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Relations between boundaries of a riemannian manifold

J.L. Schiff

For a noncompact riemannian manifold R, let $M_P(R)$ be the P-algebra, and R_P^* the P-compactification, with the assumption that $\int_R PdV = \infty$. If s is the P-singular point of the P-harmonic boundary Δ_P, and Δ is the harmonic boundary of Royden's compactification R^*, we construct a continuous mapping $\pi : R^* \to R_P^*$ such that $\pi(\Delta) = \Delta_P$ or $\pi(\Delta) = \Delta_P - s$. In the former case, $\pi(p) = s$ if and only if $\int_{U \cap R} PdV = \infty$ for every neighborhood U of p. For the open set

$$\Delta^P = \left\{p \in \Delta \mid \int_{U \cap R} PdV < \infty \text{ for some neighborhood } U \text{ of } p\right\}$$

considered by Kwang-nan Chow ["Representing measures on the Royden boundary for solutions of $\Delta u = Pu$ on a Riemannian manifold", (Doctoral dissertation, California Institute of Technology, Pasadena, 1970)], we have $\pi : \Delta^P \to \Delta_P - s$ is a homeomorphism if Δ^P is closed. If U_{HD} (U_{PE}) denotes the class of Riemann surfaces which carry HD-minimal (PE-minimal) functions, then $\Delta = \Delta^P$ implies $U_{HD} = U_{PE}$.

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1.

Let R be an arbitrary noncompact riemannian manifold, and $P \not\equiv 0$ a nonnegative density function in $C^2(R)$. Denote by $M(R)$ Royden's algebra and by R^* Royden's compactification. By means of the subalgebra $M_P(R)$ of bounded energy-finite Tonelli functions on R, one constructs the P-compactification R_P^* of R_P (Nakai and Sario [8], Kwon, Sario and Schiff [5, 6]).

Consider the continuous open mapping

$$\pi : R^* \to R_P^* \subset \prod_{f \in M_P(R)} I_f \ , \quad I_f = \left[-\|f\|_\infty, \|f\|_\infty\right] ,$$

defined by

$$\left(\pi(p)\right)_f = f(p) \ , \quad p \in R^* \ , \quad f \in M_P(R) \ .$$

Let Δ be the Royden harmonic boundary, and Δ_P the P-harmonic boundary of R (*loc. cit.*). We shall assume that $\int_R PdV = \infty$. Then Δ_P contains a unique point s, called the P-singular point, at which all functions in $M_P(R)$ vanish (*loc. cit.*).

2.

It was shown by Wang [10] that either $\pi(\Delta) = \Delta_P$ or $\pi(\Delta) = \Delta_P - s$, with the latter a homeomorphism. We shall call the pair (R, P) *singular* or *nonsingular* according as the former or latter alternative occurs.

THEOREM 1. *$\pi(p) = s$ if and only if $\int_{U \cap R} PdV = \infty$ for every neighborhood U of p.*

We remark that in the terminology of Glasner and Katz [3], $\pi(p) = s$ if and only if " P has infinite integral at p ".

Proof. Suppose $\pi(p) = s$ for some $p \in \Gamma = R^* - R$. For every $f \in M_P(R)$, $f(p) = f\left(\pi(p)\right) = 0$. Let U be a neighborhood of p. By Urysohn's property for Royden's compactification, there exists a function

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$g \in M(R)$ such that $0 \le g \le 1$ on R^*, $g(p) = 1$, and $\operatorname{supp} g \subset U$. Since $g(p) \neq 0$, $g \notin M_P(R)$. Therefore $E_R(g) = D_R(g) + \int_R Pg^2 dV = \infty$. But $g \in M(R)$ implies $D_R(g) < \infty$, and thus $\int_R Pg^2 dV = \infty$. It follows that

$$\int_{U \cap R} PdV \ge \int_{U \cap R} Pg^2 dV = \int_R Pg^2 dV = \infty .$$

Conversely, suppose $\int_{U \cap R} PdV = \infty$ for every neighborhood U of p. We claim that $\pi(p) = s$. In fact, if $\pi(p) = q \neq s$, then there exists a neighborhood N of q for which $\int_{N \cap R} PdV < \infty$ (Kwon and Sario [4]). Since π is continuous, we can choose a neighborhood U of p such that $\pi(U) \subset N$. In view of the fact that $U \cap R \subset \pi(U) \cap R$, we have

$$\int_{U \cap R} PdV \le \int_{\pi(U) \cap R} PdV \le \int_{N \cap R} PdV < \infty ,$$

a contradiction. This proves the theorem.

3.

Suppose (R, P) is nonsingular. Then for any $p \in \Delta$, $\pi(p) \neq s$, and we conclude as above that there exists a neighborhood U of p with $\int_{U \cap R} PdV < \infty$. In the general case consider the open set

$$\Delta^P = \left\{ p \in \Delta \;\middle|\; \int_{U \cap R} PdV < \infty \text{ for some neighborhood } U \text{ of } p \right\}$$

(Chow [1]). A point $p \in \Delta^P$ has been termed a *P-nondensity point* by Nakai [7].

Since π is an open mapping we obtain:

THEOREM 2. *If s is a point of accumulation of Δ_P, then $\pi(\Delta) = \Delta_P$ and $\pi(\Delta \backslash \Delta^P) = s$.*

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4.

The following result relates the set Δ^P to Δ_P (*cf.* Wang [10]):

THEOREM 3. *If* Δ^P *is closed in* R^* *, then* $\pi : \Delta^P \to \Delta_P - s$ *is a homeomorphism.*

Proof. Clearly $\pi(\Delta^P) \subset \Delta_P$. Suppose $\pi(p) = s$ for some point $p \in \Delta^P$. Then as above, we have for every neighborhood U of p

$$\int_{U \cap R} P dV = \infty ,$$

a contradiction. Therefore $\pi(\Delta^P) \subset \Delta_P - s$.

To show that π is surjective, assume there existed a point $q \in \Delta_P - s \backslash \pi(\Delta^P)$. Since Δ^P is closed in R^* , $\pi(\Delta^P)$ is compact in R_P^* . In analogy with the proof in Kwon, Sario and Schiff [6] of the Urysohn-type property for $M_P(R)$, we conclude that there exists a function $f \in M_P(R)$ with $0 \le f \le 1$ on R_P^* , $f | \pi(\Delta^P) = 0$, and $f(q) = 1$. Let u be the P-harmonic projection of f . Then $u | \Delta^P = 0$, and therefore $u \equiv 0$ on R (Chow [1]). But this is in violation of $u(q) = 1$.

On the other hand, π will be injective if for any two points $p \neq q$ in Δ^P we can find a function $f \in M_P(R)$ such that $f(p) \neq f(q)$. Since π is continuous and surjective, $\Delta_P - s$ and $\{s\}$ are disjoint compact subsets of R_P^* . Thus there exists a function $g \in M_P(R)$ with $0 \le g \le 1$ on R_P^* , $g | \Delta_P - s = 1$, and therefore $g | \Delta^P = 1$. Let $h \in M(R)$ be such that $h(p) \neq h(q)$, and $0 \le h \le 1$ on R^* . On setting $f = gh$ we obtain $f | \Delta^P = h | \Delta^P$, and consequently $f(p) \neq f(q)$. Moreover, $0 \le f \le g$ implies $f \in M_P(R)$ as desired.

5.

The following is an immediate consequence of Theorem 3.

THEOREM 4. *(R, P) is nonsingular if and only if* $\Delta = \Delta^P$.

COROLLARY 1 (*cf.* Glasner and Katz [3]). $\overline{\overline{\Delta}} = \overline{\overline{\Delta^P}} = n$ *if and only if* $\dim HBD = \dim PBE = n$ *, for* $1 \le n < \infty$.

We denote by U_{HD} (U_{PE}) the class of Riemann surfaces on which there exist *HD*-minimal (*PE*-minimal) functions.

COROLLARY 2. *If* $\Delta = \Delta^P$ *, there exists a one-to-one correspondence between the HD-minimal functions on* R *and the PE-minimal functions on* R . *Hence* $U_{HD} = U_{PE}$ *in this case.*

This can be seen by observing that the isolated points of Δ are in a one-to-one correspondence with the *HD*-minimal functions on R (*cf.* Sario and Nakai [9]), and similarly for the isolated points of $\Delta_P - s$ and the *PE*-minimal functions on R (Kwon, Sario and Schiff [5]).

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University of Auckland,
Auckland,
New Zealand.