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ON THE FIRST CONJUGATE POINT FUNCTION FOR NONLINEAR DIFFERENTIAL EQUATIONS

BY
ALLAN C. PETERSON AND DWIGHT V. SUKUP

ABSTRACT. We are concerned with the nth order differential equation $y^{(n)}=f(x, y, y', \ldots, y^{(n-1)})$, where it is assumed throughout that f is continuous on $[\alpha, \beta)\times R^n$, $\alpha<\beta\leq\infty$, and that solutions of initial value problems are unique and exist on $[\alpha, \beta)$. The definition of the first conjugate point function $\eta_1(t)$ for linear homogeneous equations is extended to this nonlinear case. Our main concern is what properties of this conjugacy function are valid in the nonlinear case.

We will be concerned with the nth order nonlinear differential equation

$$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}) \tag{1}$$

where $x \in [\alpha, \beta)$, $\alpha<\beta\leq\infty$. We will always assume that f satisfies

(A) f is continuous on $[\alpha, \beta)\times R^n$, and

(B) solutions of initial value problems (IVP's) are unique and extend to $[\alpha, \beta)$.

Sometimes we will further assume that

$$f_k(x, y, y', \ldots, y^{(n-1)}) \equiv \frac{\partial f(x, y, \ldots, y^{(n-1)})}{\partial y^{(k)}}, \qquad k = 0, 1, \ldots, n-1$$

is continuous on $[\alpha, \beta)\times R^n$. In this case we will be interested in the so called [3] variational equation along a solution $y_0(x)$ of (1):

$$z^{(n)} = \sum_{k=0}^{n-1} f_k(x, y_0(x), y_0(x), \ldots, y_0^{(n-1)}(x))z^{(k)}. \tag{2}$$

Let $t \in [\alpha, \beta)$ and $R(t)=\{r>t$: there exist distinct solutions u, v of (1) such that $u-v$ has an $(i_1, \ldots, i_m)$-distribution of zeros on $[t, r]\}$. If $R(t)\neq\phi$ set $r_{i_1\ldots i_m}(t)=\inf R(t)$. If $R(t)=\phi$, set $r_{i_1\ldots i_m}(t)=\infty$. The first conjugate point $\eta_1(t)$ for equation (1) of $x=t$ is defined by

$$\eta_1(t) = \min\left\{r_{i_1\ldots i_m}(t): \sum_{k=1}^{m} i_k = n\right\}.$$

We will let $\eta_1(t; y_0(x))$ denote the first conjugate point for equation (2) of $x=t$. We say (1) is disconjugate on a subinterval I of $[\alpha, \beta)$ provided there do not exist distinct solutions u, v of (1) such that $u-v$ has at least n zeros, counting multiplicities on I.

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Recent results show that the conjugacy function $\eta_1(t)$ for the nonlinear equation (1) has analogous properties as in the linear case. For example it is well known [2], [10] that for linear differential equations that

$$\eta_1(t) = r_{1\ldots 1}(t) \tag{3}$$

$$\eta_1(t) = \min\{r_{ij}(t): i+j = n\}. \tag{4}$$

Jackson [6] has recently proved (3) for the nonlinear case (1). For $n=3$, Jackson [7] proved (4) for the nonlinear case, while for $n=4$, D. Peterson [9] proved (4) for the nonlinear case.

In the linear case it is well known that $\eta_1(t)>t$. This is not true in general in the nonlinear case (see Theorem 2 and examples 3 and 4). However, if f satisfies a uniform Lipschitz condition with respect to $y, y', \ldots, y^{(n-1)}$ on compact subintervals of $[\alpha, \beta)$, then using bounds for the Green's function [5] and the fact that $\eta_1(t)=r_{1\ldots 1}(t)$ one can use standard fixed point arguments to prove $\eta_1(t)>t$ for all $t \in [\alpha, \beta)$. We will sometimes assume $\eta_1(t)>t$. If $f(x, y, \ldots, y^{(n-1)})= f(x, y)$ satisfies (A) and (B) and

$$|f(x, y)-f(x, z)| \leq K\,|y-z|$$

for (x, y), $(x, z) \in [\alpha, \beta)\times R$ then using the contraction mapping principle and Beesack's inequality [1] one can show that

$$\eta_1(t) \geq t+\left\{\frac{n!}{K}\right\}^{1/n}\left\{\frac{n}{n-1}\right\}^{1-(1/n)}.$$

The following lemma is useful for giving examples and for actually calculating $\eta_1(t)$.

LEMMA 1. *Assume $n=2$, $\eta_1(t)>t$, and $y(x, m)$ is the solution of the IVP* (1), *$y(t, m)=A$, $y'(t, m)=m$. Then*

$$\lim_{m\to\pm\infty} y(x, m) = \pm\infty,$$

respectively, uniformly on compact subsets of $(t, \eta_1(t))$.

Proof. Since (1) is disconjugate on $[t, \eta_1(t))$ we have for $m_2>m_1$ that $y(x, m_2)> y(x, m_1)$, $x \in (t, \eta_1(t))$. We claim that $\lim_{m\to\infty} y(x, m)=\infty$ for each x in $(t, \eta_1(t))$. To see this let $x_0 \in (t, \eta_1(t))$ and $K>0$ be given. But then there is [8] an m_0 such that $y(x_0, m_0)=K$. Hence $y(x_0, m)\geq K$ for all $m\geq m_0$. Since K and x_0 are arbitrary our claim is verified. It then follows by use of Dini's theorem that $\lim_{m\to\infty} y(x, m)= \infty$ uniformly on compact subsets of $(t, \eta_1(t))$. Similarly $\lim_{m\to-\infty} y(x, m)=-\infty$ uniformly on compact subsets of $(t, \eta_1(t))$.

THEOREM 2. *Assume $n=2$, $f(x, y, y')=f(x, y)$, the partial derivative $f_0(x, y)$ is continuous on* $[\alpha, \beta)\times R$ *and* $\lim_{y\to\infty} f_0(x, y)=-\infty$ (*or* $\lim_{y\to-\infty} f_0(x, y)=-\infty$) *uniformly on compact subsets of* (α, β), *then $\eta_1(t)=t$ for $t \in [\alpha, \beta)$.*

Proof. Assume there is a $t \in [\alpha, \beta)$ such that $\eta_1(t) > t$. Let $t < t_1 < t_2 < \eta_1(t)$. Let $y(x, m)$, $m \in R^1$ be the solution of (1) with $y(t, m) = A$, $y'(t, m) = m$. Then by Lemma 1 $\lim_{m\to\infty} y(x, m) = \infty$ uniformly on $[t_1, t_2]$. By Corollary 1.14 [12]

$$\eta_1(t_1) = \inf \eta_1(t_1; y_0(x))$$

where the infimum is taken over solutions $y_0(x)$ of (1). Assume $\lim_{y\to\infty} f_0(x, y) = -\infty$ uniformly on compact subsets of (α, β), then by applying Sturm's comparison theorem to the equation

$$z'' - f_0(x, y_m(x))z = 0$$

we get that for m sufficiently large

$$\eta_1(t_1; y(x, m)) < t_2.$$

It follows that $\eta_1(t) < t_2$ which is a contradiction.

It is well known [10] that in the linear case, $\eta_1(t)$ is strictly increasing. Using Theorem 2 one can easily verify the following example which shows that $\eta_1(t)$ need not be strictly increasing in the nonlinear case. (The reader should verify that (A) and (B) are satisfied.)

EXAMPLE 3.

$$y'' = \begin{cases} -ty^3, & t \geq 0 \\ 0, & t \leq 0. \end{cases}$$

In this case $\eta_1(t) = 0$ for $t \leq 0$ and $\eta_1(t) = t$ for $t \geq 0$.

In the linear case it is well known [10] that $\eta_1(t)$ is a continuous function of t. The following example shows that this need not be true in the nonlinear case. Again one can verify this example by using Theorem 2. Later we will see that $\eta_1(t)$ is an upper semicontinuous function of t.

EXAMPLE 4.

$$y'' = \begin{cases} ty^3, & t \leq 0 \\ -ty, & t \geq 0. \end{cases}$$

In this case $\eta_1(t) = t$, $t < 0$ and $0 < \eta_1(0) < \infty$.

Assume $n = 2$, $f(x, y, y') = f(x, y)$ and in addition to (A) and (B) being satisfied, $f_0(x, y)$ is continuous on $[\alpha, \beta) \times R$ and

$$f_0(x, y) \geq p(x) \qquad x \in [\alpha, \beta)$$

where $p(x)$ is continuous on $[\alpha, \beta)$. Let $\bar{\eta}_1(t)$ be the first conjugate point of $x = t$ for the differential equation

$$y'' - p(x)y = 0.$$

It is well known (see sections 6 and 7 in [4]) that $\eta_1(t) \geq \bar{\eta}_1(t)$. The novelty in Remark 5 is that we give conditions under which $\eta_1(t) = \bar{\eta}_1(t)$. We are not trying to be as general as possible but we want to show how to construct examples.

REMARK 5. Assume in addition to the previous discussion that

$$\lim_{y\to\infty} f_0(x, y) = p(x)$$

uniformly on compact subsets of $[\alpha, \beta)$. Then

$$\eta_1(t) = \bar{\eta}_1(t).$$

Proof. It suffices to show that the assumption $\eta_1(t)<\bar{\eta}_1(t)$ leads to a contradiction. Pick $t_1>t$ sufficiently close that $\bar{\eta}_1(t_1)<\eta_1(t)$. Let $y(x, m)$ be the solution of (1) with $y(a, m)=A$, $y'(a, m)=m$, $m \in R^1$. It follows from Lemma 1 that

$$\lim_{m\to\infty} f_0(x, y(x, m)) = p(t)$$

uniformly on $[t_1, \bar{\eta}_1(t_1)]$. By [11] the continuity of η_1 for linear equations with respect to coefficients

$$\lim_{m\to\infty} \eta_1(t, y(x, m)) = \bar{\eta}_1(t_1).$$

Hence there is an m_1 such that

$$\eta_1(t_1, y_{m_1}(t)) < \eta_1(t)$$

which leads to a contradiction.

Remark 5 gives us immediately the following example which appears in [4] and [13].

EXAMPLE 6. $y''=-y+\operatorname{Arctan} y$. In this case $\eta_1(t)=t+\pi$ for $t \in (-\infty, \infty)$.

In Example 6, $f_0(t, y)=-1+(1/(1+y^2))>-1$ for all y. Hence this equation is disconjugate on $[t, \eta_1(t)]$. This is unlike the linear case where it is well known that there always exists distinct solutions whose difference has at least n zeros on $[t, \eta_1(t)]$.

Sherman [11] showed that if equation (1) is a linear homogeneous differential equation then for every $\varepsilon>0$ there exists a solution z of (1) such that z has at least n zeros in $[t, \eta_1(t)+\varepsilon)$ the first n of which are simple zeros and the first being at $x=t$. Since $\eta_1(t)=r_{1\ldots1}(t)$, given $\varepsilon>0$ there exist distinct solutions y, z of (1) such that $y-z$ has at least n distinct zeros in $[t, \eta_1(t)+\varepsilon)$. With the aid of certain additional hypotheses we will show for the nonlinear case (1) that for every $\varepsilon>0$ there exist distinct solutions y, z of (1) such that $y-z$ has at least n simple zeros in $[t, \eta_1(t)+\varepsilon)$ the first n zeros of $y-z$ in $[t, \eta_1(t)+\varepsilon)$ being simple zeros. In the case $n=2$ we further show that the first of these zeros is at $x=t$. In the case $n=3$ if we assume that f has continuous partial derivatives with respect to y, y', y'' we also show that the first of these zeros occurs at $x=t$ by working with the linear equations of variation along solutions of equation (1).

A somewhat lengthy proof of the following lemma is given in the second author's doctoral thesis [14]. To save journal space we merely state this result.

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LEMMA 7. *If equation* (1) *satisfies* (A) *and* (B) *and* $\eta_1(t)<\infty$, *then for every* $\varepsilon>0$ *there exist distinct solutions* $y(x)$ *and* $z(x)$ *of* (1) *such that* $y(x)-z(x)$ *has at least n odd ordered zeros in* $[t, \eta_1(t)+\varepsilon)$.

THEOREM 8. *Assume that equation* (1) *satisfies* (A) *and* (B). *Then* $\eta_1(t)$ *is upper semicontinuous.*

Proof. Since $\eta_1(t)$ is a nondecreasing function of t it suffices to show that there does not exist a sequence $\{t_k\}$ and an $\varepsilon>0$ such that $t_k>t$, $k\geq 1$, and $\lim_{k\to\infty} t_k=t$ with $\eta_1(t_k)\geq\eta_1(t)+\varepsilon$ for every k. Assume that on the contrary there exists such a sequence $\{t_k\}$ and an $\varepsilon>0$ such that $\eta_1(t_k)\geq\eta_1(t)+\varepsilon$ for all k. For this ε it follows by Lemma 7 that there must exist distinct solutions $y(x)$ and $z(x)$ of (1) such that $y(x)-z(x)$ has at least n exact odd ordered zeros in $[t, \eta_1(t)+\varepsilon)$.

If the first zero of $y(x)-z(x)$ were to occur at x_1 where $x_1>t$, then $t_k<x_1$ for k sufficiently large and hence $y(x)-z(x)$ must have at least n distinct odd ordered zeros in $[t_k, \eta_1(t_k))$ which is not possible. Hence the first zero of $y(x)-z(x)$ must occur at t. Let $y_\delta(x)$ be the solution of (1) such that

$$y_\delta(t+\delta) = z(t+\delta)$$
$$y_\delta^{(l)}(t+\delta) = y^{(l)}(t+\delta), \qquad l = 1,\ldots,n-1.$$

By the continuous dependence of solutions of (1) on initial conditions, if δ is sufficiently small, then $y(x)-z(x)$ must have a zero at $t+\delta$ together with at least $n-1$ other odd ordered zeros in $(t+\delta, \eta_1(t)+\varepsilon)$. However, again we can choose k large enough that $t_k<t+\delta$ and have that $y(x)-z(x)$ must have at least n distinct zeros in $[t_k, \eta_1(t_k))$. Again this is clearly not possible and we conclude that $\eta_1(t)$ must be upper semicontinuous.

THEOREM 9. *Assume that* $t<\eta_1(t)<\eta_1(\eta_1(t))\leq\infty$ *and that equation* (1) *satisfies* (A) *and* (B). *Then for every* $\varepsilon>0$ *there exist distinct solutions* $y(x)$ *and* $z(x)$ *of* (1) *such that* $y(x)-z(x)$ *has at least n distinct zeros in* $[t, \eta_1(t)+\varepsilon)$ *the first n of which are simple zeros.*

Proof. Let $\varepsilon>0$ be given such that $\eta_1(t)+\varepsilon<\eta_1(\eta_1(t))$. By Lemma 7 there exist distinct solutions $y(x)$ and $z(x)$ of (1) such that $y(x)-z(x)$ has at least n odd ordered zeros in $[t, \eta_1(t)+\varepsilon)$. Assume that $y(x)-z(x)$ has a multiple zero in $[t, \eta_1(t)+\varepsilon)$. We apply Lemma 2.11 [13] to obtain a solution $u_1(x)$ of (1) such that $u_1(x)-y(x)$ has at least $n+1$ odd ordered zeros in $(t, \eta_1(t)+\varepsilon)$. If $u_1(x)-y(x)$ has a multiple zero in $[t, \eta_1(t)+\varepsilon)$ we apply the same argument to $u_1(x)-y(x)$ to obtain a solution $u_2(x)$ of (1) such that $u_2(x)-y(x)$ has at least $n+2$ odd ordered zeros in $[t, \eta_1(t)+\varepsilon)$. Since the difference of two distinct solutions of (1) cannot have more than $2n-1$ zeros, counting multiplicities, on $[t, \eta_1(t)+\varepsilon)$ there is a k, $0<k<n$, such that $u_k(x)-y(x)$ has at least $n+k$ simple zeros in $[t, \eta_1(t)+\varepsilon)$ and no other zeros.

THEOREM 10. *If* $n=2$ *and* (A) *and* (B) *are satisfied then for every* $\varepsilon>0$ *there exist distinct solutions* $y(x)$ *and* $z(x)$ *of* (1) *such that* $y(x)-z(x)$ *has at least two zeros* (*which must be simple zeros*) *in* $[t, \eta_1(t)+\varepsilon)$ *the first being at* $x=t$.

Proof. Suppose that for some $\varepsilon>0$ the conclusion of the theorem is not true. Let $y(x)$ and $z(x)$ be distinct solutions of (1) such that $y(x)-z(x)$ has a zero at $x_1 \in (t, \eta_1(t)+\varepsilon)$ and a zero at $x_2 \in (x_1, \eta_1(t)+\varepsilon)$. Without loss of generality we will assume that $y(x)<z(x)$ for $x \in (x_1, x_2)$. Let $u_\lambda(x)$ be the solution of (1) such that

$$u_\lambda(t) = y(t)$$
$$u_\lambda'(t) = y'(t)+\lambda$$

where $\lambda>0$. It follows that $u_\lambda(x)>y(x)$ for $x \in [t, \eta_1(t)+\varepsilon)$ by the way that ε was chosen. If λ is sufficiently small then there are points $s_\lambda, \tau_\lambda \in (x_1, x_2)$ such that $s_\lambda<\tau_\lambda$, $u_\lambda(s_\lambda)=z(s_\lambda)$, $u_\lambda(\tau_\lambda)=z(\tau_\lambda)$ and $u_\lambda(x)<z(x)$ for $x \in (s_\lambda, \tau_\lambda)$. Furthermore, if s_λ exists then τ_λ exists and s_μ exists for $0<\mu<\lambda$. Let $\Lambda=\sup\{\lambda>0 : s_\lambda \text{ exists}\}$. If $0<\lambda_1<\lambda_2<\Lambda$ then $s_{\lambda_1}<s_{\lambda_2}<\tau_{\lambda_2}<\tau_{\lambda_1}$. Let $s_\Lambda=\sup\{s_\lambda : \lambda \in (0, \Lambda)$ and $\tau_\Lambda=\inf\{\tau_\lambda : \lambda \in (0, \Lambda)$. It is clear that $s_\Lambda\leq\tau_\Lambda$.

If $\Lambda<+\infty$ and $s_\Lambda<\tau_\Lambda$ then by the continuous dependence of solutions of (1) on initial conditions and continuity we have that $u_\Lambda(s_\Lambda)=z(s_\Lambda)$, $u_\Lambda(\tau_\Lambda)$ and $u_\Lambda(x)\leq z(x)$ for $x \in (s_\Lambda, \tau_\Lambda)$. Hence by the uniqueness of solutions of IVP's for (1) it follows that $u_\Lambda(x)<z(x)$ for $x \in (s_\Lambda, \tau_\Lambda)$. Again by continuous dependence of solutions of (1) on initial conditions it follows that s_λ exists for $\lambda>\Lambda$ and $\lambda-\Lambda$ sufficiently small. But this contradicts the definition of Λ.

If $\Lambda<+\infty$ and $s_\Lambda=\tau_\Lambda$ it follows by continuous dependence of solutions of (1) on initial conditions and continuity that $u_\Lambda(s_\Lambda)=z(s_\Lambda)$ and $u_\Lambda'(s_\Lambda)=z'(s_\Lambda)$. But this contradicts the uniqueness of solutions of IVP's for (1).

If $\Lambda=+\infty$ we let $\{\lambda_k\}$ be a sequence of real numbers such that $\lim_{k\to\infty} \lambda_k=+\infty$. By Rolle's Theorem there exists for each $k>0$ a $\mu_k \in (s_{\lambda_k}, \tau_{\lambda_k})$ such that $u_{\lambda_k}'(\mu_k)=z'(\mu_k)$. The sequence $\{u_{\lambda_k}(\mu_k)\}$ is a bounded sequence of numbers hence we will assume without loss of generality that $\lim_{k\to\infty} \mu_k=\mu$ where $\mu \in [s_\Lambda, \tau_\Lambda]$ and $\lim_{k\to\infty} u_{\lambda_k}(\mu_k)=u_0$. By the continuity of $z'(x)$ we have that $\lim_{k\to\infty} u_{\lambda_k}'(\mu_k)=z'(\mu)$. Let $u(x)$ be the solution of (1) such that

$$u(\mu) = \mu_0$$
$$\mu'(\mu) = z'(\mu).$$

By the continuous dependence of solutions of (1) on inital conditions it follows that $\lim_{k\to\infty} u_{\lambda_k}'(x)=u'(x)$ uniformly on compact subintervals of $[\alpha, \beta)$. This clearly is not possible since $\lim_{k\to\infty} u_{\lambda_k}'(t)=+\infty\neq u'(t)$.

In the next theorem we will make the following compactness assumption on solutions of (1)

(C) If $\{y_k(x)\}$ is a sequence of solutions of (1) which is uniformly bounded on a compact subinterval $[c, d]$ of $[\alpha, \beta)$, then there is a subsequence

$\{y_{k_j}(x)\}$ such that $\{y_{k_j}^{(i)}(x)\}$ converges uniformly on each compact subinterval of $[\alpha, \beta)$ for each $i=0, 1, \ldots, n-1$.

THEOREM 11. *Let $n=3$ and assume that in addition to* (A) *and* (B), *condition* (C) *is satisfied. If $\eta_1(t)<r_{12}(t)$ then for every $\varepsilon>0$ there exist distinct solutions $y(x)$ and $z(x)$ of* (1) *such that $y(x)-z(x)$ has at least* 3 *zeros in $[t, \eta_1(t)+\varepsilon)$, the first three of which are simple zeros and the first of these being at $x=t$.*

Proof. Let $\varepsilon>0$ be given such that $\eta_1(t)+\varepsilon<r_{12}(t)$. By Lemma 7 there exist distinct solutions $y(x)$ and $z(x)$ of (1) such that $y(x)-z(x)$ has at least three odd ordered zeros in $[t, \eta_1(t)+\varepsilon)$. Let $t\le x_1<x_2<x_3<\eta_1(t)+\varepsilon$ and assume that $y(x_j)=z(x_j), j=1, 2, 3$. It is clear that $y(x)-z(x)$ has simple zeros at $x_j, j=1, 2, 3$ if these zeros are odd ordered zeros. If $x_1=t$ we have the desired result so assume that $x_1>t$. We will consider only the case where $y(t)<z(t)$ and assume that $y(x)>z(x)$ for $x \in (x_1, x_2)$ and that $y(x)<z(x)$ for $x \in (x_2, x_3)$. Let $u_\mu(x)$ be the solution of (1) such that

$$u_\mu^{(l)}(x_3) = y^{(l)}(x_3), \qquad l = 0, 1$$
$$u_\mu''(x_3) = y''(x_3)+\mu.$$

By Theorem 2.7 [15], $\{u_\mu(t):\mu \in R^1\}$ is an open interval, say (λ, γ). If $z(t) \in (\lambda, \gamma)$ then there must exist $\mu_1>0$ such that $u_{\mu_1}(t)=z(t)$. Furthermore $u_{\mu_1}(x)-z(x)$ has a simple zero at some point $s \in (x_2, x_3)$ and a simple zero at x_3. If $u_{\mu_1}(x)-z(x)$ has a simple zero at t we have the desired result. If the zero of $u_{\mu_1}(x)-z(x)$ at t is a double zero then a simple application of Lemma 2.11 [13] produces the desired result.

If $z(t)>\gamma$ then there must exist sequences $\{\overset{1}{s}_k\}$ and $\{\overset{3}{s}_k\}$ such that $t<\overset{1}{s}_k<\overset{3}{s}_k<x_3$, $\lim_{k\to\infty} \overset{1}{s}_k=t$, $\lim_{k\to\infty} \overset{3}{s}_k=x_3$, $u_k(\overset{1}{s}_k)=z(\overset{1}{s}_k)$ and $u_k(\overset{3}{s}_k)=z(s)$, $k\ge1$. By Rolle's Theorem there exists $\overset{2}{s}_k \in (\overset{3}{s}_k, x_3)$ for each k such that $u_k'(\overset{2}{s}_k)=z'(\overset{2}{s}_k)$. Furthermore, $\lim_{k\to\infty} u_k(\overset{2}{s}_k)=z(x_3)$. Since $t<\overset{1}{s}_k<\overset{2}{s}_k<r_{12}(t)$ for each $k\ge1$ it follows from Corollary 2.5 [15] that $\lim_{k\to\infty} u_k^{(\ell)}(x)=z^{(\ell)}(x)$ uniformly on compact subintervals of $[\alpha, \beta)$, $\ell=0, 1, 2$. This is clearly not possible since $\lim_{k\to\infty} u_k''(x_3)=+\infty$.

DEFINITION. Let $y(x; x_0, \overset{1}{y}_0, \overset{2}{y}_0, \ldots, \overset{n}{y}_0)$ be the solution of equation (1) which satisfies the initial conditions

$$y^{(l-1)}(x_0) = \overset{l}{y}_0, \qquad l = 1, \ldots, n.$$

DEFINITION. Let $\{u_j(x; x_0, y_0(x))\}_{j=1}^n$ be the fundamental set of solutions of equation (2) which satisfy the conditions

$$u_j^{(i-1)}(x_0; x_0, y_0(x)) = \delta_{ij}, \qquad i, j = 1, \ldots, n$$

where δ_{ij} is the Kronecker delta. We will sometimes denote the solution $u_j(x; x_0, y_0(x))$ of the variational equation (2) by $u_j(x; x_0, \overset{1}{y}_0, \ldots, \overset{n}{y}_0)$ where $y_0(x)$ is the solution of (1) such that $y_0^{(\ell-1)}=\overset{\ell}{y}_0$, $\ell=1, \ldots, n$.

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THEOREM 12. *Let $n=3$ and assume that $f(x, y, y', y'')$ has continuous partial derivatives with respect to y, y', and y'' on $[\alpha, \beta)\times R^3$. Then for every $\varepsilon>0$ there exist distinct solutions $y(x)$ and $z(x)$ of equation* (1) *such that $y(x)-z(x)$ has at least* 3 *zeros in $[t, \eta_1(t)+\varepsilon)$, the first three being simple zeros and the first occurring at $x=t$.*

Proof. Since

$$\eta_1(t) = \inf_{y_0(x)} \eta_1(t; y_0(x))$$

where $y_0(x)$ is a solution of (1), we have given $\varepsilon>0$. There is a solution $y_0(x)$ of (1) such that $\eta_1(t; y_0(x))<\eta_1(t)+\varepsilon$. Let $v(x)$ [11] be a solution of the variational equation (2) which has consecutive zeros at s_i, $i=1, 2, 3$, which are simple and where $t=s_1<s_2<s_3<\eta_1(t)+\varepsilon$. There are constants A, B such that

$$v(x) = Au_2(x; t, y_0(x))+Bu_3(x; t, y_0(x)).$$

Define $H_h(x)$ by

$$H_h(x) \equiv \frac{1}{h}[y(x; t, y_0^1, y_0^2+hA, y_0^3+hB)-y(x; t, y_0^1, y_0^2, y_0^3)].$$

Then

$$\begin{aligned} H_h(x) &= \frac{1}{h}[y(x; t, y_0^1, y_0^2+hA, y_0^3+hB)-y(x; t, y_0^1, y_0^2, y_0^3+hB) \\ &\quad +y(x; t, y_0^1, y_0^2, y_0^3+hB)-y(x; t, y_0^1, y_0^2, y_0^3)] \\ &= Au_2(x; t, y_0^1, y_0^2+\mu, y_0^3+hB)+Bu_3(x; t, y_0^1, y_0^2, y_0^3+\nu) \end{aligned}$$

where μ is between 0 and hA and ν is between 0 and hB. It follows that $H_h^{(i)}(x)$ converges uniformly to $v^{(i)}(x)$ as $h\to 0$ on compact subintervals of $[t, \eta_1(t)+\varepsilon)$. Hence for h sufficiently small it follows that

$$y(x; t, y_0^1, y_0^2+hA, y_0^3+hB)-y(x; t, y_0^1, y_0^2, y_0^3)$$

has a simple zero at $x=t$ together with an odd ordered zero in a neighborhood of each of the points s_2 and s_3. Pick $\delta>0$ sufficiently small that $\delta<\{\eta_1(t)+\varepsilon-s_3, \frac{1}{2}\min\{s_2-t, s_3-s_2\}\}$ and such that $v'(x)\neq 0$ for $x\in K\equiv[t, t+\delta]\cup[s_2-\delta, s_2+\delta]\cup[s_3-\delta, s_3+\delta]$. We can pick h sufficiently small that $H_h'(x)\neq 0$ for $x\in K$ and $H_h(x)\neq 0$ for $x\in[t+\delta, s_2-\delta]\cup[s_2+\delta, s^3-\delta]$. Hence for h sufficiently small $H_h(x)$ has a simple zero at $x=t$, a simple zero in $[s_2-\delta, s_2+\delta]$, a simple zero in $[s_3-\delta, s_3+\delta]$ and no other zeros in $[t, s_3+\delta]$.

REMARK 13. It is easy to see that if for $n>3$ it is true that

$$\eta_1(t) = \inf_{y_0(x)} \eta_1(t; y_0(x))$$

as conjectured in [12], then Theorem 12 generalizes to the nth order case.

BIBLIOGRAPHY

1. P. Beesack, *On the Green's function of an N-point boundary value problem*, Pac. J. of Math. **12** (1962), 801–812.

2. P. Hartman, *Unrestricted n-parameter families*, Rend. Circ. Mat. Palermo **7** (1958), 123–142.

3. P. Hartman, *Ordinary Differential Equations*, John Wiley, New York, 1964.

4. L. Jackson, *Subfunctions and second-order ordinary differential inequalities*, Advances in Math. **2** (1968), 307–363.

5. L. Jackson, Lecture notes given at the Univ. of Utah, 1971–2.

6. L. Jackson, *Uniqueness of solutions of boundary value problems for ordinary differential equations*, SIAM J. on Applied Math., **24** (1973), 535–538.

7. L. Jackson, *Uniqueness and existence of solutions of boundary value problems for third order ordinary differential equations*, J. of Diff. Eqs. **13** (1973), 432–437.

8. A. Lasota and Z. Opial, *On the existence and uniqueness of solutions of a boundary value problem for an ordinary second-order differential equation*, Colloquium Math. **18** (1967), 1–5.

9. D. Peterson, *Uniqueness, existence, and comparison theorems for ordinary differential equations*, Ph.D. thesis, Univ. of Nebraska, 1973.

10. T. Sherman, *Properties of solutions of Nth order linear differential equations*, Pac. J. of Math. **15** (1965), 1045–1060.

11. T. Sherman, *Conjugate points and simple zeros for ordinary linear differential equations*, Trans. AMS **146** (1969), 397–411.

12. J. Spencer, *Relations between boundary value functions for a nonlinear differential equation and its variational equations*, Canad. Math. Bull., to appear.

13. J. Spencer, *Boundary value functions for nonlinear differential equations*, J. of Diff. Eqs., to appear.

14. D. Sukup, *Boundary value problems for nonlinear differential equations*, Ph.D. thesis, Univ. of Nebraska, 1974.

15. D. Sukup, *On the existence of solutions of multipoint boundary value problems*, R. M. J. of Math, to appear.

University of Nebraska-Lincoln
Lincoln, Nebraska 68508

University of South Dakota
Vermillion, South Dakota 57069