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SOME RESULTS ON COMPARING TWO INTEGRAL MEANS FOR ABSOLUTELY CONTINUOUS FUNCTIONS AND APPLICATIONS

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Abstract

Some better estimates for the difference between the integral mean of a function and its mean over a subinterval are established. Various applications for special means and probability density functions are also given.

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1. Introduction

The classical Ostrowski integral inequality [19] stipulates a bound between a function evaluated at an interior point and the average of the function over an interval. More precisely,

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f(x)\right| \le \left(\frac{1}{4} + \frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right)(b-a)\|f'\|_\infty \tag{1.1}$$

for all $x \in [a,b]$, where $f' \in L_\infty(a,b)$, that is,

$$\|f'\|_\infty = \operatorname{ess}\sup_{t\in[a,b]} |f'(t)| < \infty,$$

and $f : [a,b] \to \mathbb{R}$ is a differentiable function on (a,b). Here, the constant $1/4$ is sharp in the sense that it cannot be replaced by a smaller constant.

It is worth noticing that this inequality plays a key role in adaptive numerical quadrature rules. For various results and generalisations concerning Ostrowski's inequality, see [1, 2, 4, 9, 11–13, 15–17, 20–24] and the references therein.

In [10], Dragomir and Wang introduced the following inequality of Ostrowski–Grüss type:

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f(x) - \left(\frac{a+b}{2}-x\right)\frac{f(b)-f(a)}{b-a}\right| \le \frac{1}{4}(b-a)(\Gamma-\gamma), \qquad (1.2)$$

where $f:[a,b]\to\mathbb{R}$, is a differentiable function on (a,b) and $\gamma \le f'(x) \le \Gamma$ for all $x\in[a,b]$. There are many improvements and refinements of the right-hand side of (1.2) in the literature. See, for instance, [6, 8, 11, 12, 14].

On the other hand, in [3], Barnett *et al.* compared the difference of two integral means as in the following Theorem 1.1 in which the function has the first derivative bounded where it is defined. The results are also a generalisation of (1.1) and were applied to probability density functions, special means, Jeffreys divergence in information theory and the sampling of continuous streams in statistics.

Theorem 1.1. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function with the property that $f'\in L_\infty[a,b]$. Then, for $a\le c<d\le b$, we have the inequalities*

$$\begin{aligned}\left|\frac{1}{b-a}\int_a^b f(u)\,du - \frac{1}{d-c}\int_c^d f(u)\,du\right| &\le \left(\frac{1}{4}+\left(\frac{\frac{a+b}{2}-\frac{c+d}{2}}{b-a-d+c}\right)^2\right)(b-a-d+c)\|f'\|_\infty \qquad (1.3)\\ &\le \frac{1}{2}(b-a-d+c)\,\|f'\|_\infty.\end{aligned}$$

The constant $1/4$ is best possible in the first inequality and $1/2$ is best in the second one.

The purpose of this article is to establish, by using a variant of the Grüss inequality, some improvements on Theorem 1.1 in which f' may not belong to L_∞. Applying these results, some new inequalities for special means and probability density functions will also be given in Sections 3 and 4, respectively.

2. Preliminary lemmas and main results

The following lemma is the known Grüss inequality; see [18, page 295].

Lemma 2.1. *Let $f,g:[a,b]\to\mathbb{R}$ be two integrable functions such that $\phi\le f(x)\le\Phi$ and $\gamma\le g(x)\le\Gamma$ for all $x\in[a,b]$ where γ,Γ,ϕ,Φ are constants. Then we have the inequality*

$$\begin{aligned}\left|\frac{1}{b-a}\int_a^b f(x)g(x)\,dx - \frac{1}{b-a}\int_a^b f(x)\,dx\frac{1}{b-a}\int_a^b g(x)\,dx\right|\\ \le \frac{1}{4}(\Phi-\phi)(\Gamma-\gamma),\end{aligned}$$

and the constant $1/4$ is the best possible.

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Further, Cheng and Sun [7] established the following variant of Grüss's inequality. For extensions in the general case of the Lebesgue integral on measurable spaces, the sharpness of the constant $1/2$ as well as the corresponding discrete version, see [5].

LEMMA 2.2. *Let $f, g : [a, b] \to \mathbb{R}$ be two integrable functions such that $\gamma \le g(x) \le \Gamma$ for all $x \in [a, b]$ where γ and Γ are constants. Then we have the inequality*

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)\,dx - \frac{1}{b-a}\int_a^b f(x)\,dx \frac{1}{b-a}\int_a^b g(x)\,dx\right| \le \frac{1}{2}(\Gamma-\gamma)\frac{1}{(b-a)}\int_a^b \left|f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right|dx.$$

The following lemma has been obtained by Barnett *et al.* in [3].

LEMMA 2.3. *Let $f : [a, b] \to \mathbb{R}$ be an absolutely continuous function and $a \le c < d \le b$. Denote by $K_{c,d} : [a, b] \to \mathbb{R}$, the kernel given by*

$$K_{c,d}(s) = \begin{cases} \dfrac{a-s}{b-a} & \text{if } s \in [a, c], \\ \dfrac{s-c}{d-c} + \dfrac{a-s}{b-a} & \text{if } s \in (c, d), \\ \dfrac{b-s}{b-a} & \text{if } s \in [d, b]. \end{cases}$$

Then we have the representation

$$\frac{1}{b-a}\int_a^b f(u)\,du - \frac{1}{d-c}\int_c^d f(u)\,du = \int_a^b K_{c,d}(s) f'(s)\,ds.$$

Our main results are as follows.

THEOREM 2.4. *Let $f : [a, b] \to \mathbb{R}$ be an absolutely continuous function such that $\gamma \le f'(x) \le \Gamma$ for all $x \in [a, b]$ where γ and Γ are constants. Then we have the inequality*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - \frac{1}{d-c}\int_c^d f(u)\,du - \frac{b-c-d+a}{2(b-a)}(f(b)-f(a))\right| \le \frac{1}{4}(b-a+c-d)(\Gamma-\gamma), \tag{2.1}$$

where $a \le c < d \le b$.

PROOF. Take $f(x) = K_{c,d}(x)$ and $g(x) = f'(x)$ in Lemma 2.1. Since $K_{c,d}(c) \le K_{c,d}(x) \le K_{c,d}(d)$ for all $x \in [a, b]$, by Lemma 2.1,

$$\left|\frac{1}{b-a}\int_a^b K_{c,d}(x) f'(x)\,dx - \frac{1}{b-a}\int_a^b K_{c,d}(t)\,dt \frac{1}{b-a}\int_a^b f'(x)\,dx\right| \le \frac{1}{4}(K_{c,d}(d) - Kc, d(c))(\Gamma-\gamma).$$

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Further, by Lemma 2.3,

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{1}{d-c}\int_c^d f(u)\,du)-\frac{1}{b-a}\int_a^b K_{c,d}(t)\,dt(f(b)-f(a))\right| \leq \frac{1}{4}(b-a)(K_{c,d}(d)-K_{c,d}(c))(\Gamma-\gamma).$$

Now, since

$$\int_a^b K_{c,d}(t)\,dt=\frac{b-c-d+a}{2},\quad K_{c,d}(c)=\frac{a-c}{b-a},\quad K_{c,d}(d)=\frac{b-d}{b-a},$$

by the above inequality we deduce the desired inequality (2.1).

This completes the proof of Theorem 2.4. □

Remark 2.5. Inequality (2.1) is a generalisation of (1.3). If we set $d=c+h$ with $c+h\in(a,b)$, then, by (2.1),

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{1}{h}\int_c^{c+h} f(u)\,du-\frac{b-2c-h+a}{2(b-a)}(f(b)-f(a))\right| \leq \frac{1}{4}(b-a+h)(\Gamma-\gamma).$$

Now letting $h\to 0^+$ yields

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-f(c)-\left(\frac{a+b}{2}-c\right)\frac{f(b)-f(a)}{b-a}\right|\leq\frac{1}{4}(b-a)(\Gamma-\gamma). \tag{2.2}$$

We note that (2.2) is the Ostrowski–Grüss type inequality obtained by Dragomir and Wang in [10].

Corollary 2.6. *Let f, f', γ and Γ be defined as in Theorem 2.4 and $a+b=c+d$. Then*

$$\left|\frac{1}{b-a}\int_a^b f(u)\,du-\frac{1}{d-c}\int_c^d f(u)\,du\right|\leq\frac{1}{2}(c-a)(\Gamma-\gamma). \tag{2.3}$$

Proof. Since $a+b=c+d$, by Theorem 2.4, Corollary 2.6 holds immediately. □

Remark 2.7. For $\Gamma\gamma>0$, (2.3) is an improvement on (1.3) provided that $a+b=c+d$.

For any $x\in(a,b)$ and some $\delta>0$, let the function $F(x,\cdot):[-\delta,\delta]\to\mathbb{R}$ be defined by

$$F(x,t)=\frac{1}{t}\int_{x-t/2}^{x+t/2} f(u)\,du.$$

We obtain the following corollary.

Corollary 2.8. *Assume that the function $f:[a,b]\to\mathbb{R}$ is absolutely continuous on $[a,b]$ and $\gamma<f'(t)<\Gamma$, for $t\in[a,b]$. Then the function $F(x,\cdot)$ is locally Lipschitzian and the Lipschitzian constant is $\frac{1}{4}(\Gamma-\gamma)$ and is independent of x.*

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Proof. Assume that $x \in (a, b), t_1, t_2 \in [-\delta, \delta]$, with $t_2 > t_1$. For $[x - t_1/2, x + t_1/2] \subset [x - t_2, x + t_2] \subset (a, b)$, by Corollary 2.6,

$$\left|\frac{1}{t_2}\int_{x-t_2/2}^{x+t_2/2} f(u)\,du - \frac{1}{t_1}\int_{x-t_1/2}^{x+t_1/2} f(u)\,du\right| \le \frac{1}{4}(t_2 - t_1)(\Gamma - \gamma)$$

which shows that

$$|F(x, t_2) - F(x, t_1)| \le \tfrac{1}{4}(t_2 - t_1)(\Gamma - \gamma),$$

Similarly, for $t_1 > t_2$,

$$|F(x, t_2) - F(x, t_1)| \le \tfrac{1}{4}(t_1 - t_2)(\Gamma - \gamma),$$

and then

$$|F(x, t_2) - F(x, t_1)| \le \tfrac{1}{4}|t_2 - t_1|(\Gamma - \gamma),$$

which proves the corollary. □

Remark 2.9. We note that, for $\Gamma\gamma > 0$, Corollary 2.8 is an improvement on Corollary 2.4 in [3].

Theorem 2.10. *Let $f : [a, b] \to \mathbb{R}$ be an absolutely continuous function. Then we have the inequality*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - \frac{1}{d-c}\int_c^d f(u)\,du - \frac{b-c-d+a}{2(b-a)}(f(b) - f(a))\right| \le \frac{b-a+c-d}{2(b-a)}\int_a^b \left|f'(x) - \frac{f(b)-f(a)}{b-a}\right| dx, \tag{2.4}$$

where $a \le c < d \le b$.

Proof. Set $f(x) = f'(x)$ and $g(x) = K_{c,d}(x)$ in Lemma 2.2. Since $K_{c,d}(c) \le K_{c,d}(x) \le K_{c,d}(d)$ for all $x \in [a, b]$, by Lemma 2.2,

$$\left|\frac{1}{b-a}\int_a^b K_{c,d}(x)f'(x)\,dx - \frac{1}{b-a}\int_a^b K_{c,d}(t)\,dt\frac{1}{b-a}\int_a^b f'(x)\,dx\right| \le \frac{1}{2(b-a)}(K_{c,d}(d) - Kc, d(c))\int_a^b \left|f'(x) - \frac{f(b)-f(a)}{b-a}\right| dx.$$

Further, by Lemma 2.3,

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - \frac{1}{d-c}\int_c^d f(u)\,du - \frac{1}{b-a}\int_a^b K_{c,d}(t)\,dt(f(b) - f(a))\right| \le \frac{1}{2}(K_{c,d}(d) - K_{c,d}(c))\int_a^b \left|f'(x) - \frac{f(b)-f(a)}{b-a}\right| dx.$$

Now, using the fact that

$$\int_a^b K_{c,d}(t)\,dt = \frac{b-c-d+a}{2}, \quad K_{c,d}(c) = \frac{a-c}{b-a}, \quad K_{c,d}(d) = \frac{b-d}{b-a},$$

by the above inequality we get the desired inequality (2.4). This completes the proof of Theorem 2.10. □

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REMARK 2.11. Inequality (2.4) is a generalisation of (1.3). If we set $d = c + h$ with $c + h \in (a, b)$, then, by (2.4),

$$\begin{aligned}\left|\frac{1}{b-a}\int_a^b f(x)\,dx - \frac{1}{h}\int_c^{c+h} f(u)\,du - \frac{b-2c-h+a}{2(b-a)}(f(b)-f(a))\right| \\ \leq \frac{1}{4}(b-a+h)\int_a^b \left|f'(x) - \frac{f(b)-f(a)}{b-a}\right| dx.\end{aligned}$$

Now letting $h \to 0^+$ yields

$$\begin{aligned}\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f(c) - \left(\frac{a+b}{2} - c\right)\frac{f(b)-f(a)}{b-a}\right| \\ \leq \frac{1}{4}(b-a)\int_a^b \left|f'(x) - \frac{f(b)-f(a)}{b-a}\right| dx.\end{aligned} \tag{2.5}$$

We note that the condition imposed upon f' in (2.5) is weaker than that in (2.2) given by Dragomir and Wang [10].

COROLLARY 2.12. *Let f and f' be defined as in Theorem 2.10 and $a + b = c + d$. Then*

$$\begin{aligned}\left|\frac{1}{b-a}\int_a^b f(u)\,du - \frac{1}{d-c}\int_c^d f(u)\,du\right| \\ \leq \frac{c-a}{b-a}\int_a^b \left|f'(x) - \frac{f(b)-f(a)}{b-a}\right| dx.\end{aligned} \tag{2.6}$$

PROOF. Since $b - d = c - a$, using (2.4), (2.6) holds immediately. This completes the proof of the corollary. □

REMARK 2.13. We note that the condition imposed upon f' in Corollary 2.12 is weaker than that in Corollary 2.6.

3. Applications to special means

In the following, we shall consider logarithmic, identric and generalised logarithmic means of two positive real numbers. We take

$$L(\alpha,\beta) = \frac{\beta-\alpha}{\log\beta - \log\alpha}, \qquad \alpha,\beta \in \mathbb{R}^+, \alpha \neq \beta,$$

$$I(\alpha,\beta) = \frac{1}{e}\left(\frac{\beta^\beta}{\alpha^\alpha}\right)^{1/(\beta-\alpha)} \qquad \alpha,\beta \in \mathbb{R}^+, \alpha \neq \beta,$$

$$L_p(\alpha,\beta) = \left(\frac{\beta^{p+1}-\alpha^{p+1}}{(p+1)(\beta-\alpha)}\right)^{1/p}, \qquad p \in \mathbb{R}\setminus\{-1,0\}, \alpha,\beta \in \mathbb{R}^+, \alpha \neq \beta,$$

where $\mathbb{R}^+$ is the set of positive real numbers.

PROPOSITION 3.1. *Let $a, b, x, y \in \mathbb{R}$, $0 < a \leq c < d \leq b$, $a + b = c + d$ and $p \in \mathbb{R}\setminus\{-1,0\}$. Then*

$$|L_p^p(a,b) - L_p^p(c,d)| \leq \tfrac{1}{2}(c-a)|p(b^{p-1}-a^{p-1})|. \tag{3.1}$$

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PROOF. The proof is immediate from Corollary 2.6 with $f(x) = x^p$, $x \in \mathbb{R}^+$, $p \in \mathbb{R} \setminus \{-1, 0\}$. □

PROPOSITION 3.2. *Suppose that* $a, b, x, y \in \mathbb{R}$, *and* $0 < a \leq c < d \leq b$ *with* $a + b = c + d$. *Then*

$$|L^{-1}(a,b) - L^{-1}(c,d)| \leq \frac{(c-a)(b^2-a^2)}{2a^2b^2}. \tag{3.2}$$

PROOF. The result follows from Corollary 2.6 with $f(x) = 1/x$. □

PROPOSITION 3.3. *Suppose that* $a, b, c, d \in \mathbb{R}$, *and* $0 < a \leq c < d \leq b$ *with* $a + b = c + d$. *Then*

$$\left|\log\left(\frac{I(a,b)}{I(c,d)}\right)\right| \leq \frac{(c-a)(b-a)}{2ab}. \tag{3.3}$$

PROOF. The result follows from Corollary 2.6 with $f(x) = \log x$. □

REMARK 3.4. We note that the upper bounds in (3.1)–(3.3) are less than those of (4.1)–(4.3) in [3], respectively.

4. Applications for PDFs

In the following, assume that $f : [a,b] \to \mathbb{R}^+$ is a probability density function of a certain random variable X and $F : [a,b] \to \mathbb{R}^+$, $F(t) = \int_a^t f(x)\,dx$ is its cumulative distribution function. Then we have the following propositions.

PROPOSITION 4.1. *Let* f *and* F *be as above. Then*

$$\left|F(t) - \frac{t-a}{b-a} + \frac{(b-t)(t-a)}{2(b-a)}(f(b) - f(a))\right| \leq \frac{(b-t)(t-a)}{4}(\Gamma - \gamma),$$

provided that $\gamma < f'(t) < \Gamma, t \in [a,b]$.

PROOF. Taking $c = a$ and $d = t$ in (2.1), we have the desired inequality immediately. □

Similarly, taking $c = a$ and $d = t$ in (2.4), we have the following proposition.

PROPOSITION 4.2. *Let* f *and* F *be as above. Then*

$$\left|F(t) - \frac{t-a}{b-a} + \frac{(b-t)(t-a)}{2(b-a)}(f(b) - f(a))\right| \leq \frac{(b-t)(t-a)}{2(b-a)} \int_a^b \left|f'(x) - \frac{f(b)-f(a)}{b-a}\right| dx.$$

REMARK 4.3. The conditions imposed upon f' in Propositions 4.1 and 4.2 are both weaker than that in Proposition 3.1 in [3].

Some other inequalities for the function $F(\cdot)$ are embodied in the following propositions.

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PROPOSITION 4.4. *Let f and F be as above and let*

$$E_t(X) = \int_a^t uf(u)\,du, \quad u \in [a,b].$$

Then

$$\left|\frac{(b-E(X))(t-a)}{b-a} + E_t(X) - tF(t) - \frac{(b-t)(t-a)}{2(b-a)}\right| \le \frac{(b-t)(t-a)}{4}(\Gamma-\gamma)$$

provided that $\gamma < f(t) < \Gamma, t \in [a,b]$.

PROOF. Taking $f = F, c = a$ and $d = t$ in (2.1),

$$\left|\frac{1}{b-a}\int_a^b F(x)\,dx - \frac{1}{t-a}\int_a^t F(u)\,du - \frac{b-t}{2(b-a)}\right| \le \frac{b-t}{4}(\Gamma-\gamma). \tag{4.1}$$

Since

$$\int_a^b F(x)\,dx = b - E(X),$$
$$\int_a^t F(x)\,dx = tF(t) - \int_a^t uF(u)\,du = tF(t) - E_t(X),$$

by (4.1), we have the desired inequality. □

Similarly, taking $f = F, c = a$ and $d = t$ in (2.4), we have the following result.

PROPOSITION 4.5. *Let f, F and $E_t(x)$ be as above. Then, for $t \in [a,b]$,*

$$\left|\frac{(b-E(X))(t-a)}{b-a} + E_t(X) - tF(t) - \frac{(b-t)(t-a)}{2(b-a)}\right|$$
$$\le \frac{(b-t)(t-a)}{2(b-a)^2}\int_a^b |(b-a)f(x) - 1|\,dx.$$

REMARK 4.6. We note that the conditions imposed upon f in Propositions 4.4 and 4.5 are both weaker than that of Proposition 3.2 in [3].

Let us consider the *beta function*

$$B(p,q) := \int_a^b t^{p-1}(1-t)^{q-1}\,dt, \quad p,q > -1$$

and the *incomplete beta function*

$$B(x;p,q) := \int_a^x t^{p-1}(1-t)^{q-1}\,dt, \quad p,q > -1.$$

If we define

$$f(t) = t^{p-1}(1-t)^{q-1},$$

we get

$$f'(t) = t^{p-2}(1-t)^{q-2}(p-1-(p+q-2)t).$$

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It is obvious that in the case $p > 1$ and $q > 1$, we obtain that $f(x)$ is increasing on $[0, (p-1)/(p+g-2)]$ and decreasing on $[(p-1)/(p+g-2), 1]$, and then

$$0 \le f(t) \le \frac{(p-1)^{p-1}(q-1)^{q-1}}{(p+q-2)^{p+q-2}}$$

for $t \in [0, 1]$; in the case $p > 1$ and $q < 1$, $f(t)$ is increasing on $[0, 1]$; and in the case $p < 1$ and $q > 1$, $f(t)$ is decreasing on $[0, 1]$.

Now, consider the random variable X having the pdf $g(t) = f(t)/B(p, q), t \in (0, 1)$. For $p \neq 1, q \neq 1, p + q \neq 0$ and $p + q \neq 2$, we have

$$\begin{aligned}
&\int_0^1 f(t)\,dt = B(p, q),\\
&\quad E(X) = \frac{1}{B(p,q)}\int_0^1 t^p(1-t)^{q-1}\,dt = \frac{B(p+1,q)}{B(p,q)} = \frac{p}{p+q},\\
&\quad E_x(X) = \frac{1}{B(p,q)}\int_0^x t^p(1-t)^{q-1}\,dt = \frac{B(x;p+1,q)}{B(p,q)},\\
&\quad F(x) = \frac{1}{B(p,q)}\int_0^x t^{p-1}(1-t)^{q-1}\,dt = \frac{B(x;p,q)}{B(p,q)};
\end{aligned}$$

for $p > 1, q < 1$, we have

$$\begin{aligned}
\int_0^1 |f(t) - B(p,q)|\,dt &= \int_0^{B(p,q)} (B(p,q) - f(t))\,dt + \int_{B(p,q)}^1 (f(t) - B(p,q))\,dt\\
&= 2B^2(p,q) - 2B(B(p,q); p, q);
\end{aligned}$$

and for $p < 1, q > 1$, we have

$$\begin{aligned}
\int_0^1 |f(t) - B(p,q)|\,dt &= \int_0^{B(p,q)} (f(t) - B(p,q))\,dt + \int_{B(p,q)}^1 (B(p,q) - f(t))\,dt\\
&= 2B(B(p,q); p, q) - 2B^2(p,q).
\end{aligned}$$

Using Propositions 4.4 and 4.5, we may state the following results.

Proposition 4.7. *Let X be a beta random variable with $p > 1$ and $q > 1$. Then we have the inequality*

$$\begin{aligned}
&\left|\left(\frac{qx}{p+q} - \frac{(1-x)x}{2}\right)B(p,q) + B(x; p+1, q) - xB(x; p, q)\right|\\
&\qquad \le \frac{(1-x)x}{4} \cdot \frac{(p-1)^{p-1}(q-1)^{q-1}}{(p+q-2)^{p+q-2}}
\end{aligned}$$

for all $x \in [0, 1]$.

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PROPOSITION 4.8. *Let X be a beta random variable. Then, for $x \in [a, b]$, we have the inequality*

$$\left|\left(\frac{qx}{p+q} - \frac{(1-x)x}{2}\right)B(p,q) + B(x; p+1, q) - xB(x; p, q)\right|$$
$$\le \begin{cases} (1-x)x(B^2(p,q) - B(B(p,q); p, q)), & \text{if } p > 1, q < 1 \\ (1-x)x(B(B(p,q); p, q) - B^2(p,q)), & \text{if } p < 1, q > 1. \end{cases}$$

REMARK 4.9. Proposition 4.7 provides a different inequality from that of Proposition 3.3 in [3]. We also note that the result from Proposition 4.8 is an extension of the result from Proposition 3.3 in [3] for the case $p > 1$ and $q < 1$, and the case $p < 1$ and $q > 1$.

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