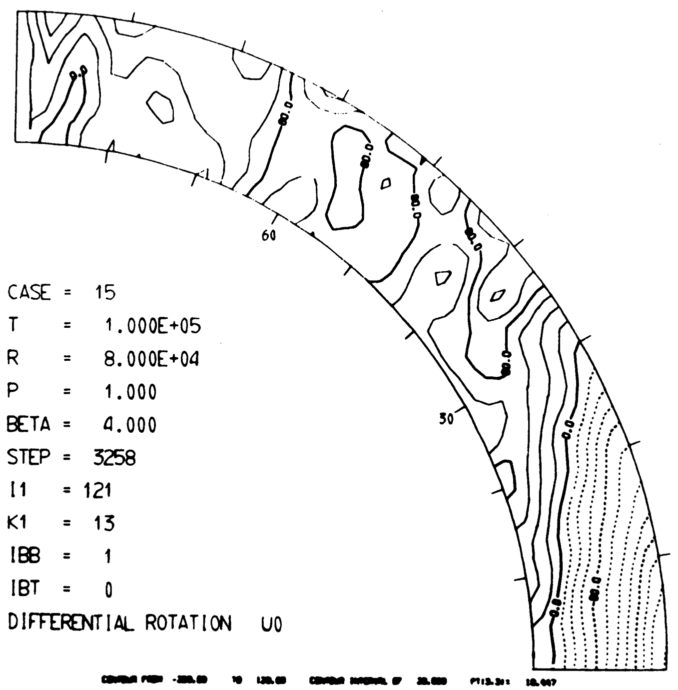

Fig. 12. Typical differential rotation for Rayleigh number R raised to 8×10^4.

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equator to about 20° north, so a slight equatorial deceleration has appeared. By $R = 2 \times 10^4$ (Figure 10) this effect is more pronounced and by $R = 4 \times 10^4$ (Figure 11) the relative velocity has actually changed sign. By $R = 8 \times 10^4$, the profile is again nearly cylindrical, but with angular velocity *increasing* with latitude up to about 60°, as well as increasing with depth.

The change in rotational velocity is so pronounced that if one plots the total angular momentum of the system (the coordinate system + flow relative to it) as a function of latitude (Figure 13), it is nearly constant with latitude up to about 30° (the curve u_0 in Figure 13). In other words, the angular momentum, originally that of solid rotation (curve c in Figure 13) has been mixed in low latitudes so much its gradients are wiped out. This is, of course, exactly the opposite of what apparently occurs on the Sun. At the lower values of R for which we do get an equatorial acceleration, its magnitude is significantly smaller than needed for the Sun, although the $R = 2 \times 10^4$ profile comes within a factor of two.

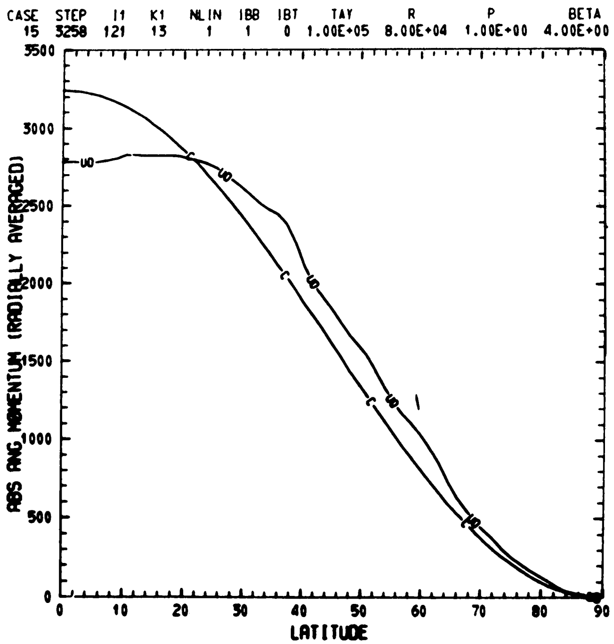

Fig. 13. Radially averaged angular momentum in convecting layer, for $T = 10^5$, $R = 8 \times 10^4$, $P = 1.0$ case. Curve c: angular momentum of rotating reference frame ($= \frac{1}{2}PT^{1/2} r^2 \cos^2 \phi$, radially averaged in which r is dimensionless radius, and ϕ is latitude). Curve u_0: same total angular momentum as curve c but redistributed according to profile of relative velocity u_0 which is built up by convection and meridional circulation ($= (\frac{1}{2}PT^{1/2} r \cos \phi + u_0) r \cos \phi$, radially averaged).

Three processes have conspired to produce this result. First, the meridional circulation has grown very strong and carries angular momentum to higher latitudes. Second, the radial angular momentum transport has changed sign as R increased, so it decelerates the outer layers near the equator by producing net inward flux. Third, the equatorward momentum flux by the cells, so dominant at low R, has been

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essentially destroyed by the nonlinear interactions. All three changes are due to nonlinear effects breaking the rotational constraints.

In so doing, the total heat flux has become much more nearly equal in latitude, but the differential rotation is now quite unsuitable for the Sun. Furthermore, the meridional circulation, which for $R = 4000$, was very small compared to the differential rotation amplitude, is of equal magnitude by $R = 4 \times 10^4$, which is much too large for the Sun.

Increasing the Taylor number does not help, because it only raises the Rayleigh number required to get convection at all latitudes, at which point the angular momentum again becomes mixed.

At smaller Taylor numbers, a similar effect simply occurs at much smaller Rayleigh number. For example, at $T = 10^3$, angular momentum is well mixed in the lowest 30° by $R = 4800$. The low Taylor number solutions have the additional disadvantage that at all R, forcing cells are very large in amplitude compared to the differential rotation they force, which is apparently not the case on the Sun. Therefore, nonlinear results obtained by Busse (1973) and others as an expansion in small amplitude and small Taylor number do not represent the dominant processes and results as R is increased significantly above what is required for convection to occur.

5.5. The search for better analogues to the Sun

Given that the solutions described above are not nearly as good an analogue of the Sun as we would like, what conditions or parameters should we change to find better ones? One possibility we have tried is to change the boundary condition on velocity at the bottom of the convecting layer to be nonslip rather than stress free. Gierasch (1975) has argued this may be a more appropriate boundary condition for the Sun. What this does is to impose the angular momentum distribution of the solidly rotating coordinate system on the flow, and in effect makes the interior below the bottom boundary an infinite source or sink of momentum, rather than allowing no momentum to pass through, which is what the stress free condition does. We have run a few cases to compare the differential rotation produced and found, as we should expect, that the nonslip condition does inhibit the angular momentum mixing in latitude somewhat, but the basic tendency is still there.

The basic problem with the $P = 1$ solutions is that in order to get convection occurring at all latitudes in the model, i.e., excite polar as well as equatorial modes, the Rayleigh number must be raised so high that the rotational constraint in low latitudes is effectively broken, and equatorial deceleration results. As indicated in Gilman (1975), however, it appears that for $P \ll 1$, convection should set in at much more nearly the same Rayleigh number at all latitudes, because the polar modes become overstable (growing oscillations) at relatively low R. Because the linear solutions oscillate, finding the linear stability boundary by numerical integrations is not easy, but nonlinear calculations can still be done. We have very recently done a few of these, for P down to 0.01 using as initial conditions results at the same T, R, but higher P. While much more careful analysis is needed, preliminary indications are that it should be possible to find some solutions which give amplitude and structure of differential rotation more nearly like those of the Sun, as well as more nearly uniform heat flux.

For example, as seen in Figure 14, solutions at $P = 0.01$ and $R = 10^4$, $T = 10^6$ (with $m = 0, 4, 8, 12, 16, 20, 24$) show an excess of angular momentum *near* the equator above that of solid rotation rather than mixed as we found for the $P = 1$ solutions. In addition, the solutions have the desirable properties (from the point of view of the Sun) that the meridional circulation remains small compared to the differential rotation, and the convective velocities themselves are also relatively small. In addition, very low longitudinal wave numbers seem to dominate in the convective spectrum, which may relate to very large scale solar features such as sectors (and may make it possible to describe the convection with a relatively small number of modes).

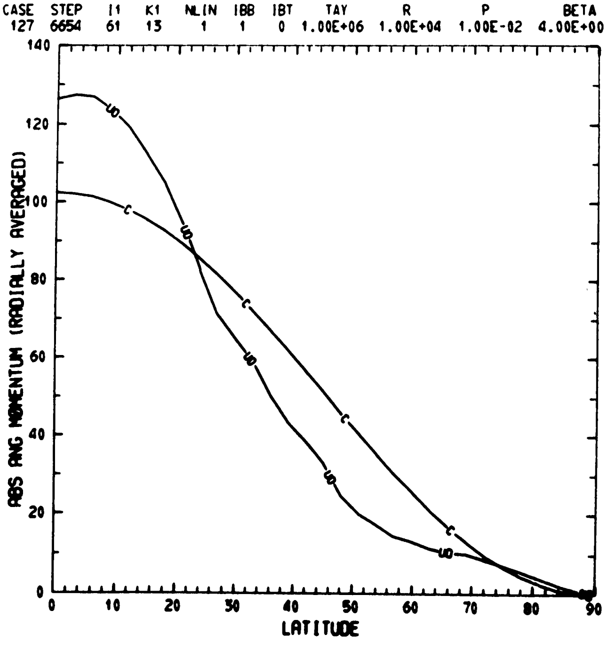

Fig. 14. Same as Figure 13 except for $P = 0.01$, $T = 10^6$, $R = 10^4$.

One caution is, however, that the uniqueness of the solutions for different initial conditions is in doubt. We did a few calculations at low P starting from a state of solid rotation and more arbitrary initial temperature perturbations. The resulting differential rotation is much smaller in amplitude than found from initial conditions from solutions at higher P; and moreover does not include an equatorial acceleration. Clearly, the presence or absence of shear in the initial conditions is important.

Now, the *molecular* Prandtl number for the Sun is very small compared to 1, but it is not clear how to justify a small P for such large-scale convection as we are dealing with, since the small scale turbulence should be taken into account. It is true that with several interacting modes present in the model at once, we are in effect including a nonlinear viscosity, which may suffice to represent this process.

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5.6. Further model studies and improvements

Clearly, the next step in the context of the Boussinesq model is to look carefully at the small Prandtl number nonlinear solutions and to also look carefully at the effect of the presence or absence of differential rotation in the initial conditions, even for $P = 1$. Assuming we find reasonable analogues for the solar problem, their dynamo properties should then be tested; we hope to generalize our model to do this within six months. Finally, of course, we need to modify the model to include a number of new physical effects, the principal one being some approximation to the large density variation with radius characteristic of the real compressible convection zone of the Sun.

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References

Busse, F.: 1970a, *Astrophys. J.* **159**, 629.
Busse, F.: 1970b, *J. Fluid Mech.* **44**, 441.
Busse, F.: 1973, *Astron. Astrophys.* **28**, 27.
Durney, B.: 1970, *Astrophys. J.* **161**, 1115.
Durney, B.: 1971, *Astrophys. J.* **163**, 353.
Gierasch, P. J.: 1975, *Astrophys. J.* **190**, 199.
Gilman, P. A.: 1972, *Solar Phys.* **27**, 3.
Gilman, P. A.: 1975, *J. Atmospheric Sci.* **32**, 1331.
Hoel, P. G.: 1947, *Introduction to Mathematical Statistics.* John Wiley, New York, 331 pp.
Howard, R. and Harvey, J.: 1970, *Solar Phys.* **12**, 23.
Roberts, P. H.: 1968, *Phil. Trans. Roy. Soc.* (London) Series A, **263**, 93.
Schwarzchild, M.: 1975, *Astrophys. J.* **195**, 137.
Simon, G. W. and Weiss, N. O.: 1968, *Z. Astrophys.* **69**, 435.
Ward, F.: 1965, *Astrophys. J.* **141**, 534.
Wolff, C. L.: 1975, *Solar Phys.* **41**, 297.
Yoshimura, H.: 1971, *Solar Phys.* **18**, 417.
Yoshimura, H. and Kato, S.: 1971. *Publ. Astron. Soc. Japan* **23**, 57.
Yoshimura, H.: 1972, *Solar Phys.* **22**, 20.
Yoshimura, H.: 1974, *Publ. Astron. Soc. Japan* **26**, 9.

DISCUSSION

Weiss: First let me comment on the dual role played by viscosity. As well as providing friction, viscosity governs diffusion of angular momentum. In the Boussinesq approximation, a low Prandtl number is necessary to permit redistribution of angular momentum and allow a finite amplitude instability. So it may be necessary to introduce a low Prandtl number in order to permit physical processes that we might expect in the Sun. Hence, it is not so terrible if you need a turbulent Prandtl number less then unity.

Secondly, have you tried using a fixed flux (rather than a fixed temperature) thermal boundary condition at the base of the convecting layer?

And how large a variation of heat flux with latitude appears in your models?

Gilman: We have not tried constant heat flux boundary conditions but what should occur is that large latitudinal temperature gradients on the boundaries appear. When equatorial modes dominate, the differential in heat flux can be as large as a factor of 3.

Stepanov: Is it possible to get accordance with observations putting convection zone depth a function of latitude?

Gilman: I do not know, but it is an interesting possibility, one that has occurred to us. It seems unlikely the unstable convection layer depth could vary much with latitude, but perhaps the amount convection overshoots into the interior does.

Vandakurov: It is known that the solar magnetic field which is extended over a great distance from the Sun and which in all probability is rooted in the deep interior of the convection zone has rigid rotation. Do you not think that this experimental evidence is in contradiction to your theory? Is the convection determined by the rotation rather than by superdiabaticity of the medium in your case?

Gilman: The model is not accurate enough yet to say under what circumstances we should expect solid rotation at the bottom of the convecting layer. Also compressibility may be very important for this particular question, since the inertia of the bottom layers becomes so high compared to the top, contrary to the incompressible case. Finally, I note that large scale magnetic and velocity disturbances can still rotate rigidly even in the presence of a differential rotation.

Superdiabaticity is required for the convection to occur, but rotation modifies it significantly.

Deinzer: Can you incorporate a core rotating faster than the convective envelope? I am asking because this is perhaps a possibility to get the observed butterfly diagram despite an angular velocity decreasing inward through the convection zone.

Are there other possibilities to get the desired butterfly diagram?

Gilman: If we impose a fast-rotating core and nonslip bottom boundary, the convecting layer will spin up to that rate. No really large radial gradient can be maintained. Weak radial gradients in angular velocity are possible. The dynamo properties of the convection and differential rotation in the model must be explicitly calculated to answer the question of whether the right butterfly diagram is produced.

Newkirk: You suggest that a way out of the dilemma of the reversal of equatorial acceleration might be to introduce a low Prandtl number into the model after the convection has been established. What would be the physical origin of such a temporal change?

Gilman: The point really is that if the initial condition contained significant differential rotation, it may remain indefinitely, and be different from what would be produced starting from initial conditions with solid rotation. Our calculations indicate this happens at low P, but it may also happen at $P = 1$. This needs to be tested.

Stenflo: I think there is strong evidence from observations of the rotation of solar magnetic fields that the angular velocity increases with depth. The magnetic fields should reflect the situation in deeper layers where they are anchored or from which they are expelled, and they systematically rotate faster than the surface layers.

Gilman: This is one reasonable interpretation of the observations but it is also true that hydromagnetic disturbances can and do propagate at different rates from the flow in which they are embedded.

Roxburgh: I would like to question the case of the Boussinesq approximation for modelling the solar convective zone. In the eddy transport approximation it is entropy that is conducted, not temperature as you assume in your calculations. If you use the convection of entropy then the Rayleigh number is not necessarily large, the convective zone may be stable against the large scale convection, or perhaps marginally unstable. Indeed Unno showed some years ago that one can interpret the mixing length theory as the convective layer being marginally unstable.

Gilman: I agree we should include compressibility when possible, but the present model allows us first to gain an understanding of rotational effects in a more simple, but still relevant system. The change to entropy from temperature is not that troublesome, really. Your comment is really an indication of the limits of mixing-length theory, rather than evidence that giant cells should not exist in the solar convection zone.

Durney: I do not think that the solution of the heat flux problem lies in the choice of the Prandtl number. Furthermore it appears reasonable to consider the Boussinesq expression for the energy flux an approximation of the compressible mixing-length expression of convective heat flux. There is then no choice of the Prandtl number.

Schröter: After this interesting discussion between theoreticians now a question from an observer.

In your Figure 8 you show an example of meridional circulation with poleward motion on the surface and equatorward motion in deeper layers. We (Dr Wöhl and myself) observed in 1974 a predominant poleward motion of Ca^+-fine mottles. Now, my question is: What is the velocity of this poleward circulation at the surface, a tenth of a m s^{-1}, several m s^{-1}, or more? I should mention that the modern computer-controlled method in tracing solar fine structures as used by us is capable of measuring systematic motions above a few m s^{-1}.

Gilman: Typically a few meters per second and virtually always poleward flow at the surface when equatorial modes dominate. Such measurements as you mention will be extremely valuable.

Mestel: You remark the axisymmetric meridional circulation driven by the Reynolds' stresses persists in the limit of zero rotation. What determines the axis if $\Omega = 0$? Is this another possible example of different initial conditions determining different asymptotic states?

Gilman: The meridional circulation persists provided a vestige of mode selectivity, present when rotation is small but not zero, remains. Indeed, initial conditions may be very important in determining the final state.

Howard: How will you handle temporal changes in the latitude dependence of the rotation within the content of your model?

Gilman: The calculations are themselves done as a function of time, so time variations, due to interactions among modes, will occur naturally. We intend to study the time dependent behavior more carefully soon.