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ADDITIVE COMPLETION OF THIN SETS

JIN-HUI FANG and CSABA SÁNDOR

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Abstract

Two sets A, B of positive integers are called *exact additive complements* if $A + B$ contains all sufficiently large integers and $A(x)B(x)/x \to 1$. For $A = \{a_1 < a_2 < \cdots\}$, let $A(x)$ denote the counting function of A and let $a^*(x)$ denote the largest element in $A \bigcap [1, x]$. Following the work of Ruzsa ['Exact additive complements', *Quart. J. Math.* **68** (2017) 227–235] and Chen and Fang ['Additive complements with Narkiewicz's condition', *Combinatorica* **39** (2019), 813–823], we prove that, for exact additive complements A, B with $a_{n+1}/na_n \to \infty$,

$$A(x)B(x) - x \geqslant \frac{a^*(x)}{A(x)} + o\Big(\frac{a^*(x)}{A(x)^2}\Big) \quad \text{as } x \to +\infty.$$

We also construct exact additive complements A, B with $a_{n+1}/na_n \to \infty$ such that

$$A(x)B(x) - x \leqslant \frac{a^*(x)}{A(x)} + (1 + o(1))\Big(\frac{a^*(x)}{A(x)^2}\Big)$$

for infinitely many positive integers x.

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1. Introduction

Two sets A, B of positive integers are called *additive complements*, if $A + B$ contains all sufficiently large integers. Let $A = \{a_1 < a_2 < \cdots\}$ be a set of positive integers. Denote by $A(x)$ the counting function of A and by $a^*(x)$ the largest element in $A \bigcap [1, x]$. If additive complements A, B satisfy

$$\frac{A(x)B(x)}{x} \to 1,$$

then we call such A, B *exact additive complements*. In 2001, Ruzsa [2] introduced the following notation which is powerful for the proof of additive complements.

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Let $m > a_1$ be an integer and let $k = A(m)$. Denote by $L(m)$ the smallest number l for which there are integers $b_1, \dots, b_l$ such that the numbers $a_i + b_j$, $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant l$, contain every residue modulo m. Obviously, $L(m) \geqslant m/k$.

THEOREM 1.1 (Ruzsa [2]). *If*

$$\frac{a_{n+1}}{na_n} \to \infty, \tag{1.1}$$

then A has an exact complement.

THEOREM 1.2 (Ruzsa [2]). *Let A be a set satisfying $A(2x)/A(x) \to 1$. The following are equivalent.*

(a) *A has an exact complement.*
(b) $A(m)L(m)/m \to 1$.
(c) *There is a sequence $m_1 < m_2 < \cdots$ of positive integers such that $A(m_{i+1})/A(m_i) \to 1$ and $A(m_i)L(m_i)/m_i \to 1$.*

THEOREM 1.3 (Ruzsa [3]). *For exact additive complements A, B with $A(2x)/A(x) \to 1$,*

$$A(x)B(x) - x \geqslant (1 + o(1))\frac{a^*(x)}{A(x)} \quad \text{as } x \to +\infty.$$

In 2019, Chen and Fang [1] improved Theorem 1.3 by removing the *exact* condition. Chen and Fang also showed in [1] that Theorem 1.3 is the best possible.

THEOREM 1.4 (Chen and Fang [1]). *There exist exact additive complements A, B with $A(2x)/A(x) \to 1$ such that*

$$A(x)B(x) - x \leqslant (1 + o(1))\frac{a^*(x)}{A(x)}$$

for infinitely many positive integers x.

In this paper, under condition (1.1) from [2], we obtain the following result.

THEOREM 1.5. *For exact additive complements A, B with (1.1),*

$$A(x)B(x) - x \geqslant \frac{a^*(x)}{A(x)} + o\Big(\frac{a^*(x)}{A(x)^2}\Big) \quad \text{as } x \to +\infty. \tag{1.2}$$

Moreover, we also show that $a^*(x)/A(x)^2$ is the best possible.

THEOREM 1.6. *There exist exact additive complements A, B with (1.1) such that*

$$\liminf_{x\to\infty} \frac{A(x)B(x) - x - \frac{a^*(x)}{A(x)}}{\frac{a^*(x)}{A(x)^2}} \leqslant 1. \tag{1.3}$$

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2. Proofs of the main results

Let

$$\sigma(x,n) = |\{(a,b) : a+b=n, a,b \leqslant x, a \in A, b \in B\}|$$

and

$$\delta(x,n) = |\{(a,b) : b-a=n, a,b \leqslant x, a \in A, b \in B\}|.$$

The ideas used in the proofs of the main results come from [1–3]. We use the following lemma of Ruzsa in the proof of Theorem 1.5.

LEMMA 2.1 [3, Lemma 2.1]. *Let U and V be finite sets of integers and let*

$$\sigma(n) = |\{(u,v) : u \in U, v \in V, u+v=n\}|$$

and

$$\delta(n) = |\{(u,v) : u \in U, v \in V, v-u=n\}|.$$

Then

$$\sum_{\sigma(n)>1} (\sigma(n)-1) \geqslant \frac{1}{|U|} \sum_{\delta(n)>1} (\delta(n)-1).$$

PROOF OF THEOREM 1.5. Assume the contrary. Suppose that (1.2) does not hold. Then there exist a positive number $\delta_0 (<1)$ and a sequence $x_1 < x_2 < \cdots < x_k < \cdots$ such that

$$A(x_k)B(x_k) - x_k \leqslant \frac{a^*(x_k)}{A(x_k)} - \delta_0 \frac{a^*(x_k)}{A(x_k)^2}. \tag{2.1}$$

We know that

$$\begin{aligned} A(x_k)B(x_k) - x_k &= \sum_{\substack{a \leqslant x_k, b \leqslant x_k \\ a \in A, b \in B}} 1 - x_k = \sum_{n=1}^{2x_k} \sigma(x_k,n) - x_k \\ &= \sum_{\substack{n=1 \\ \sigma(x_k,n)\geqslant 1}}^{x_k} (\sigma(x_k,n)-1) + \sum_{\substack{n=x_k+1 \\ \sigma(x_k,n)\geqslant 1}}^{2x_k} \sigma(x_k,n) \\ &= \sum_{\substack{n=1 \\ \sigma(x_k,n)\geqslant 1}}^{2x_k} (\sigma(x_k,n)-1) + \sum_{\substack{n=x_k+1 \\ \sigma(x_k,n)\geqslant 1}}^{2x_k} 1 \\ &= \sum_{\substack{n=1 \\ \sigma(x_k,n)> 1}}^{2x_k} (\sigma(x_k,n)-1) + \sum_{\substack{n=x_k+1 \\ \sigma(x_k,n)\geqslant 1}}^{2x_k} 1. \end{aligned}$$

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Since $a^*(x_k) \in A$ and $a^*(x_k) + b > x_k$ for all $b \in B$ with $x_k - a^*(x_k) < b \leqslant x_k$,

$$\sum_{\substack{n=x_k+1 \\ \sigma(x_k,n)\geqslant 1}}^{2x_k} 1 \geqslant B(x_k) - B(x_k - a^*(x_k)).$$

Thus,

$$A(x_k)B(x_k) - x_k \geqslant \sum_{\substack{n \\ \sigma(x_k,n)>1}} (\sigma(x_k, n) - 1) + B(x_k) - B(x_k - a^*(x_k)).$$

From Ruzsa's Lemma 2.1,

$$A(x_k)B(x_k) - x_k \geqslant \frac{1}{A(x_k)} \sum_{\substack{n \\ \delta(x_k,n)>1}} (\delta(x_k, n) - 1) + B(x_k) - B(x_k - a^*(x_k)). \tag{2.2}$$

Let

$$D = \{(a, b) : a \in A, b \in B, a \leqslant b \leqslant x_k - a^*(x_k)\}.$$

Then

$$\sum_{\substack{n \\ \delta(x_k,n)>1}} (\delta(x_k, n) - 1) = \sum_{\substack{n \\ \delta(x_k,n)\geqslant 1}} (\delta(x_k, n) - 1) \geqslant |D| - (x_k - a^*(x_k) + 1). \tag{2.3}$$

Now we need a lower bound for $|D|$. We consider the following two cases.

Case 1: $a^*(x_k) > \frac{1}{2}x_k$ for infinitely many k. By (1.1),

$$A\Big(\frac{\delta_0}{5}\frac{a^*(x_k)}{A(x_k)}\Big) = A(x_k) - 1 \quad \text{for all sufficiently large integers } k.$$

Thus, in this case, by Theorem 1.3 and $A(x)B(x)/x \to 1$,

$$\begin{aligned}
|D| &\geqslant \sum_{\substack{\frac{\delta_0}{5}\frac{a^*(x_k)}{A(x_k)} \leqslant b \leqslant x_k - a^*(x_k) \\ b\in B}} A(b) \geqslant A\Big(\frac{\delta_0}{5}\frac{a^*(x_k)}{A(x_k)}\Big)\Big(B(x_k - a^*(x_k)) - B\Big(\frac{\delta_0}{5}\frac{a^*(x_k)}{A(x_k)}\Big)\Big) \\
&= (A(x_k) - 1)B(x_k - a^*(x_k)) - A\Big(\frac{\delta_0}{5}\frac{a^*(x_k)}{A(x_k)}\Big)B\Big(\frac{\delta_0}{5}\frac{a^*(x_k)}{A(x_k)}\Big) \\
&= A(x_k)B(x_k) + A(x_k)(B(x_k - a^*(x_k)) - B(x_k)) - B(x_k - a^*(x_k)) \\
&\quad - A\Big(\frac{\delta_0}{5}\frac{a^*(x_k)}{A(x_k)}\Big)B\Big(\frac{\delta_0}{5}\frac{a^*(x_k)}{A(x_k)}\Big) \\
&\geqslant x_k + \Big(1 - \frac{\delta_0}{4}\Big)\frac{a^*(x_k)}{A(x_k)} + A(x_k)(B(x_k - a^*(x_k)) - B(x_k)) - B(a^*(x_k)) - \frac{\delta_0}{4}\frac{a^*(x_k)}{A(x_k)} \\
&\geqslant x_k + \Big(1 - \frac{\delta_0}{4}\Big)\frac{a^*(x_k)}{A(x_k)} + A(x_k)(B(x_k - a^*(x_k)) - B(x_k))
\end{aligned}$$

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$$
\begin{aligned}
&-\Big(1+\frac{\delta_0}{4}\Big)\frac{a^*(x_k)}{A(x_k)}-\frac{\delta_0}{4}\frac{a^*(x_k)}{A(x_k)}\\
&=x_k-\frac{3\delta_0}{4}\frac{a^*(x_k)}{A(x_k)}+A(x_k)(B(x_k-a^*(x_k))-B(x_k))
\end{aligned}
$$

for sufficiently large k. It follows from (2.2) and (2.3) that

$$
\begin{aligned}
A(x_k)B(x_k)-x_k &\geqslant \frac{x_k}{A(x_k)}-\frac{3\delta_0}{4}\frac{a^*(x_k)}{A(x_k)^2}+B(x_k-a^*(x_k))-B(x_k)-\frac{x_k-a^*(x_k)+1}{A(x_k)}\\
&\quad +B(x_k)-B(x_k-a^*(x_k))\\
&> \frac{a^*(x_k)}{A(x_k)}-\delta_0\frac{a^*(x_k)}{A(x_k)^2}
\end{aligned}
$$

for sufficiently large k, which is in contradiction with (2.1).

Case 2: $a^*(x_k)\leqslant \frac{1}{2}x_k$ for infinitely many k. By (1.1),

$$
A\Big(\frac{\delta_0}{4}\frac{a^*(x_k)}{A(x_k)}\Big)=A(x_k)-1 \quad \text{for all sufficiently large integers } k.
$$

Thus, in this case, by Theorem 1.3 and $A(x)B(x)/x\to 1$,

$$
\begin{aligned}
|D| &\geqslant \sum_{\substack{\frac{\delta_0}{2}\frac{a^*(x_k)}{A(x_k)}<b\leqslant x_k-a^*(x_k)\\ b\in B}} A\Big(b-\frac{\delta_0}{4}\frac{a^*(x_k)}{A(x_k)}\Big)\\
&= \sum_{\substack{\frac{\delta_0}{2}\frac{a^*(x_k)}{A(x_k)}<b<a^*(x_k)+\frac{\delta_0}{4}\frac{a^*(x_k)}{A(x_k)}\\ b\in B}} A\Big(b-\frac{\delta_0}{4}\frac{a^*(x_k)}{A(x_k)}\Big)+\sum_{\substack{a^*(x_k)+\frac{\delta_0}{4}\frac{a^*(x_k)}{A(x_k)}\leqslant b\leqslant x_k-a^*(x_k)\\ b\in B}} A\Big(b-\frac{\delta_0}{4}\frac{a^*(x_k)}{A(x_k)}\Big)\\
&= (A(x_k)-1)\Big(B\Big(a^*(x_k)+\frac{\delta_0}{4}\frac{a^*(x_k)}{A(x_k)}\Big)-B\Big(\frac{\delta_0}{2}\frac{a^*(x_k)}{A(x_k)}\Big)\Big)\\
&\quad +A(x_k)\Big(B(x_k-a^*(x_k))-B\Big(a^*(x_k)+\frac{\delta_0}{4}\frac{a^*(x_k)}{A(x_k)}\Big)\Big)\\
&= A\Big(a^*(x_k)+\frac{\delta_0}{4}\frac{a^*(x_k)}{A(x_k)}\Big)B\Big(a^*(x_k)+\frac{\delta_0}{4}\frac{a^*(x_k)}{A(x_k)}\Big)-B\Big(a^*(x_k)+\frac{\delta_0}{4}\frac{a^*(x_k)}{A(x_k)}\Big)\\
&\quad -A\Big(\frac{\delta_0}{2}\frac{a^*(x_k)}{A(x_k)}\Big)B\Big(\frac{\delta_0}{2}\frac{a^*(x_k)}{A(x_k)}\Big)+A(x_k)B(x_k)+A(x_k)(B(x_k-a^*(x_k))-B(x_k))\\
&\quad -A\Big(a^*(x_k)+\frac{\delta_0}{4}\frac{a^*(x_k)}{A(x_k)}\Big)B\Big(a^*(x_k)+\frac{\delta_0}{4}\frac{a^*(x_k)}{A(x_k)}\Big)\\
&= A(x_k)B(x_k)+A(x_k)(B(x_k-a^*(x_k))-B(x_k))-B\Big(a^*(x_k)+\frac{\delta_0}{4}\frac{a^*(x_k)}{A(x_k)}\Big)\\
&\quad -A\Big(\frac{\delta_0}{2}\frac{a^*(x_k)}{A(x_k)}\Big)B\Big(\frac{\delta_0}{2}\frac{a^*(x_k)}{A(x_k)}\Big)
\end{aligned}
$$

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$$\begin{aligned}
&\geqslant x_k + \left(1 - \frac{\delta_0}{10}\right)\frac{a^*(x_k)}{A(x_k)} + A(x_k)(B(x_k - a^*(x_k)) - B(x_k)) \\
&\quad - \left(1 + \frac{\delta_0}{10}\right)\frac{a^*(x_k) + \frac{\delta_0}{4}\frac{a^*(x_k)}{A(x_k)}}{A(x_k)} - \frac{3\delta_0}{5}\frac{a^*(x_k)}{A(x_k)} \\
&\geqslant x_k - \frac{9\delta_0}{10}\frac{a^*(x_k)}{A(x_k)} + A(x_k)(B(x_k - a^*(x_k)) - B(x_k)),
\end{aligned}$$

for sufficiently large k. It follows from (2.2) and (2.3) that

$$\begin{aligned}
A(x_k)B(x_k) - x_k &\geqslant \frac{x_k}{A(x_k)} - \frac{9\delta_0}{10}\frac{a^*(x_k)}{A(x_k)^2} + B(x_k - a^*(x_k)) - B(x_k) - \frac{x_k - a^*(x_k) + 1}{A(x_k)} \\
&\quad + B(x_k) - B(x_k - a^*(x_k)) \\
&> \frac{a^*(x_k)}{A(x_k)} - \delta_0\frac{a^*(x_k)}{A(x_k)^2}
\end{aligned}$$

for sufficiently large k, which is in contradiction with (2.1).

This completes the proof of Theorem 1.5. □

PROOF OF THEOREM 1.6. Let $a_1 = 1$ and $a_2 = 4$. We construct the sequence $a_3, a_4, \ldots$ with

$$a_{n+1} \gg n^4 a_n \tag{2.4}$$

and a sequence $n_1, n_2, \ldots$ such that $a_1, a_2, \ldots, a_{n_k}$ form a complete residue system modulo n_k and $n_k \mid a_{n_k}$. We get such a sequence by a greedy algorithm: let $n_1 = 2$, and if $n_1, n_2, \ldots, n_k$ are already defined, then let $n_{k+1} = a_{n_k}$. Since $a_1, \ldots, a_{n_k}$ are distinct residues modulo a_{n_k}, we can choose $a_{n_k+1}, \ldots, a_{n_{k+1}}$ such that $a_{m+1} \gg m^4 a_m$ for $m = n_k, \ldots, n_{k+1} - 1$, $a_{n_k} \mid a_{a_{n_k}}$ and $a_1, \ldots, a_{n_{k+1}}$ form a complete residue system modulo n_{k+1}.

For every positive integer k, we further take

$$b_1 = n_k, \quad b_2 = 2n_k, \ldots, b_{a_{n_k}/n_k} = \frac{a_{n_k}}{n_k} \cdot n_k.$$

Then $a_i + b_j$ for $1 \leqslant i \leqslant p, 1 \leqslant j \leqslant a_{n_k}/n_k$, form a complete residue system modulo a_{n_k}. From the definition of $L(a_{n_k})$,

$$L(a_{n_k}) = \frac{a_{n_k}}{n_k}. \tag{2.5}$$

For the set $A = \{a_k\}_{k=1}^{\infty}$ and every positive integer k, define q_k by

$$q_k = \left\lfloor \frac{a_{k+1}}{k^4 a_k} \right\rfloor, \quad \text{that is,} \quad q_k \cdot k^4 a_k < a_{k+1} \leqslant (q_k + 1) \cdot k^4 a_k. \tag{2.6}$$

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Define the same sets A, B as in [2, Theorem 3] (replacing m_k by a_k) and write $A_k = A \bigcap [1, a_k]$. Take $U_k \subseteq [1, a_k]$ so that $|U_k| = L(a_k)$ and $A_k + U_k$ contains every residue module a_k. Let

$$V_k = U_k + \left\{(q_k - 1)a_k, q_k a_k, (q_k + 1)a_k, \ldots, \left\lfloor \frac{q_{k+1}a_{k+1}}{a_k} \right\rfloor a_k\right\} \quad \text{and} \quad B = \bigcup_{k=1}^{\infty} V_k.$$

Let $q_k a_k \leqslant x \leqslant q_{k+1}a_{k+1}$. The sequence $\{q_k\}_{k=1}^{\infty}$ defined in (2.6) is increasing to infinity by (2.4) and $A(q_k a_k) \sim A(a_k)$. (In fact, $A(q_k a_k) = k = A(a_k)$ by (2.6).) By the same proof as in [2, Theorem 3], A, B are additive complements and $A(x)B(x) \sim x$. Thus, the set A with (2.4) has an exact complement B. Obviously, such an A with (2.4) satisfies (1.1).

Finally, we prove that (1.3) holds for infinitely many x_k. For x with $q_k a_k \leqslant x < (q_{k+1} - 1)a_{k+1}$, we have $k \leqslant A(x) \leqslant k + 1$ and

$$B(x) \leqslant \left(\left\lfloor \frac{x}{a_k} \right\rfloor - q_k + 2\right)L(a_k) + \sum_{i=2}^{k}\left(\left\lfloor \frac{q_i a_i}{a_{i-1}} \right\rfloor - q_{i-1} + 2\right)L(a_{i-1}). \tag{2.7}$$

By Theorem 1.2(b), $L(a_{k-1}) \leqslant 2a_{k-1}/(k-1)$ for large k. From (2.6),

$$\sum_{i=2}^{k}\left(\left\lfloor \frac{q_i a_i}{a_{i-1}} \right\rfloor - q_{i-1} + 2\right)L(a_{i-1}) \leqslant (k-1)\frac{2q_k a_k}{k-1} = O(q_k a_k) = O\left(\frac{a_{k+1}}{(k+1)^4}\right).$$

It is easy to see that, for large k,

$$(q_k - 2)L(a_k) \leqslant 2\frac{q_k a_k}{k} = O\left(\frac{a_{k+1}}{(k+1)^5}\right).$$

It follows from (2.7) that

$$B(x) \leqslant \frac{x}{a_k}L(a_k) + O\left(\frac{a_{k+1}}{(k+1)^4}\right). \tag{2.8}$$

Choose $x_k = a_{n_k+1}$, where n_k is the index satisfying (2.5). Then by (2.8),

$$\begin{aligned} A(x_k)B(x_k) - x_k - \frac{a^*(x_k)}{A(x_k)} &\leqslant (n_k + 1)\frac{x_k}{n_k} - x_k - \frac{x_k}{n_k + 1} + O\left(\frac{x_k}{(n_k+1)^3}\right) \\ &= \frac{x_k}{A(x_k)^2} + O\left(\frac{x_k}{A(x_k)^3}\right). \end{aligned}$$

This completes the proof of Theorem 1.6. □

References

[1] Y. G. Chen and J. H. Fang, ‘Additive complements with Narkiewicz’s condition’, *Combinatorica* **39** (2019), 813–823.

[2] I. Z. Ruzsa, ‘Additive completion of lacunary sequences’, *Combinatorica* **21** (2001), 279–291.

[3] I. Z. Ruzsa, ‘Exact additive complements’, *Q. J. Math.* **68** (2017), 227–235.

JIN-HUI FANG, School of Mathematical Sciences,
Nanjing Normal University, Nanjing 210023, PR China
e-mail: fangjinhui1114@163.com

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CSABA SÁNDOR, Institute of Mathematics,
Budapest University of Technology and Economics,
Egry József utca 1, 1111 Budapest, Hungary
and
Department of Computer Science and Information Theory,
Budapest University of Technology and Economics,
Műegyetem rkp. 3., H-1111 Budapest, Hungary
and
MTA-BME Lendület Arithmetic Combinatorics Research Group, ELKH,
Műegyetem rkp. 3., H-1111 Budapest, Hungary
e-mail: csandor@math.bme.hu

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