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The Cauchy problem for a second-order nonlinear hyperbolic equation with initial data on a line of parabolicity

John M.S. Rassias

In this paper we study the Cauchy problem for the second order nonlinear hyperbolic partial differential equation

$$(*)\qquad Lu = k(y)\cdot H^2(x, y, u, u_x, u_y)\cdot u_{xx} - u_{yy} = f(x, y, u, u_x, u_y) ,$$

with initial conditions

$$(**)\qquad u(x, 0) = r(x) , \quad u_y(x, 0) = \nu(x) ,$$

where

$$x \in I = [a, b] ,$$

$$k(y) = y^{\alpha} \quad (\alpha > 0) ,$$

$$H = H(x, y, u, u_x, u_y) \in C^2(\cdot) ,$$

$$f = f(x, y, u, u_x, u_y) \in C^2(\cdot) ,$$

and $|u|, |u_x|, |u_y| < \infty$, $y \geq 0$, $r = r(x) \in C^4(\cdot)$, $\nu = \nu(x) \in C^4(\cdot)$.

These conditions on k, H, f, r, and ν are assumed to be satisfied in some sufficiently small neighborhood of the segment I, $y = 0$, in the upper half-plane $y > 0$.

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This paper generalizes the results obtained by N.A. Lar'kin (*Differencial'nye Uravnenija* 8 (1972), 76-84), who has treated the special case $H = H(x, y, u)$; that is, the quasi-linear hyperbolic equation (*).

Introduction

In this paper we study the Cauchy problem for the second order non-linear hyperbolic partial differential equation

$$Lu \equiv K(y)\cdot H^2(x, y, u, u_x, u_y)\cdot u_{xx} - u_{yy} = f(x, y, u, u_x, u_y) ,$$

with initial conditions

$$u(x, 0) = r(x) , \quad u_y(x, 0) = \nu(x) , \quad x \in I = [a, b] ,$$

where $K = K(y) = y^{\alpha}$ $(\alpha > 0)$, $H = H(x, y, u, u_x, u_y) \neq 0$, $f = f(x, y, u, u_x, u_y)$ are all twice continuously differentiable functions defined for $x \in I$ (an interval), $y \geq 0$, $|u|, |u_x|, |u_y| < \infty$, and $r = r(x)$, $\nu = \nu(x)$ are given functions having continuous derivatives up to the fourth order inclusive. These conditions on K, H, f, r , and ν are assumed to be satisfied in some sufficiently small neighborhood of the segment I , $y = 0$, in the upper half-plane $y > 0$.

Frankl [8] solved the Cauchy problem for the equation

$$y\cdot u_{xx} - u_{yy} + a\cdot u_x + b\cdot u_{yy} + c\cdot u = 0 ,$$

$$a = a(x, y) , \quad b = b(x, y) , \quad c = c(x, y) ,$$

under the assumption that the coefficients are analytic.

Berezin [1] treated the same problem for the equation

$$h(x, y)\cdot y^{\alpha}\cdot u_{xx} - u_{yy} + a\cdot u_x + b\cdot u_y + c\cdot u + f = 0 ,$$

with restrictions on the coefficients similar to those for $Lu = f$, but with the condition $\alpha \in (0, 2)$. Starting from a different point of view Bers [2] solved the Cauchy problem for the equation $K(y)\cdot u_{xx} - u_{yy} = 0$, where $K = K(y)$ is a continuous monotone increasing function of y with

$K(0) = 0$. A solution to the same problem has been obtained for the equation $K(y) \cdot u_{xx} - u_{yy} = 0$ by Germain and Bader [9]. They make the additional assumption that $K(y) \sim c \cdot y$ as $y \to 0$ and thus make use of Riemann's method. The result of Bers shows that if the lower order terms are absent in an equation such as

$$h(x, y) \cdot y^{\alpha} \cdot u_{xx} - u_{yy} + a \cdot u_x + b \cdot u_y + c \cdot u + f = 0 ,$$

there is no restriction on the rate of growth of the coefficient of u_{xx} . On the other hand, Berezin gives an example to show that for $\alpha > 2$ the Cauchy problem is not correctly set for the equation

$$h(x, y) \cdot y^{\alpha} \cdot u_{xx} - u_{yy} + a \cdot u_x + b \cdot u_y + c \cdot u + f = 0 .$$

Conti [6] has shown that the Cauchy problem for the equation

$$h(x, y) \cdot y^{\alpha} \cdot u_{xx} - u_{yy} = f(x, y, u, u_x, u_y)$$

is correctly set for the range $\alpha \in (0, 2)$, if

$$y \cdot f_{u_x}(x, y, u, u_x, u_y) / \sqrt{K} \to 0$$

as $y \to 0$.

Protter [18] showed that the Cauchy problem for the equation

$$K(y) \cdot h(x, y) \cdot u_{xx} - u_{yy} + a(x, y) \cdot u_x + b(x, y) \cdot u_y + c(c, y) \cdot u + f(x, y) = 0$$

is correctly set, if $y \cdot a(x, y) / \sqrt{K} \to 0$ as $y \to 0$.

Lick [11], [12], [13] showed that the Cauchy problem for the equation

$$y^{2\alpha} \cdot r^2(x, y) \cdot u^{2\beta} \cdot u_x^{2\gamma} \cdot u_{xx} - u_{yy} + a(x, y) \cdot u_x + b(x, y) \cdot u_y + c(x, y) \cdot u = 0 ,$$

with homogeneous initial conditions, has only the trivial solution. Besides, he showed that the Cauchy problem for

$$u_x^{2\gamma} \cdot u_{xx} - u_{yy} + f(x, y, u, u_x, u_y) = 0 \quad (y > 0) ,$$

with initial conditions $u(x, 0) = 0$, $u_y(x, 0) = \phi(x) : x \in I = [a, b]$, is correctly set whenever $f_p(x, y, u, p, q) = O(y^{\gamma-1})$ as $y \to 0$,

$p = u_x$, $q = u_y$.

Ogawa [15], [16], [17] showed that the Cauchy problem for the equation

$$r^2(x, y)\cdot u^{2\beta}\cdot u_{xx} - u_{yy} + f = 0 , \quad f = f(x, y, u, u_x, u_y) \quad (y > 0) ,$$

with the initial conditions $u(x, 0) = 0$,

$$u_y(x, 0) = \phi(x) : x \in I = [a, b]$$

is well-posed if $f_p(x, y, u, p, q) = O(y^{\beta-1})$ as $y \to 0$.

It has also been proved recently by Singer [20] that the (singular) Cauchy problem for the second order, quasi-linear, hyperbolic partial differential equation

$$y^{2\alpha}\cdot r^2(x, y)\cdot u^{2\beta}\cdot u_x^{2\gamma}\cdot u_{xx} - u_{yy} + f(x, y, u, u_x, u_y) = 0 \quad (y > 0) ,$$

with initial conditions $u(x, 0) = 0$, $u_y(x, 0) = \phi(x) : x \in I$, has one and only one solution in a neighborhood $(y > 0)$ of $I = [a, b]$, if α, β , and γ are non-negative real numbers with $\alpha + \beta + \gamma > 0$, I is any finite interval on the x-axis, and $f_p(x, y, u, p, q) = O(y^{\alpha+\beta+\gamma-1})$ as $y \to 0$, $\beta\gamma < 1$.

Lar'kin [10] showed that the Cauchy problem for the second-order quasi-linear hyperbolic equation with initial data on a line of parabolicity, namely, for the equation

$$u_{yy} - y^m\cdot K^2(x, y, u)\cdot u_{xx} + \Phi(x, y, u) = 0 , \quad y > 0 , \quad m > 0 ,$$

where $K(x, y, u) \neq 0$ and $\Phi(x, y, u)$ are twice continuously differentiable for $x \in [a, b]$, $y \geq 0$, $|u| < \infty$, with initial data $u(x, 0) = \phi(x)$, $u_y(x, 0) = \psi(x) : x \in [a, b] = I$, $y = 0$, ϕ and ψ continuously differentiable up to the fourth order inclusive, has a unique regular solution.

In the developments mentioned above the authors have applied mainly Schauder's fixed point theorem to a system of integral equations, the Picard method of iteration, and the Ascoli-Arrela theorem, as well as the classical mean value theorem.

THEOREM

Let us consider the non-linear hyperbolic equation of second order

(1) $$Lu \equiv K(y) \cdot H^2(x, y, u, u_x, u_y) \cdot u_{xx} - u_{yy} = f(x, y, u, u_x, u_y) ,$$

with initial data on a line of parabolicity; namely,

(2) $$\begin{cases} u(x, 0) = r(x) , \\ u_y(x, 0) = \nu(x) , \end{cases}$$

where $H = H(x, y, u, u_x, u_y) \neq 0$, $K(y) = y^\alpha$ $(\alpha > 0)$, $f = f(x, y, u, u_x, u_y)$ *are twice continuously differentiable functions defined for* $x \in [a, b]$, $y \geq 0$, $|u|, |u_x|, |u_y| < \infty$, $r = r(x)$, $\nu = \nu(x)$ *are given functions having continuous derivatives up to the fourth order inclusive.*

Equation (1) *is hyperbolic for* $y > 0$ *and is parabolically degenerate for* $y = 0$.

If $H = H(x, y, u, u_x, u_y)$, $f = f(x, y, u, u_x, u_y)$, $r = r(x)$, *and* $\nu = \nu(x)$ *satisfy the above conditions in some sufficiently small neighborhood of the segment* $a \leq x \leq b$, $y = 0$ *in the half-plane* $y > 0$, *then the Cauchy problem* (1) *and* (2) *has a unique regular solution* $u = u(x, y)$ *in some sufficiently small neighborhood of the segment* $a \leq x \leq b$, $y = 0$ *in the half-plane* $y > 0$, *which is twice continuously differentiable for* $y > 0$ *and continuous for* $y \geq 0$.

Proof

Let us introduce the new unknown function

(3) $$v = u - y \cdot \nu - r ,$$

for which the initial conditions (2) become homogeneous, and for which obviously there is no loss of generality.

By (3), conditions (1) and (2) become

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(1)' $\overline{L}v \equiv K(y)\cdot H^2(x,\ y,\ v,\ v_x,\ v_y)\cdot v_{xx} - v_{yy} = F(x,\ y,\ v,\ v_x,\ v_y)$,

(2)' $$\begin{cases} v(x,\ 0) = 0\ , \\ v_y(x,\ 0) = 0\ , \end{cases}$$

where $H = H(x,\ y,\ v,\ v_x,\ v_y)$, and $F = F(x,\ y,\ v,\ v_x,\ v_y)$ are known functions. For convenience, let $u = v$, $H = K = K(x,\ y,\ u,\ u_x,\ u_y)$ in (1)' and (2)', such that

(1)" $$\overline{\overline{L}}u \equiv y^{\alpha}\cdot K^2\cdot u_{xx} - u_{yy} = F(x,\ y,\ u,\ u_x,\ u_y)\ ,$$

(2)" $$u(x,\ 0) = u_y(x,\ 0) = 0\ .$$

(I). At first, we reduce the non-linear hyperbolic partial differential equation of second order, (1)", to a system of integral equations, as follows.

Let us introduce the new unknown functions

(4) $$\begin{cases} u_1 = u_1(x,\ y) = z_1(x,\ y)\ , \\ u = z_1\ , \\ u_2 = u_2(x,\ y) = H(x,\ y,\ u,\ u_x,\ u_y)\cdot y^{\alpha/2}\cdot z_2 + z_3\ , \\ u_x = z_2\ ,\quad u_y = z_3 \\ u_3 = u_3(x,\ y) = -H(x,\ y,\ u,\ u_x,\ u_y)\cdot y^{\alpha/2}\cdot z_2 + z_3\ , \\ H = K = K(x,\ y,\ u,\ u_x,\ u_y)\ . \end{cases}$$

It is clear that

$$(z_1)_y = z_3\ ,\quad (z_2)_y = (z_3)_x\ ,$$

(5) $$(z_3)_y = y^{\alpha}\cdot K^2(x,\ y,\ u,\ u_x,\ u_y)\cdot(z_2)_x - F(x,\ y,\ u,\ u_x,\ u_y)\ ,$$

(6) $$z_1(x,\ 0) = z_2(x,\ 0) = z_3(x,\ 0) = 0\ .$$

By (4),

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$$(4)' \quad \begin{cases} z_1 = z_1(x,\ y) = u_1 \ , \\ z_2 = z_2(x,\ y) = [u_2(x,\ y)-u_3(x,\ y)]/\left(2y\cdot^{\alpha/2}K\right) \ , \\ z_3 = z_3(x,\ y) = [u_2(x,\ y)+u_3(x,\ y)]/2 \ ; \end{cases}$$

$$(7) \qquad \left(u_2\right)_y - K\cdot y^{\alpha/2}\cdot\left(u_2\right)_x = \left(A_2/y\right)\cdot\left(u_2-u_3\right) + B_2 \ ,$$

where

$$(8) \quad \begin{cases} A_2 = \Big[\alpha-2y^{(\alpha+2)/2}\cdot K_x+2y\cdot K^{-1}\cdot\big(K_y+K_{z_1}\cdot u_3+K_{z_2}\cdot\left(z_2\right)_y \\ \qquad\qquad +K_{z_3}\cdot\left(z_3\right)_y\big)-2\cdot y^{(\alpha+2)/2}\cdot\big(K_{z_2}\cdot\left(z_2\right)_x+K_{z_3}\cdot\left(z_3\right)_x\big)\Big]/4 \ , \\ \text{and} \\ B_2 = -F \ ; \end{cases}$$

$$(9) \qquad \left(u_3\right)_y + K\cdot y^{\alpha/2}\cdot\left(u_3\right)_x = \left(A_3/y\right)\cdot\left(u_2-u_3\right) + B_3 \ ,$$

where

$$(10) \quad \begin{cases} A_3 = -\Big[\alpha+2\cdot y^{(\alpha+2)/2}\cdot K_x+2y\cdot K^{-1}\cdot\big(K_y+K_{z_1}\cdot u_2+K_{z_2}\cdot\left(z_2\right)_y \\ \qquad\qquad +K_{z_3}\cdot\left(z_3\right)_y\big)+2\cdot y^{(\alpha+2)/2}\cdot\big(K_{z_2}\cdot\left(z_2\right)_x+K_{z_3}\cdot\left(z_3\right)_x\big)\Big]/4 \ , \\ \text{and} \\ B_3 = -F \ . \end{cases}$$

Hence (1)" may be written equivalently as the following system of three equations, namely:

$$u_{1y} = [u_2(x,\ y)+u_3(x,\ y)]/2 \ ,$$

$$u_{2y} - \left(K\cdot y^{\alpha/2}\right)\cdot u_{2x} = \frac{A_2}{y}\cdot [u_2(x,\ y)-u_3(x,\ y)] + B_2 \ ,$$

$$u_{3y} + K\cdot y^{\alpha/2}\cdot u_{3x} = \frac{A_3}{y}\cdot [u_2(x,\ y)-u_3(x,\ y)] + B_3 \ ;$$

or

$$(11)\qquad \begin{cases} u_{1y} = (u_2+u_3)/2 \,, \\ u_{iy} - (-1)^i \cdot (K\cdot y^{\alpha/2}) \cdot u_{ix} = \dfrac{A_i}{y}\,(u_2-u_3) + B_i \quad (i = 1,\, 2)\,. \end{cases}$$

It is worth noting that

$$(12)\qquad B_2 = B_3 = -F = y^{\alpha}\cdot H^2\cdot (y\cdot \nu_{xx}+r_{xx}) - f\,.$$

In fact, by (3), $u = v + y\cdot\nu + r$, whence

$$u_x = v_x + y\cdot\nu_x + r_x\,, \quad u_{xx} = v_{xx} + y\cdot\nu_{xx} + r_{xx}\,,$$

$$u_y = v_y + \nu\,, \quad u_{yy} = v_{yy}\,.$$

Therefore,

$$\begin{aligned} K(y)\cdot H^2\cdot u_{xx} - u_{yy} &= K(y)\cdot H^2\cdot (v_{xx}+y\cdot\nu_{xx}+r_{xx}) - v_{yy} \\ &= \{K(y)\cdot H^2\cdot v_{xx}-v_{yy}\} + K(y)\cdot H^2\cdot (y\cdot\nu_{xx}+r_{xx}) \\ &= F + K(y)\cdot H^2\cdot (y\cdot\nu_{xx}+r_{xx}) = f\,, \end{aligned}$$

and thus

$$F = f(x,\, y,\, u,\, u_x,\, u_y) - y^{\alpha}\cdot H^2(x,\, y,\, u,\, u_x,\, u_y)\cdot (y\cdot\nu_{xx}+r_{xx})\,.$$

The characteristics of (11) are the lines $x = \text{const}$ and the two families of curves given by $dy/dx = \pm y^{-\alpha/2}\cdot K^{-1}$.

Let $P = P(x,\, y)$ be a point in D and construct the three characteristics of (11) passing through the point P .

The left side of each of the equations in (11) represents a derivative in a characteristic direction.

If we denote by S_2 the member of the family

$$\frac{dy}{dx} = -y^{-\alpha/2}\cdot K^{-1}$$

passing through P , and by S_3 the member of the family

$$\frac{dy}{dx} = +y^{-\alpha/2}\cdot K^{-1}$$

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passing through P, we can write (11) in the form

$$(13)\qquad \begin{cases} \dfrac{du_1}{dy} = (u_2+u_3)/2 \,, \\ \dfrac{du_i}{dS_i} = \dfrac{\tilde{A}_i}{y} \cdot (u_2-u_3) + \tilde{B}_i \quad (i = 2,\ 3) \,, \end{cases}$$

where

$$\tilde{A}_i = A_i/[1+y^\alpha \cdot K^2]^{\frac{1}{2}} \quad (i = 1,\ 2) \,, \quad \tilde{B}_i = -F/[1+y^\alpha \cdot K^2]^{\frac{1}{2}} \,.$$

In fact

$$\frac{du_i}{dS_i} = (u_{ix}dx+u_{iy}dy)/dS_i = \left(u_{iy} + u_{ix}\frac{dx}{dy}\right) \cdot \frac{dy}{dS_i} = \left[u_{iy}-u_{ix}\cdot K\cdot y^{\alpha/2}\right]\cdot\left(\frac{dy}{dS_i}\right) \,,$$

where

$$\frac{dy}{dS_i} = \left|\frac{du_i}{dS_i}\right| = [1+y^\alpha\cdot K^2]^{\frac{1}{2}} \,.$$

By integrating (13) along the characteristics, we obtain the required system of non-linear singular integral equations, equivalent to (1)":

$$u_1(\xi,\ \eta) = \tfrac{1}{2} \cdot \int_0^\eta [u_2(x,\ y)+u_3(x,\ y)]\cdot dy \,,$$

$$(14)\quad u_i(\xi,\ \eta) = \int_0^\eta \{[\tilde{A}_i(x_i,\ y,\ u_1,\ u_2,\ u_3)/y]\cdot[u_2(x_i,\ y)-u_3(x_i,\ y)] + \tilde{B}_i(x_i,\ y,\ u_1,\ u_2,\ u_3)\}\cdot dy \quad (i = 2,\ 3) \,,$$

where

$$(15)\quad x_i(y;\ \xi,\ \eta) = \xi - (-1)^i \cdot \int_\eta^y t^{\alpha/2}\cdot K(x,\ t,\ u_1,\ u_2,\ u_3)\cdot dt \qquad (i = 2,\ 3) \,,$$

and subject to the initial conditions

$$(16)\qquad u_i(x,\ 0) = 0 \,, \quad i = 1,\ 2,\ 3 \,, \quad x \in [a,\ b] \,.$$

(II). By applying the above reduction (I), we achieve the main part of the proof of our theorem, as follows.

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Let $\overline{D} = \{(x, y) \mid (x \in [a, b]) \text{ and } (y \in [0, y_0])\}$ be a closed domain in $\mathbb{R}^2$, where y_0 is arbitrarily chosen.

Let us define in $\overline{D}$ the following set of continuous functions, namely:

$$C(\overline{D}) = \{\phi \mid \phi = \phi(x, y) \text{ is continuous in } \overline{D} \text{ such that } \phi_x \text{ is equicontinuous with respect to } x \text{ and } \phi(x, 0) = 0;\ \max\{|\phi|, |\phi_x|, |\phi_y|\} \le M, \text{ where } M \text{ is a fixed number}\} .$$

Let $\overline{D}_j = \bigcap_{i=1}^{3} \overline{D}_{ji}$, where $\overline{D}_{ji}$ are the closed domains of definition of arbitrary functions $\phi_{ji} \in C(\overline{D})$ replacing the functions u_i $(i = 1, 2, 3)$ and such that these domains are bounded by

$$y = \varepsilon < y_0 , \quad y = 0 ,$$

$$x_2 = a + \int_0^y t^{\alpha/2} \cdot K(x, t, \phi_{j1}(x, t), \phi_{j2}(x, t), \phi_{j3}(x, t)) \cdot dt ,$$

and

$$x_3 = b + \int_0^y (-t)^{\alpha/2} \cdot K(x, t, \phi_{j1}(x, t), \phi_{j2}(x, t), \phi_{j3}(x, t)) \cdot dt ,$$

where $i = 1, 2, 3$ and $j = 1, 2, 3, \ldots$, and the function $K = K(x, y, \phi_1, \phi_2, \phi_3)$ is bounded in the closed parallelepiped

$$\overline{\Pi} = \{(x, y, \phi) \mid (x \in [a, b]) \text{ and } (y \in [0, y_0]) \text{ and } (\phi = \phi_1, \phi_2, \phi_3) \in [-M, M])\} \subseteq \mathbb{R}^3 .$$

LEMMA 1. *Let* $\max_{\overline{D}} |K| \le \widetilde{M}$ *where* $K = K(x, y, \phi_1, \phi_2, \phi_3)$, $|\phi_i| \le M$ $(i = 1, 2, 3)$.

Let $\overline{D}_\varepsilon = \bigcap_{j=1}^{\infty} D_j$, *and* $S_\varepsilon = \{\phi \mid \phi = \phi(x, y)$ *is defined in* $\overline{D}_\varepsilon\}$, *where* $\varepsilon > 0$ *is sufficiently small.*

We define the norm of continuously differentiable functions

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$\phi = \phi(x, y)$ *in* S_ε *as* $\|\phi\|_{C^1} = \max_{\overline{D}_\varepsilon} \{|\phi|, |\phi_x|\}$.

We prove that the following inequalities hold, namely:

(1.1) $$\left|x_k^\phi(y;\ \xi,\ \eta)-x_k^\psi(y;\ \xi,\ \eta)\right| \le \lambda\cdot\|\phi-\psi\| \quad (k = 2,\ 3)\ ,$$

where

$$x_k^\phi(y;\ \xi,\ \eta) = \xi - (-1)^k \cdot \int_\eta^y t^{\alpha/2}\cdot K\left[x^\phi,\ t,\ \phi_1,\ \phi_2,\ \phi_3\right]\cdot dt\ ,$$

$$\phi_i = \phi_i(x,\ t) \quad (i = 1,\ 2,\ 3)\ , \quad (k = 2,\ 3)\ ,$$

$$x_k^\psi(y;\ \xi,\ \eta) = \xi - (-1)^k \cdot \int_\eta^y t^{\alpha/2}\cdot K\left[x^\psi,\ t,\ \psi_1,\ \psi_2,\ \psi_3\right]\cdot dt\ ,$$

$$\psi_i = \psi_i(x,\ t) \quad (i = 1,\ 2,\ 3)\ , \quad (k = 2,\ 3)\ ;$$

(1.2) $$|x_2(y;\ \xi,\ \eta)-x_3(y;\ \xi,\ \eta)| \le \mu\cdot\eta^{(\alpha+2)/2}\ , \quad 1 \le |x_\xi(y;\ \xi,\ \eta)| < 2\ ,$$

(1.3) $$\left|\frac{\partial x_2}{\partial \xi}(y;\ \xi,\ \eta) - \frac{\partial x_3}{\partial \xi}(y;\ \xi,\ \eta)\right| \le \Lambda\cdot\eta^{(\alpha+2)/2}\ ,$$

such that

$$\mu = 4\overline{M}/(\alpha+2)\ , \quad \lambda < 1\ , \quad 0 \le y \le \eta \le \varepsilon\ ,$$

and

$$\max_{\overline{D}_\varepsilon} \left\{|K_x|,\ |K_y|,\ |K_\phi|,\ |K^{-1}|;\ |\tilde{B}|,\ |\tilde{B}_x|,\ |\tilde{B}_\phi|\right\} \le \overline{M}\ ,$$

where

$$\tilde{B} = \tilde{B}_2 = \tilde{B}_3\ , \quad \tilde{B} = \tilde{B}(x,\ y,\ \phi)\ , \quad K = K(x,\ y,\ \phi)\ .$$

Proof. The proof of this lemma is an immediate application of the classical mean-value theorem. We prove only the inequalities (1.1) and (1.2) and observe that (1.3) is clear. In fact,

(1.1) $$x_k^\phi(y;\ \xi,\ \eta) - x_k^\psi(y;\ \xi,\ \eta) \le \overline{M}\cdot\left|\int_\eta^y t^{\alpha/2}\cdot\left[|x^\phi - x^\psi| + |\phi-\psi|\right]\cdot dt\right|$$
$$\le l \le \lambda\cdot\|\phi-\psi\|\ , \quad \lambda < 1\ ,$$

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because

$$l = \max_{\overline{D}_\varepsilon} \left| x_k^\phi - x_k^\psi \right| , \quad (l + |\phi-\psi|) \cdot |y-\eta|^{(\alpha+2)/2} \cdot (2\overline{M}/\alpha+2) \geq l ,$$

and therefore,

$$l \cdot [1-(2\overline{M}/\alpha+2) \cdot |y-\eta|^{(\alpha+2)/2}] \leq (2\overline{M}/\alpha+2) \cdot |y-\eta|^{(\alpha+2)/2} \cdot \|\phi-\psi\| ,$$

and by letting ε be so that $0 \leq y \leq \eta \leq \varepsilon$,

$$(1.2) \quad |x_2(y;\ \xi,\ \eta) - x_3(y;\ \xi,\ \eta)| \leq 2 \cdot \int_0^y |t^{\alpha/2} \cdot K| \cdot dt \leq 2\overline{M} \cdot \int_0^y t^{\alpha/2} \cdot dt$$

$$\leq 2\overline{M} \cdot \int_0^\eta t^{\alpha/2} \cdot dt$$

$$= (4\overline{M}/\alpha+2) \cdot \eta^{(\alpha+2)/2} = \mu \cdot \eta^{(\alpha+2)/2} ,$$

where $\mu = 4\overline{M}/\alpha + 2$.

The rest of the inequalities is clear. //

LEMMA 2. *For all* n *the following inequalities hold in* S_ε *, namely:*

$$(2.1) \qquad \left| u_i^{(n)}(\xi,\ \eta) \right| \leq \overline{M} \cdot \sum_{j=0}^{n} \delta^j \cdot \eta \quad (i = 1,\ 2,\ 3) ,$$

$$(2.2) \qquad \left| u_2^{(n)}(\xi,\ \eta) - u_3^{(n)}(\xi,\ \eta) \right| \leq \overline{M} \cdot \sum_{j=0}^{n} \delta^j \cdot \eta^{(\alpha+2)/2} ,$$

$$(2.3) \qquad \left| u_i^{(n)}(x_2,\ y) - u_i^{(n)}(x_3,\ y) \right| \leq \overline{M} \cdot \sum_{j=0}^{n} \delta^j \cdot \eta^{(\alpha+2)/2} \quad (i = 1,\ 2,\ 3) ,$$

where δ *is taken sufficiently close to* 1 .

Proof. To establish the *existence of a solution* of the system (14) we proceed by iterations. We define $u_i^{(0)}(\xi,\ \eta) = 0 \ (i = 1,\ 2,\ 3)$, and the quantities $u_i^{(n)}(x,\ y)$ by the relations

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$$(i7)\quad \begin{cases} u_1^{(n)} = \frac{1}{2} \cdot \int_0^\eta \left[u_2^{(n-1)} + u_3^{(n-1)}\right] \cdot dy \ , \\ u_i^{(n)} = \int_0^\eta \left\{ |\tilde{A}_i(x_i,\ y,\ \phi_1,\ \phi_2,\ \phi_3)/y| \cdot \left[u_2^{(n-1)}(x_i,\ y) - u_3^{(n-1)}(x_i,\ y)\right] \right. \\ \qquad \left. + \tilde{B}_i(x_i,\ y,\ \phi_1,\ \phi_2,\ \phi_3) \right\} \cdot dy \quad (i = 2,\ 3) \ . \end{cases}$$

We proceed by induction on n ; that is, we show that all the inequalities (2.1), (2.2), and (2.3) hold for $n = 1$, and then by assuming they all hold for $n = k$, we establish each inequality for $n = k + 1$.

We establish all inequalities simultaneously.

Case 1. $n = 1$

$$(2.1)\quad \left|u_i^{(1)}(\xi,\ \eta)\right| \le \int_0^\eta |\tilde{B}_i(x_i,\ y,\ \phi_1,\ \phi_2,\ \phi_3)| \cdot dy \le \overline{M} \cdot \eta \le \overline{M} \cdot \sum_{j=0}^{1} \delta^j \cdot \eta \qquad (i = 1,\ 2) \ ,$$

where δ is taken sufficiently close to 1 . The case $i = 1$ is trivially true.

$$(2.2)\quad \left|u_2^{(1)}(\xi,\ \eta) - u_3^{(1)}(\xi,\ \eta)\right| \le \int_0^\eta |\tilde{B}_2(x_2,\ y,\ \phi) - \tilde{B}_3(x_3,\ y,\ \phi)| \cdot dy$$

$$= \int_0^\eta |\tilde{B}(x_2,\ y,\ \phi) - \tilde{B}(x_3,\ y,\ \phi)| \cdot dy$$

$$\le \int_0^\eta [\|\tilde{B}_x\| + \|\tilde{B}_\phi\| \cdot \|\phi_x\|] \cdot |x_2(y;\ \xi,\ \eta) - x_3(y;\ \xi,\ \eta)| \cdot dy$$

$$\le 2\overline{M} \cdot \mu \cdot M \cdot \eta^{(\alpha+2)/2} \cdot \eta \le \overline{M} \cdot \sum_{j=0}^{1} \delta^j \cdot \eta^{(\alpha+2)/2} \ ,$$

where $\tilde{B} = \tilde{B}_i$ $(i = 2,\ 3)$, $\delta < 1$, ε is such that $0 \le \eta \le \varepsilon$ (in S_ε), and $2\mu \cdot M \cdot \eta < 1$. Besides, we have applied Lemma 1 through this proof. Similarly

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$$(2.3)\quad \left|u_i^{(1)}(x_2,\ y)-u_i^{(1)}(x_3,\ y)\right| \le \int_0^\eta |\tilde{B}(x_i(t;\ x_2,\ y),\ t,\ \phi) -\tilde{B}(x_i(t;\ x_3,\ y),\ t,\ \phi)|\cdot dt \le \overline{M}\cdot\sum_{j=0}^{1}\delta^j\cdot\eta^{(\alpha+2)/2}\ .$$

The case $i = 1$ is trivially true.

Case 2. $n = k + 1$

We suppose that Lemma 2 holds for $n = k$.

$$(2.1)\quad \left|u_1^{(k+1)}(\xi,\ \eta)\right| \le \tfrac{1}{2}\int_0^\eta \left[|u_2|^k+|u_3|^{\bar{k}}\right]\cdot dy \le \tfrac{1}{2}\int_0^\eta\left[2\overline{M}\cdot\sum_{j=0}^{k}\delta^j y\right]dy = \overline{M}\cdot\sum_{j=0}^{k}\delta^j\cdot\tfrac{1}{2}\eta^2 \le \overline{M}\cdot\sum_{j=0}^{k+1}\delta^j\cdot\eta\ ,$$

$$\left|u_i^{(k+1)}(\xi,\ \eta)\right| \le \int_0^\eta\left\{[|\tilde{A}_i(x_i,\ y,\ \phi)/y|]\cdot\left[\left|u_2^{(k)}(x_i,\ y)-u_3^{(k)}(x_i y)\right|\right] + |\tilde{B}_i(x_i,\ y,\ \phi)|\right\}\cdot dy\ .$$

But

$$|\tilde{A}_i| = |A_i\cdot|1+y^\alpha\cdot k^2|^{-\frac{1}{2}}| \le \frac{\alpha}{4}+\gamma\cdot y \to \frac{\alpha}{4}\ ,\ \text{as}\quad y\to 0\quad (i = 2,\ 3)\ .$$

Therefore,

$$\left|u_i^{(k+1)}(\xi,\ \eta)\right| \le \int_0^\eta\left[\left(\frac{\alpha}{4}+\gamma\cdot y\right)\cdot\overline{M}\cdot\sum_{j=0}^{k}\delta^j\cdot y^{\alpha/2}+\overline{M}\right]\cdot dy \le \overline{M}\cdot\left[1+\left(\sum_{j=0}^{k}\delta^j\right)\cdot\delta\right]\cdot\eta \le \overline{M}\cdot\sum_{j=0}^{k+1}\delta^j\cdot\eta\quad (i = 2,\ 3)\ ;$$

by choosing the width ε of the strip so that for $0 \le y \le \eta \le \varepsilon$,

$$\left(\frac{\alpha}{4}+\gamma\cdot y\right)\cdot y^{\alpha/2} \le \delta < 1\ ,\ \text{and}\quad \gamma\cdot y\to 0\ ,\ \text{as}\quad y\to 0\ .$$

$$(2.2)\quad \left|u_2^{(k+1)}(\xi,\ \eta)-u_3^{(k+1)}(\xi,\ \eta)\right|$$

$$\le \int_0^\eta \left|\left\{\tilde{A}_2(x_2,\ y,\ \phi)\cdot\left[u_2^{(k)}(x_2,\ y)-u_3^{(k)}(x_3,\ y)\right]\right.\right.$$

$$\left.-\tilde{A}_3(x_3,\ y,\ \phi)\cdot\left[u_2^{(k)}(x_3,\ y)-u_3^{(k)}(x_3,\ y)\right]\right\}/y$$

$$\left.+\ \tilde{B}_2(x_2,\ y,\ \phi)-\tilde{B}_3(x_3,\ y,\ \phi)\right|\cdot dy$$

$$\le \int_0^\eta \left\{\left(\frac{\alpha}{4}+\gamma\cdot y\right)\cdot\overline{M}\cdot\sum_{j=0}^{k}\delta^j\cdot y^{\alpha/2}+2\overline{M}\cdot M\cdot|x_2-x_3|\right\}\cdot dy$$

$$\le \overline{M}\cdot\eta^{(\alpha+2)/2}\cdot\left\{2\mu\cdot M\cdot\eta+[4/(\alpha+2)]\cdot\left(\frac{\alpha}{4}+\gamma\cdot y\right)\cdot\sum_{j=0}^{k}\delta^j\right\}$$

$$\le \overline{M}\cdot\sum_{j=0}^{k+1}\delta^j\cdot\eta^{(\alpha+2)/2}\ ,$$

by choosing ε such that $0\le\eta\le\varepsilon$, and

$$2\mu\cdot M\cdot\eta = 2\cdot\frac{4\overline{M}}{\alpha+2}\cdot M\cdot\eta = 8\cdot\frac{\overline{M}\cdot M}{\alpha+2}\cdot\eta < 1\ ,$$

$$\frac{4}{\alpha+2}\cdot\left(\frac{\alpha}{4}+\gamma\cdot y\right)\le\delta<1\ ,\quad \lim_{y\to 0}\gamma\cdot y = 0\ .$$

Similarly,

$$(2.3)\quad \left|u_i^{(k+1)}(x_2,\ y)-u_i^{(k+1)}(x_3,\ y)\right|$$

$$\le \int_0^\eta \left|\left\{[\tilde{A}_i(x_i(t;\ x_2,\ y),\ y,\ \phi)/y]\cdot\left[u_2^{(i)}(x_i(t;\ x_2,\ y),\ y)\right.\right.\right.$$

$$\left.\left.-u_3^{(k)}(x_i(t;\ x_2,\ y),\ y)\right]+\tilde{B}_i(x_i(t;\ x_2,\ y),\ y,\ \phi)\right\}$$

$$-\left\{[\tilde{A}_i(x_i(t;\ x_3,\ y),\ y,\ \phi)/y]\cdot\left[u_2^{(k)}(x_i(t;\ x_3,\ y),\ y)\right.\right.$$

$$\left.\left.\left.-u_3^{(k)}(x_i(t;\ x_3,\ y),\ y)\right]+\tilde{B}_i(x_i(t;\ x_3,\ y),\ y,\ \phi)\right\}\right|\cdot dy$$

$$\le \overline{M}\cdot\sum_{j=0}^{k+1}\delta^j\cdot\eta^{(\alpha+2)/2}\quad (i = 2,\ 3)\ .$$

The case $i = 1$ is trivial. //

LEMMA 3. *For all* n *the following inequalities hold:*

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$$(3.1)\quad \left|u_i^{(n+1)}(\xi,\ \eta)-u_i^{(n)}(\xi,\ \eta)\right| \le \overline{M}\cdot\delta^n\cdot\eta \quad (i = 1,\ 2,\ 3)\ ;$$

$$(3.2)\quad \left|u_2^{(n+1)}(\xi,\ \eta)-u_3^{(n+1)}(\xi,\ \eta)-u_2^{(n)}(\xi,\ \eta)+u_3^{(n)}(\xi,\ \eta)\right| \le \overline{M}\cdot\delta^n\cdot\eta^{(\alpha+2)/2}\ ;$$

$$(3.3)\quad \left|u_i^{(n+1)}(x_2,\ y)-u_i^{(n+1)}(x_3,\ y)-u_i^{(n)}(x_2,\ y)+u_i^{(n)}(x_3,\ y)\right| \le \overline{M}\cdot\delta^n\cdot\eta^{(\alpha+2)/2} \quad (i = 1,\ 2,\ 3)\ .$$

Proof. We prove only (3.1) and (3.2), while (3.3) can be proved in the same way as in the cases (3.1) and (3.2). In fact, by induction on n,

$$(3.1)\quad \left|u_i^{(n+1)}(\xi,\ \eta)-u_i^{(n)}(\xi,\ \eta)\right| \le \int_0^\eta \left\{\left|\tilde{A}_i(x_i,\ y,\ \phi)/y\right|\cdot\left[\left|u_2^{(n)}(x_i,\ y)-u_2^{(n-1)}(x_i,\ y)\right| + \left|u_3^{(n)}(x_i,\ y)-u_3^{(n-1)}(x_i,\ y)\right|\right]\right\}\cdot dy$$

$$\le \int_0^\eta 2\left(\frac{\alpha}{4}+\gamma\cdot y\right)\cdot\frac{1}{y}\,\overline{M}\cdot\delta^{n-1}\cdot y\cdot dy$$

$$\le \overline{M}\cdot\delta^n\cdot\eta\ ,\quad \lim_{y\to 0}\gamma\cdot y = 0\ ;$$

$$(3.2)\quad \left|u_2^{(n+1)}(\xi,\ \eta)-u_3^{(n+1)}(\xi,\ \eta)-u_2^{(n)}(\xi,\ \eta)+u_3^{(n)}(\xi,\ \eta)\right|$$

$$\le \int_0^\eta \Big|\left[\tilde{A}_2(x_2,\ y,\ \phi)/y\right]\cdot\left[u_2^{(n)}(x_2,\ y)-u_3^{(n)}(x_2,\ y)\right] - \left[\tilde{A}_3(x_3,\ y,\ \phi)/y\right]\cdot\left[u_2^{(n)}(x_3,\ y)-u_3^{(n)}(x_3,\ y)\right] - \left[\tilde{A}_2(x_2,\ y,\ \phi)/y\right]\cdot\left[u_2^{(n-1)}(x_2,\ y)-u_3^{(n-1)}(x_2,\ y)\right] + \left[\tilde{A}_3(x_3,\ y,\ \phi)/y\right]\cdot\left[u_2^{(n-1)}(x_3,\ y)-u_3^{(n-1)}(x_3,\ y)\right]\Big|\cdot dy$$

$$\le \int_0^\eta \left(\frac{\alpha}{4}+\gamma\cdot y\right)\cdot\frac{1}{y}\cdot\left\{\left|u_2^{(n)}(x_2,\ y)-u_3^{(n)}(x_2,\ y) -u_2^{(n-1)}(x_2,\ y)+u_3^{(n-1)}(x_2,\ y)\right| + \left|u_2^{(n)}(x_3,\ y) -u_3^{(n)}(x_3,\ y)-u_2^{(n-1)}(x_3,\ y)+u_3^{(n-1)}(x_3,\ y)\right|\right\}\cdot dy$$

$$\le 2\cdot\left(\frac{\alpha}{4}+\tilde{\alpha}\right)\cdot\overline{M}\cdot\delta^{n-1}\cdot\eta^{\alpha/2} \le \overline{M}\cdot\delta^n\cdot\eta^{(\alpha+2)/2}\ ;$$

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$$\tilde{\alpha} = \tilde{\alpha}(y) = \gamma \cdot y \ , \quad \lim_{y \to 0} \tilde{\alpha}(y) = 0 \ .$$

(3.3) follows similarly, as in the above case. //

Then by applying Lemmas 1, 2, 3, and Ascoli-Arzela's theorem we get the required result. (See also [10].)

In fact, by Lemma 3 it is clear that the sequences $\left\{u_i^{(n)}(x, y)\right\}$ $(i = 2, 3)$ converge uniformly in S_ε . Since each $u_i^{(n)} = u_i^{(n)}(x, y)$ is continuous in S_ε , so are the limits, which we denote by $u_i = u_i(x, y)$.

The resulting linear system is solvable, and the solutions satisfy the following inequalities uniformly in S_ε :

$$|u_i(x, y)| \le \bar{M} \cdot \sum_{j=0}^{\infty} \delta^j \cdot y \ ;$$

$$(*) \quad |u_2(x, y) - u_3(x, y)| \le M \cdot \sum_{j=0}^{\infty} \delta^j \cdot y^{(\alpha+2)/2} \ , \text{ where } \ \delta < 1, \ i = 1, 2, 3 .$$

For an appropriate choice of ε , $|u_i(\xi, \eta)| \le M$ for all $\phi \in S_\varepsilon$.

Similar inequalities hold for the derivatives $(u_i)_\xi(\xi, \eta)$ and $(u_i)_\eta(\xi, \eta)$ in S_ε $(i = 1, 2, 3)$. Moreover, if ε is sufficiently small, $|(u_i)_\xi| \le M$, $|(u_i)_\eta| \le M$ in S_ε $(i = 1, 2, 3)$.

Then by taking into account (14), (15), and (16), we are done.

As a matter of fact to prove that the system of integral equations (14) has a unique solution in a neighborhood $(y > 0)$ of $I = [\alpha, \beta]$, we apply the well-known Schauder Fixed Point Theorem (namely: a continuous mapping of a convex, compact subset of a Banach space into itself has a fixed point).

Let the continuous operator $T : S_\varepsilon \xrightarrow{\text{into}} S_\varepsilon$ be defined as follows:

$$\begin{cases} T_1(\phi) = \frac{1}{2} \cdot \int_0^\eta [u_2(x, y)+u_3(x, y)] \cdot dy , \\ T_i(\phi) = \int_0^\eta \left\{\left[\tilde{A}_i\left(x_i^\phi, x, \phi\right)/y\right] \cdot \left[u_2\left(x_i^\phi, y\right) - u_3\left(x_i^\phi, y\right)\right] + \tilde{B}_i\left(x_i^\phi, y, \phi\right)\right\} \cdot dy \\ \qquad\qquad (i = 2, 3) , \end{cases}$$

where $\phi = \phi\left(x_i^\phi, y\right)$.

By applying the classical mean value theorem and the above lemmas, we get

$$\|T_i(\phi)-T_i(\psi)\| \leq \lambda \cdot \eta \cdot \|\phi-\psi\| ,$$

$$\|T_1(\phi)-T_1(\psi)\| \leq \lambda \cdot \frac{\eta^2}{2} \cdot \|\phi-\psi\| , \quad i = 2, 3 , \quad \lambda = \text{const.}$$

Therefore

(19) $$\|T(\phi)-T(\psi)\| \leq \bar{\lambda} \cdot \|\phi-\psi\| ,$$

where $\bar{\lambda} = \lambda \cdot (4\eta+\eta^2)/2$; and now we choose ε sufficiently small, such that $0 \leq y \leq \eta \leq \varepsilon \Rightarrow \bar{\lambda} < 1$; and hence by (19),

(20) $$T : S_\varepsilon \xrightarrow{\text{into}} S_\varepsilon \quad \text{is a contraction operator;}$$

and from Schauder's Fixed Point Theorem it follows that T has a unique fixed point in S_ε .

We note the uniform convergence of $\left\{u_i^{(j)}\right\}$ $(i = 1, 2, 3)$, $(j = 1, 2, 3, \ldots)$, and of the derivatives of $\left\{u_i^{(j)}\right\}$, namely $\left\{u_{i\xi}^{(j)}\right\}$ and $\left\{u_{i\eta}^{(j)}\right\}$, in S_ε is a consequence of Ascoli-Arzela's Theorem. For a more detailed proof see also [10]. //

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11 Dervenakion Str,
Daphne,
Athens,
Greece.