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WEIGHTED CONVOLUTIONS OF CERTAIN POLYNOMIALS

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For $\alpha \geqslant 0$ and $\beta \geqslant 0$, let $K(\alpha, \beta)$ consist of those functions $f(z)$, analytic and non-zero in $|z| < 1$, such that for $\theta_1 < \theta_2 < \theta_1 + 2\pi$ and $0 < r < 1$, $-\alpha\pi \leqslant \arg f\left(re^{i\theta_2}\right) - \arg f\left(re^{i\theta_1}\right) + 1/2(\alpha - \beta)(\theta_1 - \theta_2) \leqslant \beta\pi$. It is conjectured that for $1 \leqslant \alpha \leqslant \beta$ and α an integer, the weighted convolution of polynomials having their zeros on $|z| = 1$ and belonging to $K(\alpha, \beta)$, also belong to $K(\alpha, \beta)$. This conjecture is known to be true for the case $\alpha = 1$, which leads to an alternative proof for the generalised Polya–Schoenberg conjecture. The case $\alpha = 2$ is also known to be true for cubic polynomials. We prove the conjecture for certain quartic polynomials when $2 \leqslant \alpha \leqslant 4$.

1. INTRODUCTION

1.1. For $\alpha \geqslant 0$ and $\beta \geqslant 0$, let the Kaplan class $K(\alpha, \beta)$, be the class of functions $f(z)$, analytic and non-zero in $|z| < 1$ such that for $\theta_1 < \theta_2 < \theta_1 + 2\pi$, and $0 < r < 1$,

$$-\alpha\pi \leqslant \arg f\left(re^{i\theta_2}\right) - \arg f\left(re^{i\theta_1}\right) + \frac{1}{2}(\alpha - \beta)(\theta_1 - \theta_2) \leqslant \beta\pi.$$

For $\alpha \geqslant 1$ and $\beta \geqslant 1$ where α is an integer, define

$$Q_n(z;\theta) = (1+z)^{\alpha-1} \prod_{j=1}^{n+1-\alpha} \left(1 + ze^{i(2j+\alpha-n-2)\theta}\right),$$

where $\theta = \pi/(n + \beta - \alpha)$ and $1 \leqslant \alpha \leqslant n$.

Recall that the Hadamard product or convolution of two power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ and } g(z) = \sum_{n=0}^{\infty} b_n z^n \text{ is } (f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

Define $Q_n^{(-1)}(z;\theta)$ so that $Q_n^{(-1)}(z;\theta) * Q_n(z;\theta) = 1/(1-z)$. The following is a special case of a conjecture given in [1].

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This work was begun while the author was visiting the Universityof Kentucky at Lexington, where he enjoyed many valuable discussions with Professor Ted J. Suffridge.

1.2. Conjecture. Let $1 \leqslant \alpha \leqslant n$ and $\alpha \leqslant \beta$ where α and n are integers. Assume that the polynomials

$$p_n(z) = \prod_{k=1}^{n} \left(1 + ze^{i\phi_k}\right) \text{ and } q_n(z) = \prod_{k=1}^{n} \left(1 + ze^{i\psi_k}\right)$$

belong to $K(\alpha, \beta)$. Then the polynomial

$$R_n(z) = (p_n * q_n)(z) * Q_n^{(-1)}(z;\theta)$$

also belongs to $K(\alpha, \beta)$.

This conjecture is proved for the case $\alpha = n - 1 = 2$ in ([**1**, Theorem 4]) and for the case $1 = \alpha \leqslant \beta$ by Suffridge ([**10**, Theorem 5]).

Since starlike functions of order $\gamma; \gamma \leqslant 1$, can be characterised as the class of limits of sequences of polynomials in $K(\alpha, \alpha + 2 - 2\gamma); \alpha \geqslant 1$ (see [**1**, Theorem 2] and [**10**, Theorem 1]), the above conjecture for $\alpha = 1$, leads to an alternative proof for the generalised Polya–Schoenberg conjecture [**3**]. The special cases $\gamma = 0$ and $1/2$ were first proved by Ruscheweyh and Sheil-Small [**6**] while the general case $\gamma \leqslant 1$ was obtained by Suffridge [**10**], and later by Lewis [**2**] and Ruscheweyh [**4**]. The truth of our conjecture for other values of α, that is $\alpha \neq 1$, may throw new light on a variety of old problems, because $K(\alpha, \beta)$ is closely related to several well-knnown classes of analytic and univalent functions. (For more details see [**5–9**]). In this note we prove the following

THEOREM 1.3. *Let $2 \leqslant \alpha \leqslant 4$ and $\alpha \leqslant \beta$ where α is an integer. Assume that the polynomials*

$$p_4(z) = \prod_{k=1}^{4} \left(1 + ze^{i\phi_k}\right) \text{ and } q_4(z) = \prod_{k=1}^{4} \left(1 + ze^{i\psi_k}\right)$$

belong to $K(\alpha, \beta)$ where $\phi_k = -\phi_{5-k}$ and $\psi_k = -\psi_{5-k}$. Then $R_4(z) = (p_4 * q_4)(z) * Q_4^{(-1)}(z;\theta)$ also belongs to $K(\alpha, \beta)$.

Note that Theorem 1.3 is true for $\alpha = 1$ by Suffridge ([**10**, Theorem 5]).

2. Some key lemmas

To prove our theorem we shall need the following lemmas, the first of which is a special case of Theorem 1 in [**1**].

LEMMA 2.1. *Let* $0 \leqslant \phi_1 \leqslant \phi_2 \leqslant \pi$. *Then for* $2 \leqslant \alpha \leqslant 4$ *and* $\alpha \leqslant \beta$, *the polynomial*

$$p_4(z) = \left(1 + ze^{i\phi_1}\right)\left(1 + ze^{i\phi_2}\right)\left(1 + ze^{i(2\pi-\phi_2)}\right)\left(1 + ze^{i(2\pi-\phi_1)}\right)$$

belongs to $K(\alpha, \beta)$ *if and only if* $0 \leqslant \phi_1 \leqslant (\beta\pi)/(4+\beta-\alpha)$ *and* $((4-\alpha)\pi)/(4+\beta-\alpha) \leqslant \phi_2 \leqslant \pi$.

LEMMA 2.2. *Let* $0 < c < 1$, $-c \leqslant x \leqslant 1$, $-c \leqslant u \leqslant 1$, $-1 \leqslant y \leqslant c$, $-1 \leqslant v \leqslant c$, $y \leqslant x$ *and* $v \leqslant u$. *Then*

$$F(x, y, u, v) = 2 + \frac{(x+y)^2(u+v)^2}{(1+c)^2} - \frac{2(1+2xy)(1+2uv)}{1+2c} \geqslant 0.$$

PROOF: The minimum of F occurs either at a critical point or on the boundary. By setting $\partial F/\partial x = \partial F/\partial y = 0$ we obtain either $1 + 2uv = 0$ or $\left((1+c)^2(1+2uv)\right)/\left((1+2c)(u+v)^2\right) = 1$. By setting $\partial F/\partial u = \partial F/\partial v = 0$ we obtain either $1 + 2xy = 0$ or $\left((1+c)^2(1+2xy)\right)/\left((1+2c)(x+y)^2\right) = 1$. If $1 + 2uv = 0$, then $F(x, y, u, v) = 2 + \left((x+y)^2(u+v)^2\right)/\left((1+c)^2\right) > 0$.

If $\left((1+c)^2(1+2uv)\right)/\left((1+2c)(u+v)^2\right) = 1$, then $x = y$ and so $F(x, y, u, v) = 2\{1 - (u+v)^2/(1+c)^2\} \geqslant 0$.

Similarly, $F(x, y, u, v) \geqslant 0$ when $1+2xy = 0$ or $(1+c)^2(1+2xy) = (1+2c)(x+y)^2$.

Now we check the values of $F(x, y, u, v)$ on the boundary. Since $F(x, y, u, v)$ is symmetric in (x, y) and (u, v), it would be enough to show that $F(x, y, u, v) \geqslant 0$ on the boundary involving x and y. Let $y = -1$. Then

$$F(x, -1, u, v) = \frac{(u+v)^2}{(1+c)^2}x^2 + 2\{\frac{2(1+2uv)}{1+2c} - \frac{(u+v)^2}{(1+c)^2}\}x + 2 + \frac{(u+v)^2}{(1+c)^2} - \frac{2(1+2uv)}{1+2c}.$$

Setting $dF(x, -1, u, v)/dx = 0$ we obtain $x_0 = 1 - 2(1+c)^2(1+2uv)/(1+2c)(u+v)^2$. But $F(x_0, -1, u, v) = 2\{1 + (1+2uv)/(1+2c)x_0\} \geqslant 0$ because we must have $-c \leqslant x_0 \leqslant 1$. Also observe that $F(-c, -1, u, v) = (u-v)^2 \geqslant 0$ and $F(1, -1, u, v) = 2\{1 + (1+2uv)/(1+2c)\} \geqslant 0$. Then $F(x, y, u, v) \geqslant 0$ when $y = -1$. If $x = 1$, then $F(1, y, u, v) = F(-y, -1, u, v) \geqslant 0$. Let $y = c$. Then

$$F(x, c, u, v) = \frac{(u+v)^2}{(1+c)^2}x^2 + 2c\{\frac{(u+v)^2}{(1+c)^2} - \frac{2(1+2uv)}{1+2c}\}x + 2 + \frac{c^2(u+v)^2}{(1+c)^2} - \frac{2(1+2uv)}{1+2c}.$$

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Letting $dF(x,c,u,v)/dx = 0$ we obtain $x_1 = c\{\left(2(1+c)^2(1+2uv)\right)/\left((1+2c)(u+v)^2\right) - 1\}$.

Observe that $F(x_1,c,u,v) \geqslant 0$ because we must have $y = c \leqslant x_1 \leqslant 1$. If $x = -c$, then by setting $dF(-c,y,u,v)/dy = 0$ we obtain $y_1 = -x_1$, and so $F(-c,y_1,u,v) = F(x_1,c,u,v) \geqslant 0$. If $x = y$, then it is easy to see that

$$F(x,x,u,v) = 2\{2[\frac{(u+v)^2}{(1+c)^2} - \frac{1+2uv}{1+2c}]x^2 + 1 - \frac{1+2uv}{1+2c}\} \geqslant 0.$$

Therefore $F(x,y,u,v)$ is never negative for the given x, y, u, v and c. ∎

The following Lemma can also be proved using a similar argument and so we omit its proof.

LEMMA 2.3. *Let $0 < c < 1$, $-c \leqslant x \leqslant 1$, $-c \leqslant u \leqslant 1$, $-1 \leqslant y \leqslant c$, $-1 \leqslant v \leqslant c$, $y \leqslant x$ and $v \leqslant u$. Then*

$$F(x,y,u,v) = 1 + \frac{(1+2xy)(1+2uv)}{1+2c} - \frac{2(x+y)(u+v)}{1+c} \geqslant 0.$$

LEMMA 2.4. *Let $0 < c < 1$, $-c \leqslant x \leqslant 1$, $-c \leqslant u \leqslant 1$, $-1 \leqslant y \leqslant c$, $1 \leqslant v \leqslant c$, $y \leqslant x$ and $v \leqslant u$. Furthermore, let*

$$-(1+c) \leqslant \frac{(x+y)(u+v)}{1+c} < -2c, \tag{2.4.1}$$

and

$$1 + 2c^2 < \frac{(1+2xy)(1+2uv)}{1+2c} \leqslant 1 + 2c. \tag{2.4.2}$$

Then

$$F(x,y,u,v) = \frac{(1+2xy)(1+2uv)}{1+2c} + \frac{2c(x+y)(u+v)}{1+c} + 2c^2 - 1 \leqslant 0.$$

PROOF: Since $(\partial^2F/\partial x^2)(\partial^2F/\partial y^2) - (\partial^2F/\partial x\partial y)^2 = -(\partial^2F/\partial x\partial y)^2 < 0$, the maximum of $F(x,y,u,v)$ occurs on the boundary. Note that $\partial^2F/\partial x\partial y$ cannot be equal to zero, because that contradicts (2.4.2). Observe that $F(x,y,u,v)$ is symmetric in (x,y) and (u,v). Therefore it would be enough to show that $F(x,y,u,v) \leqslant 0$ for the boundary values involving x and y. Since $(x+y)(u+v) < 0$, without loss of generality we assume that $x + y < 0$ and $u + v > 0$.

(i) If $x = -c$, then

$$F(-c,y,u,v) = 2c\{\frac{u+v}{1+c} - \frac{1+2uv}{1+2c}\}y + \frac{1+2uv}{1+2c} - \frac{2c^2(u+v)}{1+c} + 2c^2 - 1,$$

where, by (2.4.1), $-1 \leqslant y \leqslant -c$.

Now we check $F(-c,y,u,v)$ for $y=-c$ and $y=-1$.

(i)(1) Let $y=-c$. Then

$$F(-c,-c,u,v)=\frac{-4c^2(u+v)}{1+c}+\frac{(1+2c^2)(1+2uv)}{1+2c}+2c^2-1\leqslant 0,$$

because $1+2uv\leqslant 1+2c$ and for $x=y=-c$, $-2c(u+v)<-2c(1+c)$, by (2.4.1).

(i)(2) Let $y=-1$. Then

$$F(-c,-1,u,v)=-2c(u+v)+\frac{2c(1+2uv)}{1+2c}+2c^2-1.$$

For $x=-c$ and $y=-1$ we obtain from (2.4.1) and (2.4.2) that $2c<u+v$ and $1+2uv\leqslant 1+2c$. Substituting these in $F(-c,-1,u,v)$, it follows that

$$F(-c,-1,u,v)\leqslant -2c^2+2c-1\leqslant 0.$$

(ii) If $x=1$, then (by (2.4.1)), $y=-1$ and $c=0$. Therefore

$$F(1,-1,u,v)=-2-2uv<0.$$

(iii) If $y=-1$, then

$$F(x,-1,u,v)=2\{\frac{c(u+v)}{1+c}-\frac{1+2uv}{1+2c}\}x+\frac{1+2uv}{1+2c}-\frac{2c(u+v)}{1+c}+2c^2-1.$$

We check $F(x,-1,u,v)$ for $x=-c$ and $x=1$. By (i)(2), $F(-c,-1,u,v)\leqslant 0$ and by (ii), $F(1,-1,u,v)\leqslant 0$.

(iv) If $y=+c$, then (by (2.4.1)), $x=-c=0$ and $u=v=0$. Therefore

$$F(-c,c,u,v)\leqslant 0 \text{ when } c=0.$$

Thus $F(x,y,u,v)$ is never positive.

$\square$

LEMMA 2.5. *If we change x to $-y$ and y to $-x$ or change u to $-v$ and v to $-u$ in Lemma 2.4, we obtain*

$$\frac{(1+2xy)(1+2uv)}{1+2c}-\frac{2c(x+y)(u+v)}{1+c}+2c^2-1\leqslant 0.$$

LEMMA 2.6. *The polynomial $R_4(z)$ in Theorem 1.3, has all its zeros on $|z| = 1$.*

PROOF: Using trigonometric identities we can write

$$Q_4(z;\theta) = 1 + 2(1 + \cos\phi)z + 2(1 + 2\cos\phi)z^2 + 2(1 + \cos\phi)z^3 + z^4$$

where $\phi = (4-\alpha)\theta = [(4-\alpha)\pi]/(4+\beta-\alpha)$. Now

$$R_4(z) = 1 + 2Az + Bz^2 + 2Az^3 + z^4 \tag{2.6.1}$$

where $A = (x+y)(u+v)/(1+c)$, $B = 2(1+2xy)(1+2uv)/(1+2c)$, $0 < c = \cos\phi < 1$, $-c \leqslant x = \cos\phi_1 \leqslant 1$, $-c \leqslant u = \cos\psi_1 \leqslant 1$, $-1 \leqslant y = \cos\phi_2 \leqslant c$, $-1 \leqslant v = \cos\psi_2 \leqslant c$, $y \leqslant x$ and $v \leqslant u$.

If $R_4(z)$ has zero in $|z| < 1$, it must have a zero in $|z| > 1$, and vice-versa. Therefore, to prove that $R_4(z)$ has all its zeros on $|z| = 1$, it is sufficient to show that $R_4(z) \neq 0$ in $|z| < 1$. Write $R_4(z) = \left(z^2 + \alpha_1 z + 1\right)\left(z^2 + \alpha_2 z + 1\right) = p(z)q(z)$ where $\alpha_1 = A - \sqrt{A^2 + 2 - B}$ and $\alpha_2 = A + \sqrt{A^2 + 2 - B}$. By Lemma 2.2., $A^2 + 2 - B \geqslant 0$, so α_1 and α_2 are real. Now we need to show that $p(z) \neq 0$ and $q(z) \neq 0$ in $|z| < 1$. For this, it is sufficient to show that $-2 \leqslant \alpha_1$ and $\alpha_2 \leqslant 2$ since $\alpha_1 \leqslant \alpha_2$.

For the first case, we observe that $-2 \leqslant \alpha_1$ because $-2 \leqslant A - \sqrt{A^2 + 2 - B}$ if and only if $2 + B + 4A \geqslant 0$, which is true by Lemma 2.3. For the second case, we observe that $\alpha_2 \leqslant 2$ because $2 + B - 4A \geqslant 0$ by Lemma 2.3. This completes the proof. ∎

3. PROOF OF THEOREM 1.3

3.1. By Lemma 2.6, $R_4(z)$ has all its zeros on $|z| = 1$ and can be written as $R_4(z) = 1 + 2Az + Bz^2 + 2Az^3 + z^4$ where A and B are as in (2.6.1). Then there exist θ_1 and θ_2 with $0 \leqslant \theta_1 \leqslant \theta_2 \leqslant \pi$ such that

$$R_4(z) = 1 + 2(cos\theta_1 + \cos\theta_2)z + 2(1 + 2\cos\theta_1\cos\theta_2)z^2 + 2(\cos\theta_1 + \cos\theta_2)z^3 + z^4$$

where

$$A = \cos\theta_1 + \cos\theta_2 \text{ and } B = 2(1 + 2\cos\theta_1\cos\theta_2). \tag{3.1.1}$$

Because $p_4(z)$ and $q_4(z)$ are in $K(\alpha,.\beta)$, then (by Lemma 2.1),

$$-(1+c) \leqslant x + y,\ u + v \leqslant 1 + c \text{ and } -1 \leqslant 1 + 2xy,\ 1 + 2uv \leqslant 1 + 2c. \tag{3.1.2}$$

So, by (2.6.1) and (3.1.1),

$$-(1+c) \leqslant \cos\theta_1 + \cos\theta_2 \leqslant 1 + c \text{ and } -1 \leqslant 1 + 2\cos\theta_1\cos\theta_2 \leqslant 1 + 2c. \tag{3.1.3}$$

To show that $R_4(z)$ belongs to $K(\alpha,\beta)$, it is sufficient (by Lemma 2.1.), to show that θ_1 is never greater than $\beta\pi/(4+\beta-\alpha)$ and θ_2 is never less than $[(4-\alpha)\pi]/(4+\beta-\alpha)$.

3.2. First we show that θ_1 is always between 0 and $\beta\pi/(4+\beta-\alpha)$. Assume that $\theta_1 = \{\beta\pi/(4+\beta-\alpha)\} + \varepsilon$ where $\varepsilon > 0$. Then

$$-1 \leqslant \cos\theta_2 \leqslant \cos\theta_1 < -c. \tag{3.2.1}$$

From (3.1.3) and (3.2.1) it follows that

$$-(1+c) \leqslant \cos\theta_1 + \cos\theta_2 < -2c \text{ and } 1 + 2c^2 < 1 + 2\cos\theta_1\cos\theta_2 \leqslant 1 + 2c.$$

Therefore, by (2.6.1) and (3.1.2),

$$-(1+c)^2 \leqslant (x+y)(u+v) < -2c(1+c),$$

and

$$(1+2c)(1+2c^2) < (1+2xy)(1+2uv) \leqslant (1+2c)^2.$$

Note that the above two inequalities are the conditions (2.4.1) and (2.4.2).

Solving the first equation of (3.1.1) for $\cos\theta_2$ and substituting this into the second equation, we obtain $\cos^2\theta_1 - A\cos\theta_1 + 1/4B - 1/2 = 0$. Then

$$\cos\theta_1 = \frac{1}{2}\{A \pm \sqrt{A^2+2-B}\}. \tag{3.2.2}$$

Now from (3.2.1) and (3.2.2) it follows that

$$-2 \leqslant 2\cos\theta_1 = A \pm \sqrt{A^2+2-B} < -2c. \tag{3.2.3}$$

We will show that (3.2.3) is never true. If $A + \sqrt{A^2+2-B} < -2c$, then $4Ac + B - 2 + 4c^2 > 0$ which is not true (by Lemma 2.4.). If $\cos\theta_1 = 1/2\bigl(A - \sqrt{A^2+2-B}\bigr) < -c$, then by (3.1.1), $\cos\theta_2 = A - \cos\theta_1 = 1/2\bigl(A + \sqrt{A^2+2-B}\bigr)$. Since $\cos\theta_2 < -c$ (by (3.2.1)), $A + \sqrt{A^2+2-B} < -2c$. But by Lemma 2.4, this cannot be true. Therefore we conclude that θ_1 is never greater than $\beta\pi/(4+\beta-\alpha)$.

3.3. Finally, we show that θ_2 is never less than $[(4-\alpha)\pi]/(4+\beta-\alpha)$. Suppose that $\theta_2 = \{(4-\alpha)\pi/(4+\beta-\alpha)\} - \delta$ where $\delta > 0$. Then

$$c < \cos\theta_2 \leqslant \cos\theta_1 \leqslant 1. \tag{3.3.1}$$

From (3.1.1)–(3.1.3) and (3.3.1) we obtain

$$2c(1+c) < (x+y)(u+v) \leqslant (1+c)^2 \text{ and}$$
$$(1+2c)(1+2c^2) < (1+2xy)(1+2uv) \leqslant (1+2c)^2.$$

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These are the conditions (2.4.1) and (2.4.2) if we change x to $-y$ and y to $-x$, or change u to $-v$ and v to $-u$.

Solving the first equation of (3.1.1) for $\cos\theta_1$ and substituting in the second one we obtain

$$\cos\theta_2 = \frac{1}{2}\{A \pm \sqrt{A^2+2-B}\}. \tag{3.3.2}$$

From (3.3.1) and (3.3.2) it follows that

$$2c < 2\cos\theta_2 = A \pm \sqrt{A^2+2-B} \leqslant 2. \tag{3.3.3}$$

We will show that (3.3.3) is never true. If $2c < A-\sqrt{A^2+2-B}$, then $B-4Ac+4c^2-2>0$. But this is impossible by Lemma 2.5.

If $2c < 2\cos\theta_2 = A+\sqrt{A^2+2-B}$, then by (3.1.1) and (3.3.1),

$$c < \cos\theta_1 = A - \cos\theta_2 = \frac{1}{2}\{A - \sqrt{A^2+2-B}\}.$$

By Lemma 2.5, this is not possible. Therefore we conclude that θ_2 is never less than $(4-\alpha)\pi/(4+\beta-\alpha)$. This completes the proof of Theorem 1.3. ∎

4. Conclusion

Our intuition is that if n is a sufficiently small positive integer, then for given polynomials $p_n(z)$ and $q_n(z)$ of the form

$$p_{2m}(z) = \prod_{k=1}^{2m}(1+ze^{i\theta_k}) \text{ or } p_{2m+1}(z) = (1+z)p_{2m}(z)$$

where $\theta_k = -\theta_{2m+1-k}$, an argument similar to that used to prove Theorem 1.3 can be used to show that if $p_n(z)$ and $q_n(z)$ belong to $K(\alpha,\beta)$ where $1\leqslant\alpha\leqslant n$ and $\alpha\leqslant\beta$, then $R_n(z) = (p_n * q_n)(z) * Q_n^{(-1)}(z;\theta)$ also belong to $K(\alpha,\beta)$. For example, when $n=5$, we obtain (from Theorem 1 of [1]) that $p_5(z)$ is in $K(2,\beta)$ if and only if $\pi/(3+\beta)\leqslant\theta_1\leqslant(1+\beta)/(3+\beta)$ and $3\pi/(3+\beta)\leqslant\theta_2\leqslant\pi$. Now (analogous to Lemma 2.6) we may show that $R_5(z)$ has all its zeros on $|z|=1$. This means that $R_5(z)$ can be written in the form $p_{2m+1}(z)$. Next (analogous to the proof of Theorem 1.3), we show that the zeros of $R_5(z)$ are located so that $R_5(z)$ belongs to $K(2,\beta)$.

For polynomials of large degree, the above method turns out to be lengthy and rather involved. Perhaps one can come up with a better technique which works for polynomials of any degree. A reader interested in pursuing this problem may find the studies in [5–10] and specially [1] (Section 4, p.56) of some use.

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