COMMENTS ON THE PRECEDING PAPER OF MICHAEL'S

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(received 6 December 1963)

It is possible, by a different approach leading to a structure theorem for the left ideal K, to prove the main result of the preceding paper more simply and at the same time relaxing the conditions considerably. In particular we may drop the stipulation that S be separable and metric and one of the conditions (A) or (B) and replace the other by one of the weaker conditions (A′) or (B′) below:

(A′) *There is an element* $a \in K$ *such that the set* $A = \{x \in K : xa = a\}^*$ *is compact* [1].

(B′) *There is an element* $a \in K$ *such that the set* $B = \{x \in K : ax = a\}^*$ *is compact and non-empty.*

We note that both condition (A′) and (B′) are implied by (A) and (B) of Michael, although (B′) is not independent of (A), whereas (A′) is contained entirely in (A).

We prove the following:

THEOREM. *Let* S *be a locally compact semigroup with minimal left ideal* K *and having one of the properties* (A′) *or* (B′). *Then* K *is closed in* S *and isomorphic to* $E \times G$ *where* E *is a locally compact semigroup with multiplication* $xy = x$ *for all* $x, y \in E$, *and* G *is a locally compact topological group. If* (B′) *is satisfied, then* E *is even compact.*

A non-trivial right invariant measure u on K is obtained then by taking the product measure of right invariant Haar measure on G and any non-trivial Borel measure on E as is done for example by Rosen [**3**: p. 1081]. One then extends u to S in the usual way by $u^*(F) = u(F \cap K)$ for any Borel set $F \subset S$. From u^* one obtains a right invariant integral for S in the case where K is unique.

PROOF OF THE THEOREM. The sets A and B are semigroups if they are non-empty. By hypothesis, B is non-empty if condition (B′) is satisfied, and A is non-empty becauxe $yx = x$ has a solution for every $x \in K$ since

[1] * denotes the closure operation.

Kx is a left ideal for all $x \in K$ and thus $Kx = K$. Hence one or the other is a compact semigroup and therefore contains an idempotent (see for example [4]). By Theorem 1.27 of Clifford and Preston [1], K is algebraically a direct product of the set E of its idempotents with multiplication $xy = x$ and any subgroup maximal with respect to containing a given idempotent e. Moreover this group is precisely $eKe = eSe$, which, since it is maximal in S with respect to e being an identity, is a closed subset of S, and hence is locally compact. It follows from Ellis' theorem [2] that eSe is a locally compact group. That E is closed is virtually trivial because $xy = x$ for $x, y \in E$. It is now sufficient to show that $\varphi : E \times eSe \to K$ defined by $\varphi(f, g) = fg$ is continuous and the projection $\pi : K \to E$ defined by $\pi(x) = e$, where e is the identity for x, is continuous for then $\varphi^{-1} = \pi \times \gamma$, where $\gamma(k) = eke$, $k \in K$. That φ is continuous follows from the continuity of multiplication. We now show that π is continuous. We must show that, if $x_n \to x$ is a convergent net, then $\pi(x) = \lim_n \pi(x_n)$. Let V be a compact neighborhood of x in K. Now $\gamma : K \to \pi(x)K = \pi(x)K\pi(x)$ is a continuous homomorphism of K onto the maximal subgroup $\pi(x)K = \pi(x)K\pi(x)$ of $\pi(x)$. Further, for each $y \in K$, $y = \pi(y)\gamma(y)$. Hence, assuming $x_n \in V$, we have $\gamma(x_n) = \pi(x)\pi(x_n)\gamma(x_n) = \pi(x)x_n \in \pi(x)V$ which is compact. Hence, upon taking subnets, we have $\gamma(x_{\alpha(n)}) \to g \in \pi(x)K$. Since inversion is continuous in $\pi(x)K$, $\gamma(x_{\alpha(n)})^{-1} \to g^{-1}$, and it follows that $\pi(x_n) = x_n\gamma(x_n)^{-1} \to xg^{-1}$. But clearly idempotents can only converge to an idempotent, so that $xg^{-1} = \pi(x)$. This being true for each convergent subnet implies $\pi(x_n) \to \pi(x)$ then.

To show that K is closed, suppose the net $\{x_n\}$ converges to $x \in S$. Each $e \in E$ is a right identity for K, so that $x_n e = x_n$, and thus $xe = x$ which implies $x \in K$.

Now if (B') is satisfied, since $E \subset B$, E is compact.

Bibliography

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