THE SURPLUS PROCESS AS A FAIR GAME—UTILITYWISE

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1. INTRODUCTION AND SUMMARY

The concept of utility is twofold. One may think of utility:

1) as a tool to describe a "fair game"
2) as a quantity that ought to be maximized.

The first line of thought was initiated by Daniel Bernoulli in connection with the St. Petersburg Paradox. In recent decades, actuaries, economists, operations researchers and statisticians (this order is alphabetical) have been concerned mostly with optimization problems, which belong to the second category. Most of the actuarial models can be found in a paper by Borch [4] as well as in the texts by Beard, Pesonen and Pentikainen [3], Bühlmann [6], Seal [15], and Wolff [17].

We shall adopt the first variant and stipulate the existence of a utility function such that the surplus process of an insurance company is a fair game in terms of utility. This condition is naturally satisfied under the following procedure: a) a utility function is selected, possibly resulting from a compromise between an insurance company and supervising authorities, b) whenever the company makes a decision that affects the surplus, it should not affect the expected utility of the surplus.

Mathematically, this simply means that the utility of the surplus is a martingale. Therefore martingale theory (that was initiated by Doob) is the natural framework in which we shall study the model. We shall utilize one of the most powerful tools provided by this theory, the Martingale Convergence Theorem.

Section 2 is devoted to the relationship between the probability of ultimate ruin and the utility function that underlies the surplus process. Theorems 1 and 2 are in the spirit of and extend results by DeFinetti, see [7] p. 58-68 and Dubourdieu, see [8] p. 163-174 and

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p. 258-260. These authors have used martingale techniques without using martingale language. Theorems 1 and 2 show us that (under mild restrictions for the surplus process) the probability of ruin is less than one if and only if the underlying utility function is bounded from above. Apart from the fair game hypothesis, no restrictions (such as stationarity or Markov property) are made about the surplus process.

In section 3 we illustrate the general theory in three special models. For example, Theorems 1 and 2 enable us to show that Ottaviani's conjecture is true. The ambitious reader should look at the end of subsection 3.1, where he will find an unsolved problem.

2. The General Theory

While the ideas do not depend on the model chosen (discrete or continuous time), the continuous time model involves considerably more technicalities. In order to prevent the latter from obscuring the ideas, we present the general theory in the discrete time model.

2.1. *Definitions and interpretations*

What follows is based on a fixed probability space, i.e. a triple (Ω, A, P). Here Ω denotes the sample space, A is the σ-algebra of all events, and P is the probability measure defined on it. Furthermore, we are given a sequence $A_1, A_2, A_3, \ldots$ of σ-subalgebras of A satisfying

$$A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots \tag{1}$$

Intuitively, A_n is the information available at time n, or more precisely, the set of all events whose occurrence (or non-noccurrence) is known at time n. We start at time zero, and therefore assume that A_0 consists only of Ω and its complement (the impossible event). We are interested in a sequence of random variables $X_1, X_2, X_3 \ldots$ such that $X_n = X_n(\omega)$, $\omega \varepsilon \Omega$, is measurable with respect to A_n. We interpret X_n as the surplus of an insurance company, measured in index adjusted monetary units. Let $X_0 = x$ be the initial surplus. The distribution of the X_n's depends on many factors, such as claims, premiums, dividends, investment return, inflation, taxes, expenses. Therefore, under realistic assumptions

A_n will be much larger than the σ-algebra generated by $(X_1, X_2, \ldots, X_n)$: Even if the actuary is primarily interested in the surplus of his company at various times, he cannot afford to ignore the information that can be obtained from the outside world.

The *time of ruin* T is defined as

$$T = \text{Minimum}\,\{n \mid X_n \leq 0\} \tag{2}$$

with the understanding that $T = \infty$ if $X_n > 0$ for all n. Thus $[T < \infty]$ is the event that ruin will ultimately occur. Let

$$\Psi(x) = P[T < \infty] \tag{3}$$

denote its probability. Here the argument x is just a reminder that this is our initial surplus; it will *not* be varied in the following.

If we stop the surplus process $\{X_n\}$ at time T, we obtain a new process $\{\tilde{X}_n\}$ where

$$\tilde{X}_n = \begin{cases} X_n \text{ if } T > n \\ X_T \text{ if } T \leq n \end{cases} \tag{4}$$

Of course $\tilde{X}_n$ is also measurable with respect to A_n.

Finally, we shall use the following definition: The process $\{\tilde{X}_n\}$ is said to be *asymptotically fluctuating*, if the probability that $T = \infty$ and $\{\tilde{X}_n\}$ converges to a finite limit is zero. In most applications it is easy to check the validity of this property.

2.2. *The fair game hypothesis*

Mathematically this crucial assumption is as follows: There exists a strictly increasing, continuous function $u(s)$, $-\infty < s < \infty$, such that $\{u(\tilde{X}_n)\}$ is a *martingale with respect to* $\{A_n\}$. This means that

$$E[u(\tilde{X}_{n+1}) \mid A_n] = u(\tilde{X}_n) \quad \text{a.s.} \tag{5}$$

or equivalently, that

$$E[u(\tilde{X}_{n+k}) \mid A_n] = u(\tilde{X}_n) \quad \text{a.s.} \tag{6}$$

for $n = 0, 1, 2, \ldots$ and $k = 1, 2 \ldots$

Intuitively, $u(s)$ is the insurance company's *utility* for a surplus of s. The interpretation of conditions (5) and (6) is: At any time, and

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under all circumstances, the company plays a *fair game*, not in terms of *monetary units* but in terms of *utility*. Thus, for given utility function, the fair game assumption sets a boundary condition to the company's decisions.

Remarks. 1) The fair game hypothesis is the dynamic extension of the *principle of zero utility*, see p. 86 in [6]. In a special case it has been introduced by Ferra, see p. 63-67 in [10].

2) Since the surplus is measured in indexed monetary units, it is not too unreasonable to assume time independence of the utility function.

3) It is customary to assume that $u(s)$ is *risk averse*, i.e. concave from below. However, the general theory does not depend on this assumption.

4) Sometimes it is easier to verify that $\{X_n\}$ is a martingale with respect to $\{A_n\}$. This condition implies that $\{\tilde{X}_n\}$ is a martingale, because stopping does not affect the martingale property.

2.3. *The ruin probability in the case of bounded utility*

The monotonicity of $u(s)$ implies that $u(\infty) = \lim u(s)$, $s \to \infty$, is well defined (possibly infinite). In this subsection we study the case of bounded (from above) utility, $u(\infty) < \infty$. Let us introduce the function $v(y)$, $-\infty < y < \infty$, that originates from $u(y)$ by the following linear transformation:

$$v(y) = \frac{u(\infty) - u(y)}{u(\infty) - u(0)} \tag{7}$$

Thus $v(y)$ is a decreasing function with $v(0) = 1$ and vanishing at infinity. The importance of this function becomes evident in the following Theorem.

Theorem 1. If the fair game hypothesis is valid, then

$$\Psi(x) \leq \frac{v(x)}{E[v(X_T) \mid T < \infty]}$$

with equality holding if and only if the process $\{\tilde{X}_n\}$ is asymptotically fluctuating.

Proof. A linear transformation of a martingale is again a martingale. Thus $\{v(\tilde{X}_n)\}$ is a positive martingale with respect to $\{A_n\}$.

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Therefore, the Martingale Convergence Theorem (see for example [2], p. 275, [5] p. 63, or [12] p. 89) is applicable. It tells us that there is a random variable, say V, such that

$$v(\tilde{X}_n) \to V \text{ for } n \to \infty, \quad \text{a.s.} \tag{8}$$

On the set $[\omega \mid T(\omega) < \infty]$, i.e. in the event that ruin occurs, this convergence is of course trivial: there V coincides with $v(X_T)$.

We can decompose V into a sum,

$$V = {}_1V + {}_2V \tag{9}$$

where the auxiliary random variables ${}_iV$ are defined by the following table.

${}_1V =$	${}_2V =$	
$v(X_T)$	0	if $T < \infty$
0	V	if $T = \infty$

Similarly, we can write

$$v(\tilde{X}_n) = {}_1V_n + {}_2V_n \tag{10}$$

where the auxiliary random variables ${}_iV_n$ are given by the table

${}_1V_n =$	${}_2V_n =$	
$v(X_T)$	0	if $T \leq n$
0	$v(X_n)$	if $T > n$

Statement (8) implies that for $i = 1, 2$

$${}_iV_n \to {}_iV \text{ for } n \to \infty, \quad \text{a.s.} \tag{11}$$

Moreover, convergence of the expected values takes place:

$$E[{}_iV_n] \to E[{}_iV] \quad \text{for} \quad n \to \infty \tag{12}$$

For $i = 1$ this follows from the Monotone Convergence Theorem (the sequence $\{{}_1V_n\}$ is increasing) and for $i = 2$ it is a consequence of the Dominated Convergence Theorem (observe that $0 < {}_2V_n < 1$).

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From the martingale property, and from formulas (10) and (12) we get

$$v(x) = E[v(\tilde{X}_n)] = E[{}_1V_n] + E[{}_2V_n]$$
$$\rightarrow E[{}_1V] + E[{}_2V] \quad \text{for} \quad n \rightarrow \infty \tag{13}$$

Thus

$$v(x) = E[{}_1V] + E[{}_2V] \tag{14}$$

Since

$$E[{}_1V] = E[v(X_T) \mid T < \infty] \, \Psi(x) \tag{15}$$

we get the inequality of Theorem 1 by omitting the last term in formula (14). Furthermore, equality holds iff $E[{}_2V] = 0$, i.e. iff ${}_2V_n \rightarrow 0$ for $n \rightarrow \infty$. Since we know that $\{{}_2V_n\}$ converges with probability one, the last condition is equivalent to the condition that $\{\tilde{X}_n\}$ is asymptotically fluctuating. q.e.d

Remark. Since $v(y) \geq 1$ for $y \leq 0$, we get as a corollary from Theorem 1 that $\Psi(x) \leq v(x)$. This inequality is due to Dubourdieu (see [8], Theorem F' on p. 262).

2.4. *Bounded versus unbounded utility functions*

We just saw that $u(\infty) < \infty$ implies that $\Psi(x) < 1$ (if $x > 0$). In many cases the converse is also true:

> *Theorem* 2. Suppose that a) $\{\tilde{X}_n\}$ is asymptotically fluctuating, and b) that supremum $E[u(\tilde{X}_n) —] < \infty$.
> Then $u(\infty) = \infty$ implies that $\Psi(x) = 1$.

Proof. Condition b) assures us that the Martingale Convergence Theorem is applicable. Thus, with probability one, $\{u(\tilde{X}_n)\}$ converges as $n \rightarrow \infty$. If $T = \infty$, convergence of $\{u(\tilde{X}_n)\}$ implies convergence of $\{\tilde{X}_n\}$, because $u(\infty) = \infty$. But condition a) makes this impossible, therefore $P[T = \infty] = 0$. q.e.d

Remark. The above result certainly speaks for bounded (from above) utility functions. However the argument loses some of its weight, because in many cases where ruin is certain, the expected time of its occurrence is infinite!

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2.5. *The surplus process as a favorable game*

As far as inequalities are concerned, the results of subsection 2.3 carry through to the more general situation where the surplus process is a favorable game, at any time and under all circumstances, in terms of utility. Mathematically, this means that $\{u(\tilde{X}_n)\}$ is a submartingale, or equivalently, that $\{v(\tilde{X}_n)\}$ is a super-martingale with respect to $\{A_n\}$. The latter condition means that

$$E[v(\tilde{X}_{n+k}) \mid A_n] \leq v(\tilde{X}_n) \qquad \text{a.s.} \tag{16}$$

for $n = 0, 1, 2 \ldots$ and $k = 1, 2 \ldots$. Actually, all that is needed is that these inequalities hold for $n = 0$. From this we get, starting with the right side,

$$\begin{aligned} v(x) &\geq E[v(\tilde{X}_k)] \\ &= E[{}_1V_k] + E[{}_2V_k] \geq E[{}_1V_k] \end{aligned} \tag{17}$$

By the Monotone Convergence Theorem the last term converges to $E[{}_1V]$ for $n \to \infty$. Therefore,

$$v(x) \geq E[{}_1V] = E[v(X_T) \mid T < \infty]\,\Psi(x) \tag{18}$$

which is the inequality contained in Theorem 1.

Remark. The assumption that the surplus process is a favorable game (utilitywise) makes a lot of sense from the insurance company's point of view. However the consumer, perhaps represented by an insurance commissioner, is likely to insist that the *favorable* game be extreme, i.e. *fair*.

3. Illustrations and Applications

The general theory was developed for an arbitrary surplus process satisfying the fair game hypothesis. In view of this, the following examples may appear rather restrictive.

3.1. *The classical claims—premium model*

Ignoring factors such as interest, inflation and expenses, we set

$$X_n = x + P_1 - S_1 + \ldots + P_n - S_n \tag{19}$$

Here $S_1, S_2, \ldots$ are independent and identically distributed random variables (the claims in subsequent periods). We wish to determine the premiums $P_1, P_2, \ldots$ such that the fair game hypothesis holds. Having chosen an appropriate utility function, P_n is obtained as the solution of the equation

$$E[u(X_{n-1} + P_n - S_n)] = u(X_{n-1}) \tag{20}$$

Thus $P_n = P_n(X_{n-1})$ is a function of X_{n-1}; observe that this quantity is known at the time when the premium P_n is due.

In the special case of an *exponential utility function*

$$\begin{aligned} u(s) &= 1/R\,(1 - e^{-Rs}) \\ v(s) &= e^{-Rs},\ R > 0 \end{aligned} \tag{21}$$

the premiums are independent of the surplus

$$P_n = 1/R \ln E[e^{RS_n}] \tag{22}$$

Since the surplus is asymptotically fluctuating, we get from Theorem 1 that

$$\Psi(x) = \frac{e^{-Rx}}{E[e^{-RX_T} \mid T < \infty]} \tag{23}$$

which appears as formula (12.14) in [3], p. 143. The parameter R is sometimes called the *adjustment coefficient*.

In this model we can vary the initial surplus, which we did not do in the general model. The famous asymptotic result of Lundberg-Cramer refers to the special case of exponential utility and is: There is a constant C such that

$$\Psi(x)\, e^{Rx} \rightarrow C \quad \text{for} \quad x \rightarrow \infty \tag{24}$$

The question remains whether a more general statement of the following kind is true: Given a bounded utility function, and certain regularity conditions, there is a constant D such that

$$\frac{\Psi(x)}{v(x)} \rightarrow D \quad \text{for} \quad x \rightarrow \infty \tag{25}$$

Note that this is equivalent to convergence of the denominator in Theorem 1. The author has not found the mathematical tools yet to prove this conjecture (which is believed to be true under mild regularity conditions for the distribution of the claims).

3.2. *The diffusion model*

In this and the following subsections the time parameter will be continuous, $t \geq 0$. Without many scruples we will adopt the results of the general discrete time model to these two continuous time models.

We assume that the surplus process $\{X_t\}$ is a diffusion process (see [9] section 10.4, or [5] section 16.2) with

$$\begin{aligned} &\text{infinitesimal drift } \mu(y) \\ &\text{infinitesimal variance } \sigma^2(y) \end{aligned} \tag{26}$$

depending on location (read surplus). It is assumed that they are continuous functions of y and that $\sigma^2(y) \geq C > 0$. The last condition excludes complications and guarantees us that the surplus process is asymptotically fluctuating.

Let us now look at the martingale condition. If $u(s)$ is a twice differentiable function, then $\{u(X_t)\}$ is also a diffusion process, namely with infinitesimal drift

$$u'(s)\,\mu(s) + \tfrac{1}{2}u''(s)\,\sigma^2(s) \tag{27}$$

and infinitesimal variance

$$u'^2(s)\,\sigma^2(s) \tag{28}$$

(see [5] p. 386 for example). But a diffusion process is a martingale iff its drift vanishes. Therefore the fair game condition becomes the following differential equation:

$$u'(s)\,\mu(s) + \tfrac{1}{2}u''(s)\,\sigma^2(s) = 0 \tag{29}$$

Thus the drift is proportional to the product of variance and risk aversion. The initial conditions can be chosen arbitrarily. If we set $u(0) = 0$, $u'(0) = 1$, and solve the differential equation, we get

$$u(y) = \int_0^y e^{-\int_0^z \frac{2\mu(s)}{\sigma^2(s)}\,ds}\,dz \tag{30}$$

which is of primary interest for $y \geq 0$. Since the samplepaths are continuous, the surplus at the time of ruin is necessarily zero. Thus if $u(\infty) < \infty$, the denominator in Theorem 1 is one, and we obtain

$$\psi(x) = v(x) = \frac{u(\infty) - u(x)}{u(\infty)}, \quad x \geq 0 \tag{31}$$

If on the other hand $u(\infty) = \infty$, Theorem 2 shows that $\psi(x) = 1$.

Example 1. If $\mu(y) = \mu > 0$, $\sigma^2(y) = \sigma^2$, formula (30) shows that the utility function is exponential, see formulas (21), with

$$R = \frac{2\mu}{\sigma^2} \tag{32}$$

This leads to $\Psi(x) = e^{-Rx}$, $x \geq 0$.

Example 2. If $\mu(y) = \mu + \delta y$ $(\delta > 0)$, $\sigma^2(y) = \sigma^2$, we get from formulas (30) and (31) that

$$\Psi(x) = \frac{1 - \Phi(ax + b)}{1 - \Phi(b)}, \quad x \geq 0 \tag{33}$$

where

$$a = \frac{\sqrt{2\delta}}{\sigma}, \quad b = \frac{2\mu}{\sigma\sqrt{2\delta}} \tag{34}$$

and $\Phi(\cdot)$ denotes the standard normal distribution. Note that δ can be interpreted as a force of interest.

Remarks. 1) Probabilists call $u(y)$ the *scale function*, and the process $\{u(X_t)\}$ is said to be on its *natural scale*.

2) If $u(\infty) < \infty$, there is a possibility that $u(\tilde{X}_t) = u(\infty)$ for a finite t with positive probability (which means that the surplus drifts to infinity in a finite time span). From proposition 16.43 in [5] we gather that this is the case, iff

$$\int_0^\infty \frac{u(\infty) - u(s)}{u'(s)\, \sigma^2(s)}\, ds < \infty \tag{35}$$

This condition is not satisfied in the examples above. In the first example this is easily verified. In the second it requires some calculations that are left to the reader.

3.3. *The Compound Poisson model*

In this context it is natural to assume a differentiable utility function. The surplus changes in time for two reasons: a) because of the claims to be paid and b) because of the premiums received. Suppose that the claim number process is Poisson (with parameter α), and that the individual claim amounts are independent and identically distributed random variables with distribution $F(y)$, $-\infty < y < \infty$. The premiums are received continuously, say at a rate $c(s)$ if the surplus equals s. The fair game condition becomes

$$c(s)\, u'(s) = \alpha\{u(s) - \int u(s-y)\, dF(y)\} \tag{36}$$

This is best seen by interpretation: The left side is the gain of utility per unit time due to the premiums received (assuming a surplus of s). And this should be offset by the expected loss of utility per unit time due to the possible occurrence of a claim. (Obviously some regularity assumptions about the claim amount distribution have to be made to make formula (36) meaningful.)

Clearly, the surplus process is asymptotically fluctuating. Let us now consider a bounded utility function and assume that the premium density is determined from (36). In two cases Theorem 1 leads to explicit expressions.

1) *Only Negative "Claims"*. Suppose $F(0) = 1$, which implies negative "premiums". The surplus at the time of ruin is necessarily zero, and therefore $\Psi(x) = v(x)$ as in (31).

2) *Exponential Claim Amounts*. Suppose that with probability p $(0 < p \leq 1)$ a claim is positive, in which case it follows an exponential distribution:

$$F(y) = 1 - pe^{-y}, \quad y > 0 \tag{37}$$

The claim amount distribution for $y \leq 0$ is arbitrary but we assume that $c(0) > 0$. Thus if ruin occurs, it is caused by a positive claim. Since the exponential distribution has a "lack of memory", the conditional distribution of X_T (given $T < \infty$) has to be exponential. Hence we can evaluate the denominator in Theorem 1 and get

$$\Psi(x) = \frac{v(x)}{\int_0^\infty v(-y)\, e^{-y}\, dy} \tag{38}$$

The case $p = 1$ will be discussed more in detail in the following subsection.

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Remarks. 1) If the utility function is *exponential*, see formulas (21), $c(s) = c$ is constant. Equation (36) now reduces to the familiar formula

$$\int e^{Ry}\, dF(y) = 1 + R(c/\alpha) \tag{39}$$

If the utility function is *quadratic* with level of saturation $L > 0$,

$$u(s) = \begin{cases} L^2 - (L - s)^2, & s < L \\ L^2 & , \ s \geq L \end{cases} \tag{40}$$

and if $F(0) = 0$ (only positive claims), equation (36) leads to

$$c(s) = \alpha \int_0^\infty y dF(y) + \frac{\alpha}{2(L - s)} \int_0^\infty y^2 dF(y) \tag{41}$$

for $s < L$. Thus the loading is proportional to the infinitesimal variance of the claims process, where the proportionality factor is an increasing function of the surplus, exploding at $s = L$: If the surplus reaches the level of saturation, the company could only lose utility by continuing business! (This curiousity is due to the fact that, contrary to our assumption, the utility function is not strictly increasing).

2) It is possible that the surplus drifts to infinity in a finite time interval (see Remark 2 in subsection 3.2.). If $F(0) = 0$, this happens with positive probability, if for some $S_0 > 0$ (therefore for all $S_0 > 0$)

$$\int_{s_0}^\infty \frac{ds}{c(s)} < \infty \tag{42}$$

Reason: This integral is the time it takes the surplus to get from S_0 to infinity in the absence of claims.

3) The model can be generalized such that the premium density at time t, the claim frequency at time t and the claim amount distribution at time t depend on $A_t (t \geq O)$. For example, if the claim number process is a renewal process, (see [1]) the claim frequency (and therefore the premium density) at any time depends on the time that has elapsed since the last claim occurred.

3.4. *Ottaviani's problem*

As indicated, we continue the discussion of exponential claim amounts (example 2 of the preceding subsection with $p = 1$), $F(y) = 1 - e^{-y}$ for $y \geq 0$. Let us assume $\alpha = 1$ (operational time).

Given is a positive and continuous premium density $c(y)$, $y \geq 0$. In [13] Ottaviani raised the question about necessary and sufficient conditions for the function $c(y)$ that imply a probability of ruin less than one. Since Theorems 1 and 2 enable us to answer this question in terms of the utility function, we simply have to construct such a utility function. Setting $z = s - y$ in formula (36) we get

$$c(s)\, u'(s) = u(s) - \int_{-\infty}^{s} u(z)\, e^{-(s-z)}\, dz \tag{43}$$

valid for $s \geq 0$, as a necessary condition. Temporarily we assume that $c(s)$ is differentiable. Taking the derivative leads to

$$c(s)\, u''(s) + c'(s)\, u'(s) = u'(s) - u(s) + \int_{-\infty}^{s} u(z)\, e^{-(s-z)}\, dz \tag{44}$$

By adding the last two equations we eliminate the integral and get the differential equation

$$cu'' + (c' + c - 1)\, u' = 0 \tag{45}$$

valid for $s \geq 0$. We solve it, setting (for example) $u(0) = 0$ $u'(0) = 1$, and get

$$u(x) = \int_0^x e^{-\int_0^y \frac{c' + c - 1}{c} ds}\, dy \tag{46}$$

and from this

$$u(x) = c(0) \int_0^x \frac{1}{c(y)}\, e^{-\int_0^y \frac{c-1}{c} ds}\, dy \tag{47}$$

Let this be the definition of our function $u(x)$ for $x > 0$. For $x \leq 0$ let $u(x)$ be any differentiable increasing function such that $u(0) = 0$, $u'(0) = 1$ and equation (43) is satisfied for $s = 0$,

$$c(0) = -\int_{-\infty}^{0} u(z)\, e^{z}\, dz. \tag{48}$$

The verification that this function $u(x)$, $-\infty < x < \infty$, satisfies the fair game condition (43) is left to the reader.

From Theorems 1 and 2 we gather: For any $x \geq 0$, $\Psi(x) < 1$ iff $u(\infty) < \infty$. Moreover, $\Psi(x)$ is given by formula (38). Making use of formulas (7), (47), and (48) this can be stated in terms of the premium density function as follows:

Theorem. For any $x \geq 0$, $\Psi(x) < 1$ iff

$$\int_0^\infty \frac{1}{c(y)} e^{-\int_0^y \frac{c-1}{c} ds} dy < \infty \tag{49}$$

Furthermore, if this condition holds, $\Psi(x)$ is given by the formula

$$\frac{\int_x^\infty \frac{1}{c(y)} e^{-\int_0^y \frac{c-1}{c} ds} dy}{1 + \int_0^\infty \frac{1}{c(y)} e^{-\int_0^y \frac{c-1}{c} ds} dy} \tag{50}$$

Examples. 1) For constant premiums, $c(y) = c$, condition (49) holds iff $c > 1$ (positive security loading). In this case formula (50) reduces to the well known expression

$$\Psi(x) = \frac{1}{c} e^{-\frac{c-1}{c} x}, \quad x \geq 0 \tag{51}$$

2) if $c(y) = c + \delta y$, where $c > 0$, $\delta > 0$, condition (49) is satisfied and formula (50) reduces to an expression that is best written in terms of the Gamma function, see [16], p. 288.

Remarks. 1) If we replace

$$\frac{1}{c(y)} \text{ by } 1 - \frac{c(y) - 1}{c(y)} \tag{52}$$

and perform the obvious integration, formulas (49) and (50) can be written in a different form. Formula (50) becomes Ottaviani's formula (10′) see [13] p. 65. In the case of nonnegative security loadings, $c(y) \geq 1$, condition (49) becomes equivalent to

$$\int_0^\infty e^{-\int_0^y \frac{c-1}{c} ds} dy < \infty \tag{53}$$

Therefore Ottaviani's conjecture, see p. 66 in [13], which was formulated for this case, turns out to be true.

2) As a further illustration to the Theorem above, the reader may verify that the validity of condition (42) implies that condition (49) holds. By way of interpretation this is clear: If the surplus becomes infinite in a finite time span (with a positive probability), the probability of ruin has to be less than one!

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