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ANALYTIC FUNCTIONS WITH DECREASING COEFFICIENTS AND HARDY AND BLOCH SPACES

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Abstract The following rather surprising result is noted.

(1) A function $f(z) = \sum a_n z^n$ such that $a_n \downarrow 0$ $(n \to \infty)$ belongs to H^1 if and only if $\sum (a_n/(n+1)) < \infty$.

A more subtle analysis is needed to prove that assertion (2) remains true if H^1 is replaced by the predual, $\mathfrak{B}^1 (\subset H^1)$, of the Bloch space. Assertion (1) extends the Hardy–Littlewood theorem, which says the following.

(2) f belongs to H^p $(1 < p < \infty)$ if and only if $\sum (n+1)^{p-2} a_n^p < \infty$.

A new proof of (2) is given and applications of (1) and (2) to the Libera transform of functions with positive coefficients are presented. The fact that the Libera operator does not map H^1 to H^1 is improved by proving that it does not map $\mathfrak{B}^1$ into H^1.

Keywords: analytic functions; Hardy spaces; Bloch type spaces; Libera operator

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1. Introduction and result

One of many important results of Hardy and Littlewood states that a function

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

that is analytic in the unit disc $\mathbb{D}$ satisfies the conditions

$$f \in H^p \implies \sum_{n=0}^{\infty} (n+1)^{p-2} |a_n|^p < \infty \quad (0 < p \leqslant 2) \tag{1.1}$$

and

$$\sum_{n=0}^{\infty} (n+1)^{p-2} |a_n|^p < \infty \implies f \in H^p \quad (2 \leqslant p < \infty). \tag{1.2}$$

Recall that the p-Hardy space H^p consists of those $f \in H(\mathbb{D})$ (i.e. the set of all functions analytic in $\mathbb{D}$) for which

$$\|f\|_p^p := \sup_{0<r<1} M_p^p(r, f) < \infty,$$

where

$$M_p^p(r, f) = \frac{1}{2\pi} \int_0^{2\pi} |f(r\mathrm{e}^{\mathrm{i}\theta})|^p \,\mathrm{d}\theta$$

(see [**3**]). In the case when $p = 2$, both the relations are consequences of (and are weaker than) Parseval's Theorem. As noted in [**14**], (1.1) can easily be deduced from the implication

$$f \in H^p \implies \int_0^1 M_2^p(r, f)(1 - r)^{-p/2} \,\mathrm{d}r < \infty \quad (0 < p < 2),$$

which is a special case of another theorem of Hardy and Littlewood [**3**, Theorem 5.11]. Then (1.2) is obtained from (1.1) by a duality argument. The converses of (1.1) and (1.2) do not hold in general, but Hardy and Littlewood showed in [**6**] (see [**23**, Chapter XII, Lemma 6.6]) that if the sequence $\{a_n\}$ decreases to zero, then the converses hold for all $p > 1$.

Theorem A. *Let $1 < p < \infty$. If $a_n \downarrow 0$ as $n \to \infty$, then the function $f(z) = \sum a_n z^n$ is in H^p if and only if*

$$\sum_{n=0}^{\infty} (n + 1)^{p-2} a_n^p < \infty. \tag{1.3}$$

The proof given in [**23**] and the proof that we present below depend heavily on the hypothesis $p > 1$, and they suggest, perhaps, that Theorem A does not hold for $p = 1$. However, we have the following.

Theorem 1.1. *If $a_n \downarrow 0$ as $n \to \infty$, then f is in H^1 if and only if*

$$\sum_{n=0}^{\infty} \frac{a_n}{n + 1} < \infty.$$

Moreover, there exists a constant C independent of $\{a_n\}$ such that

$$C^{-1}\|f\|_1 \leqslant \sum_{k=0}^{\infty} \frac{a_k}{k + 1} \leqslant C\|f\|_1. \tag{1.4}$$

Proof. The 'only if' part is contained in (1.1). To prove the 'if' part, write f as

$$f(z) = \sum_{k=0}^{\infty} (a_k - a_{k+1}) \sum_{j=0}^{k} z^j.$$

Hence, for $0 < r < 1$,

$$
\begin{aligned}
2\pi M_1(r,f) &= \int_0^{2\pi} |f(re^{i\theta})|\,d\theta \\
&\leqslant \sum_{k=0}^{\infty}(a_k - a_{k+1}) \int_0^{2\pi} \left| \sum_{j=0}^{k} r^j e^{ij\theta} \right| d\theta \\
&\leqslant \sum_{k=0}^{\infty}(a_k - a_{k+1}) \int_0^{2\pi} \left| \sum_{j=0}^{k} e^{ij\theta} \right| d\theta \\
&\leqslant C \sum_{k=0}^{\infty}(a_k - a_{k+1}) \log(k+2),
\end{aligned}
$$

where we have used the following well-known estimate:

$$
\int_0^{2\pi} \left| \sum_{j=0}^{k} e^{ij\theta} \right| d\theta \leqslant C \log(k+2),
$$

where C is an absolute constant. Next, we use the inequality

$$
\log(k+2) \leqslant C_1 \sum_{j=0}^{k} \frac{1}{j+1}
$$

to obtain

$$
\begin{aligned}
M_1(r,f) &\leqslant C_2 \sum_{k=0}^{\infty}(a_k - a_{k+1}) \sum_{j=0}^{k} \frac{1}{j+1} \\
&= C_2 \sum_{j=0}^{\infty} \frac{1}{j+1} \sum_{k=j}^{\infty}(a_k - a_{k+1}) \\
&= C_2 \sum_{j=0}^{\infty} \frac{1}{j+1} a_j,
\end{aligned}
$$

as desired. (The existence of a constant C satisfying (1.4) follows from the proof.) $\square$

Remark 1.2. When writing the sum in (1.3) as

$$
\left(\sum_{n=0}^{\infty} ((n+1)a_n)^p (n+1)^{-2} \right)^{1/p}
$$

and letting $p \to \infty$, we get $\sup_n (n+1)a_n$. In this limiting case we have the following.

Let BMOA denote the space of analytic functions of bounded mean oscillation.

Theorem B (Xiao [22, Corollary 3.3.1]). *If $a_n \downarrow 0$, then f belongs to BMOA if and only if $\sup_n (n+1)a_n < \infty$.*

2. Proof of Theorem A

Let

$$\Delta_n(z) = \sum_{k\in I_n} z^k,$$

where

$$I_n = \{k\colon 2^{n-1} \leqslant k < 2^{n+1}\} \quad \text{for } n \geqslant 1, \qquad I_0 = 0.$$

Let $\Delta_n f$ denote the Hadamard product of Δ_n and f:

$$\Delta_n f(z) = \sum_{k\in I_n} a_k z^k.$$

The following useful fact was proved in [**14**]. The proof is relatively easy and is based on the Riesz Projection Theorem and a theorem on L^p-integrability of power series with positive coefficients [**13**, Theorem 1].

Theorem C. *Let $1 < p < \infty$ and $\alpha > -1$. The following quantities are then equivalent for $g \in H(\mathbb{D})$:*

$$Q_1(g) = \int_{\mathbb{D}} |g(z)|^p (1-|z|)^\alpha \,\mathrm{d}A(z),$$

$$Q_2(g) = \sum_{n=0}^{\infty} 2^{-n(\alpha+1)} \|\Delta_n g\|_p^p.$$

'Equivalent' means that $Q_1(g) < \infty \iff Q_2(g) < \infty$, and that $C^{-1}Q_1(g) \leqslant Q_2(g) \leqslant CQ_1(g)$, where C is independent of g. The latter is denoted as $Q_1(g) \asymp Q_2(g)$.

We need the following consequence of the Riesz Projection Theorem.

Lemma A. *Let $\lambda = \{\lambda_n\}$ be a monotone non-negative sequence, and let $\lambda g = \sum_{n=0}^\infty \lambda_n b_n z^n$, where $g(z) = \sum b_n z^n$. Then*

$$C^{-1}\lambda_{2^{n-1}} \|\Delta_n f\|_p \leqslant \|\Delta_n \lambda f\|_p \leqslant C\lambda_{2^n} \|\Delta_n f\|_p \qquad \text{if } \lambda \text{ is non-decreasing}$$

and

$$C^{-1}\lambda_{2^n} \|\Delta_n f\|_p \leqslant \|\Delta_n \lambda f\|_p \leqslant C\lambda_{2^{n-1}} \|\Delta_n f\|_p \quad \text{if } \lambda \text{ is non-increasing.}$$

This fact is familiar from the theory of Schauder bases. In our case the sequence $e_n(z) = z^n$ is, by the Riesz Theorem, a Schauder basis of H^p for $1 < p < \infty$ (but not for $p = 1$). The proof of the lemma is easy and is based on summation by parts and we therefore omit it (see [**11**, Proposition 1.a.3] concerning this point).

In [**17**], a short elementary proof, based on a simple version of Green's formula, of the following theorem of Littlewood and Paley [**12**] was given.

Theorem D. *If $p \geqslant 2$, and $f \in H^p$, then there is a constant C such that*

$$\int_{\mathbb{D}} |f'(z)|^p (1-|z|)^{p-1} \,\mathrm{d}A(z) \leqslant C\|f\|_p^p, \tag{2.1}$$

where $\mathrm{d}A$ is the Lebesgue measure in the plane.

In fact, in [**17**] a seemingly different result is proved: if $u = \operatorname{Re} f$, then

$$\int_{\mathbb{D}} |\nabla u(z)|^p (1-|z|)^{p-1} \, \mathrm{d}A(z) \leqslant C\|u\|_p^p,$$

but this is the same as (2.1) since $|\nabla u| = |f'|$, and $\|f\|_p \leqslant C\|u\|_p$ (by the Riesz Theorem).

Another useful fact is the following.

Lemma B (Mateljević and Pavlović [14, Lemma 3.1]). *If* $g(z) = \sum_{k=m}^n c_k z^k$, *then*

$$r^n \|g\|_p \leqslant M_p(r, g) \leqslant r^m \|g\|_p, \quad 0 < r < 1, \ 0 < p \leqslant \infty.$$

Proof of Theorem A. It follows from Theorem D, via Theorem C and Lemma A with $\lambda_k = k$, that

$$\sum_{n=0}^{\infty} \|\Delta_n f\|_p^p \leqslant C\|f\|_p^p, \quad 2 < p < \infty. \tag{2.2}$$

This holds for all $f \in H^p$. If $a_n \downarrow 0$, then we again appeal to Lemma A, with $\lambda_n = a_n$, to obtain

$$C^{-1} a_{2^n} \|\Delta_n\|_p \leqslant \|\Delta_n f\|_p \leqslant C a_{2^{n-1}} \|\Delta_n\|_p.$$

In order to finish the proof (for $p > 2$) we need to estimate $\|\Delta_n\|_p$. First we have $\|\Delta_n\|_\infty = 2^{n-1}$ for $n \geqslant 1$. Then we use the inequality

$$M_\infty(r, g) \leqslant C(1-r)^{-1/p} M_p(r, g), \quad 0 < r < 1,$$

together with Lemma B, with $r = 1 - 2^{-n-1}$, to find that $\|\Delta_n\|_\infty \leqslant C 2^{n/p} \|\Delta_n\|_p$, which implies $\|\Delta_n\|_p \geqslant C^{-1} 2^{n(1-1/p)}$.

In the other direction, let $g(z) = (1-z)^{-1}$ and $g_r(z) = g(rz)$, $0 < r < 1$. By the Riesz Theorem, we have

$$\|\Delta_n g_r\|_p^p \leqslant C\|g_r\|_p^p \leqslant C(1-r)^{1-p}.$$

On the other hand, we have $r^{2^n} \|\Delta_n g\|_p = r^{2^n} \|\Delta_n\|_p \leqslant \|\Delta_n g_r\|_p$, from which we obtain, by taking $r = 1 - 2^{-n-1}$, that $\|\Delta_n\|_p \leqslant C 2^{n(1-1/p)}$. Thus,

$$\|\Delta_n\|_p \asymp 2^{n(1-1/p)}, \quad p > 1. \tag{2.3}$$

Combining this with (2.2) and (2) we get

$$\sum_{n=1}^{\infty} a_{2^n}^p 2^{n(p-1)} \leqslant C\|f\|_p^p,$$

which, together with (1.2), proves Theorem A in the case $p > 2$.

If $1 < p < 2$, then we use the Riesz Projection Theorem and a duality argument to show that

$$\|f\|^p \leqslant C \sum_{n=0}^{\infty} \|\Delta_n f\|_p^p, \quad 1 < p < 2.$$

Then proceed as in the case $p > 2$ to complete the proof of Theorem A. $\square$

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3. On the predual of the Bloch space: Hardy–Bloch spaces*

Let $\mathfrak{B}^p$ $(1 \leqslant p \leqslant \infty)$ denote the space of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathfrak{B}_p} := |f(0)| + \biggl(\int_{\mathbb{D}} |f'(z)|^p (1-|z|)^{p-1}\,\mathrm{d}A(z)\biggr)^{1/p} < \infty.$$

In the case $p = \infty$ this quantity is interpreted as

$$\|f\|_{\mathfrak{B}^\infty} = |f(0)| + \sup_{z\in\mathbb{D}} |f'(z)|(1-|z|),$$

and hence $\mathfrak{B}^\infty$ coincides with the Bloch space $\mathfrak{B}$.

From Lemmas B and A we get

$$\|f\|_{\mathfrak{B}^p}^p \asymp \sum_{n=0}^{\infty} \|\Delta_n f\|_p^p, \quad 1 < p < \infty.$$

From this, Lemma A and (2.3) we obtain the following.

Theorem 3.1. *Let $1 < p < \infty$ and $a_n \downarrow 0$. The function $f \in \mathfrak{B}^p$* (if and only if $f \in H^p$) *if and only if*

$$\sum_{n=0}^{\infty} (n+1)^{p-2} a_n^p < \infty.$$

It is easily shown that if $a_n \downarrow 0$, then

$$f \in \mathfrak{B} \iff \sup_n (n+1)a_n < \infty,$$

and so $f \in \mathfrak{B}$ if and only if $f \in \mathrm{BMOA}$. Therefore, it is natural to ask what is happening with the space

$$\mathfrak{B}^1 = \biggl\{f\colon \int_{\mathbb{D}} |f'(z)|\,\mathrm{d}A(z) < \infty\biggr\}$$

normed by

$$\|f\|_{\mathfrak{B}^1} = |f(0)| + \int_{\mathbb{D}} |f'|\,\mathrm{d}A.$$

This space is the predual of the ordinary Bloch space with respect to the duality pairing

$$\langle f, g\rangle = \lim_{r\to 1^-} \sum_{n=0}^{\infty} \hat{f}(n)\hat{g}(n) r^n$$

(see, for example, [**19**]). It turns out that the situation is the same as in the case of H^1.

Theorem 3.2. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$, where $a_n \downarrow 0$. Then $f \in \mathfrak{B}^1$ if and only if*

$$\sum_{n=0}^{\infty} \frac{a_n}{n+1} < \infty,$$

* This term was introduced in [**5**].

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and we have

$$\|f\|_{\mathfrak{B}^1} \asymp \sum_{k=0}^{\infty} \frac{a_k}{k+1}.$$

The 'only if' part is true because of Theorem 1.1 and the inclusion $\mathfrak{B}^1 \subset H^1$.

In order to prove the 'if' part we need a decomposition of $\mathfrak{B}^1$. In [**8**], a sequence $\{V_n\}_0^\infty$ was constructed in the following way.

Let ω be a C^∞-function on $\mathbb{R}$ such that

(1) $\omega(t) = 1$ for $t \leqslant 1$,

(2) $\omega(t) = 0$ for $t \geqslant 2$,

(3) ω is decreasing and positive on the interval $(1, 2)$.

Let $\varphi(t) = \omega(\frac{1}{2}t) - \omega(t)$, let $V_0(z) = 1 + z$ and, for $n \geqslant 1$, let

$$V_n(z) = \sum_{k=0}^{\infty} \varphi(k/2^{n-1}) z^k = \sum_{k=2^{n-1}}^{2^{n+1}-1} \varphi(k/2^{n-1}) z^k.$$

These polynomials have the following properties:

$$g(z) = \sum_{n=0}^{\infty} V_n g(z) \quad \text{for } g \in H(\mathbb{D}), \tag{3.1}$$

$$\|V_n g\|_p \leqslant C \|g\|_p \quad \text{for } g \in H^p,\ p > 0, \tag{3.2}$$

$$\|V_n\|_p \asymp 2^{n(1-1/p)} \quad \text{for all } p > 0. \tag{3.3}$$

In [**9**, Lemma 2.1], the following analogue of Lemma B was proved.

Lemma C. *Let $0 < p < \infty$, let $\alpha > -1$ and let ν be a non-negative integer. A function $g \in H(\mathbb{D})$ satisfies the condition*

$$K_1(g) := \sum_{j=0}^{\nu-1} |g^{(j)}(0)| + \int_{\mathbb{D}} |g^{(\nu)}(z)|^p (1-|z|)^\alpha \, \mathrm{d}A(z) < \infty$$

if and only if

$$K_2(g) := \sum_{n=0}^{\infty} 2^{n(\nu p - \alpha - 1)} \|V_n g\|_p^p < \infty,$$

and we have $K_1(g) \asymp K_2(g)$.

(In the case $\nu = 0$, the sum $\sum_{j=0}^{-1}$ is interpreted as zero.)

Proof of Theorem 3.2. We have to prove that

$$\|f\|_{\mathfrak{B}^1} \leqslant C a_0 + C \sum_{n=1}^{\infty} a_{2^{n-1}}.$$

By Lemma C,

$$\|f\|_{\mathfrak{B}^1} \asymp |a_0| + \sum_{n=1}^{\infty} \|V_n f\|_1.$$

Let $n \geqslant 1$, let $m = 2^{n-1}$ and let $Q_k = \sum_{j=m}^{k} \varphi(j/m)e_j$. Since $Q_{4m-1} = V_n$, we have

$$\begin{aligned} V_n f &= \sum_{k=m}^{4m-1} \varphi(k/m) a_k e_k \\ &= \sum_{k=m}^{4m-1} (a_k - a_{k+1})Q_k + a_{4m}Q_{4m-1} \\ &= \sum_{k=m}^{4m-1} (a_k - a_{k+1})Q_k + a_{4m}V_n. \end{aligned}$$

On the other hand, $Q_k = V_n \Delta_{n,k}$, where

$$\Delta_{n,k} = \sum_{j=2^{n-1}}^{k} z^k, \quad 2^{n-1} \leqslant k \leqslant 2^{n+1}.$$

By (3.2), with $g = \Delta_{n,k}$, we have

$$\|Q_k\|_1 \leqslant C\|\Delta_{n,k}\|_1 \leqslant C \log(k + 1 - 2^{n-1}) \leqslant C(n + 1).$$

Combining these inequalities we get

$$\begin{aligned} \|V_n f\|_1 &\leqslant C \sum_{k=m}^{4m-1} (a_k - a_{k+1})(n + 1) + Ca_{4m}\|V_n\|_1 \\ &\leqslant C(n + 1)(a_m - a_{4m}) + Ca_{4m} \\ &= C(n + 1)(a_{2^{n-1}} - a_{2^{n+1}}) + Ca_{2^{n+1}}. \end{aligned}$$

Here we have used the relation $\|V_n\|_1 \leqslant C$ (see (3.3))! Thus,

$$\|V_n f\|_1 \leqslant C(n + 1)(a_{2^{n-1}} - a_{2^n}) + C(n + 1)(a_{2^n} - a_{2^{n+1}}) + Ca_{2^{n+1}},$$

and therefore it remains to compute the sums

$$S_1 = \sum_{n=1}^{\infty}(n + 1)(a_{2^{n-1}} - a_{2^n}) \quad \text{and} \quad S_2 = \sum_{n=1}^{\infty}(n + 1)(a_{2^n} - a_{2^{n+1}}).$$

In order to compute S_1, observe that from the convergence of $\sum(a_k/(k + 1))$ and the monotonicity of $\{a_n\}$ it follows that

$$\begin{aligned} 0 \xleftarrow[n\to\infty]{} \sum_{2^{n/2}<k\leqslant 2^n} \frac{a_k}{k + 1} &\geqslant a_{2^n} \sum_{2^{n/2}<k\leqslant 2^n} \frac{1}{k + 1} \\ &\geqslant c(n + 1)a_{2^n} \quad (c = \text{const.} > 0), \end{aligned}$$

and hence

$$\lim_{n\to\infty}(n+1)a_{2^n} = 0.$$

Using this and the formula

$$\sum_{n=1}^{\infty}(c_n - c_{n+1}) = c_1 \quad \text{if } c_n \to 0,$$

with $c_n = na_{2^{n-1}}$, we get

$$S_1 = \sum_{n=1}^{\infty}(na_{2^{n-1}} - (n+1)a_{2^n}) + a_{2^{n-1}} = a_1 + \sum_{n=1}^{\infty} a_{2^{n-1}}.$$

In a similar manner we get

$$S_2 = \sum_{n=1}^{\infty}(na_{2^n} - (n+1)a_{2^{n+1}}) + a_{2^n} = a_2 + \sum_{n=1}^{\infty} a_{2^n},$$

which completes the proof. □

Remark 3.3. The main points in the above proof are the relations $Q_{2^{n+1}-1} = V_n$ and $\|V_n\|_1 \asymp 1$. It is interesting to try to use the inequality

$$\|f\|_{\mathfrak{B}^1} \leqslant C\sum_{n=0}^{\infty}\|\Delta_n f\|_1.$$

Then, working as above, we get

$$\Delta_n f = \sum_{k=2^{n-1}}^{2^n-1}(a_k - a_{k+1})\sum_{j=2^{n-1}}^{k} e_k + a_{2^n}\Delta_n.$$

Now, instead of V_n we have Δ_n at the end, and application of the triangle inequality only yields

$$\|\Delta_n f\|_p \leqslant C(n+1)(a_{2^{n-1}} - a_{2^n}) + Ca_{2^n}(n+1).$$

The extra factor $n+1$ makes this attempt unsuccessful. Thus, in a sense, the above proof of Theorem 3.2 is accidental.

Remark 3.4. Lemma C was deduced in [**9**] from the case $\nu = 0$ (which is relatively easy to discuss) by using some non-trivial results of Hardy and Littlewood [**7**] and of Flett [**4**]. A simpler deduction is possible: see [**16**, Exercise 7.3.5].

Remark 3.5. The space $\mathfrak{B}^1$ is closely related to H^1 in that $H^1 \otimes H^1 = \mathfrak{B}^1$, where $X \otimes Y$ denotes the set of all $g \in H(\mathbb{D})$, which can be represented as

$$g = \sum_{n=0}^{\infty} h_n * k_n, \quad h_n \in X,\ k_n \in Y,$$

with $\sum\|h_n\|_X\|k_n\|_Y < \infty$ (see [**1**]).

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4. Libera transform of functions with positive coefficients

In [**10**], Libera introduced the operator

$$g(z) \mapsto \frac{2}{z} \int_0^z g(\zeta)\,\mathrm{d}\zeta$$

and demonstrated its importance in the theory of univalent functions. In particular, it was shown in [**10**] that this operator transforms the class of starlike functions into itself. The 'generalized' Libera operator

$$\Lambda_a g(z) = \frac{1}{a-z} \int_z^a g(\zeta)\,\mathrm{d}\zeta, \quad \text{where } |a| \leqslant 1,$$

was introduced and studied from the functional analytic point of view by Siskakis in [**20**, **21**], and was then studied further in [**2**, **15**, **18**] and other papers. The case $|a| < 1$ is not interesting because Λ_a is then defined on $H(\mathbb{D})$ and, on classical spaces, has almost the same linear topological properties as the integration operator $f(z) \mapsto \int_0^z f(\zeta)\,\mathrm{d}\zeta$. Therefore, we can assume that $a = 1$. Define $\mathcal{L} = \Lambda_1$, i.e.

$$\mathcal{L}f(z) = \frac{1}{1-z} \int_z^1 f(\zeta)\,\mathrm{d}\zeta.$$

The integral is not defined on $H(\mathbb{D})$: for example, $f(z) = 1/(1-z)$. However, if $f(z) = \sum a_n z^n$ and

$$\sum_{n=0}^{\infty} \frac{|a_n|}{n+1} < \infty, \tag{4.1}$$

then the integral exists as

$$\lim_{\mathbb{D} \ni w \to 1} \int_z^w f(\zeta)\,\mathrm{d}\zeta$$

(see [**15**]) and we have

$$\mathcal{L}f(z) = \sum_{n=0}^{\infty} b_n z^n,$$

where

$$b_n = \sum_{k=n}^{\infty} \frac{a_k}{k+1}.$$

Condition (4.1) is satisfied if $f \in H^1$ (Hardy's inequality) and consequently if $f \in H^p$, $p > 1$. Since $\mathfrak{B}^1 \subset H^1$, (4.1) is satisfied if $f \in \mathfrak{B}^1$.

If $a_n \geqslant 0$, then $b_n \downarrow 0$, and we can apply Theorems 1.1 and 3.2 to $\mathcal{L}f$.

Theorem 4.1. *Let $a_n \geqslant 0$ for all n. The following three conditions are then equivalent.*

$$\mathcal{L}f \in H^1, \tag{4.2}$$

$$\mathcal{L}f \in \mathfrak{B}^1, \tag{4.3}$$

$$\sum_{n=0}^{\infty} \frac{a_n \log(n+2)}{n+1} < \infty. \tag{4.4}$$

The proof is straightforward and is omitted here.

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It is known (and is easy to check) that $\mathcal{L}$ does not map H^1 into H^1 (see, for example, [**15**]). Moreover, we have the following corollary.

Corollary 4.2. *The operator $\mathcal{L}$ does not map $\mathfrak{B}^1$ ($\subsetneqq H^1$) into H^1.*

Proof. By Theorem 3.2, the function

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad \text{where } a_n = \frac{1}{\log^2(n+2)},$$

belongs to $\mathfrak{B}^1$. On the other hand, by Theorem 4.1, the function $\mathcal{L}f$ is not in H^1. □

Theorem 4.3. *Let $a_n \geqslant 0$, and let $1 < p < \infty$. Then $\mathcal{L}f$ is in H^p if and only if*

$$\sum_{n=0}^{\infty} 2^{-n} \biggl(\sum_{k \in I_n} a_k \biggr)^p < \infty. \tag{4.5}$$

For the proof we need a lemma.

Lemma 4.4. *Let $\{d_n\}_0^\infty$ be a non-negative sequence, let $\beta > 0$ and let $\gamma \geqslant 1$. Then*

$$\sum_{n=0}^{\infty} 2^{n\beta} |d_n|^\gamma \leqslant \sum_{n=0}^{\infty} 2^{n\beta} \biggl| \sum_{k=n}^{\infty} d_k \biggr|^\gamma \leqslant C \sum_{n=0}^{\infty} 2^{n\beta} |d_n|^\gamma,$$

where C is a constant independent of $\{d_n\}$.

Proof. The left inequality is clear. Let $\gamma \geqslant 1$. The other inequality is equivalent to the following:

$$\sum_{n=0}^{\infty} 2^{n\alpha\gamma} \biggl| \sum_{k=n}^{\infty} 2^{-k\alpha} s_k \biggr|^\gamma \leqslant C \sum_{n=0}^{\infty} |s_n|^\gamma, \tag{†}$$

where $\alpha > 0$ and $\{s_n\}$ is a sequence of complex numbers.

To prove this inequality we define the operator T by

$$T(\{s_n\}_0^\infty) = \{2^{\alpha n} t_n\}_0^\infty, \quad \text{where } t_n = \sum_{k=n}^{\infty} 2^{-\alpha k} s_k,$$

and consider the action of T on the spaces $L^\gamma(\mu, \mathbb{N})$, where $\mathbb{N}$ is the set of all positive integers and $\mu(\{n\}) = 2^{n\alpha}$. It is easy to show that T acts as a bounded operator from $L^\gamma(\mathbb{N}, \mu)$ to ℓ^γ, for $\gamma = 1$. The same holds for $\gamma = \infty$. Therefore, by the Riesz–Thorin Theorem, T maps ℓ^γ into ℓ^γ for $1 < \gamma < \infty$. Since the norms are independent of $\{c_n\}$, we get (†), which completes the proof. □

Remark 4.5. If $\sum_n |s_n|^\gamma < \infty$, then $\sum_n 2^{-n\alpha} |s_n| < \infty$ by Hölder's inequality.

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Proof of Theorem 4.3. By Theorem A, the function $\mathcal{L}f$ belongs to H^p if and only if

$$\sum_{n=0}^{\infty} 2^{n(p-1)} b_{2^n}^p < \infty. \tag{‡}$$

Since

$$b_{2^n} = \sum_{k=2^n}^{\infty} \frac{a_k}{k+1} = \sum_{j=n}^{\infty} c_j,$$

where

$$c_j = \sum_{k\in I_j} \frac{a_k}{k+1},$$

condition (‡) is, by Lemma 4.4 with $\beta = p-1$ and $\gamma = p$, equivalent to

$$\sum_{n=0}^{\infty} 2^{n(p-1)} c_n^p < \infty,$$

which is equivalent to

$$\sum_{n=0}^{\infty} 2^{n(p-1)} \biggl(2^{-n} \sum_{k\in I_n} a_k\biggr)^p < \infty.$$

This is obviously equivalent to (4.5), which was to be proved. □

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