Canad. Math. Bull. Vol. 33 (1), 1990

PSEUDOUMBILICAL 2-TYPE SURFACES IN SPHERES

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ABSTRACT. It is proved that a pseudoumbilical 2-type surface in a sphere has constant mean curvature. Moreover, the dimension of the sphere is greater than four.

0. **Introduction.** Let M be an n-dimensional manifold immersed in an $(m+1)$-dimensional Euclidean space, E^{m+1}. Denote by H the mean curvature vector of that immersion. If there exists a function λ on M such that $\langle H, \sigma(X,Y)\rangle = \lambda\langle X, Y\rangle$, where σ is the second fundamental form, and $\langle\ ,\ \rangle$ is the scalar product in E^{m+1}, then M is called a pseudoumbilical submanifold of E^m. Note that λ is the square of the mean curvature.

On the other hand, for an isometric immersion $x : M \longrightarrow E^{m+1}$ of a compact Riemannian manifold M into E^{m+1}, we can get a spectral decomposition of the position vector x in the following way: $x = x_0 + \sum_{t>0} x_t$, where x_0 is the center of mass of M and x_t's are $(m+1)$-valued eigenfunctions of Δ, the Laplacian of $M : \Delta x_t = \lambda_t x_t$. If there are exactly k nonzero x_t's in the decomposition of x, we say that (M, x) is of k-type (see [3]).

Pseudoumbilical submanifolds in the Euclidean space with the mean curvature vector parallel in the normal bundle are precisely those submanifolds which are minimal in hyperspheres [5], so that, by Takahashi's theorem [9], this gives also a characterization of 1-type Euclidean submanifolds. We know also that a 2-type spherical Chen surface whose center of mass coincides with the center of the sphere in which it is immersed is either pseudoumbilical or flat [7]. Moreover, we constructed in that work examples of pseudoumbilical 2-type surfaces immersed in spheres. In this note, we want to gain more information about the relationship between pseudoumbilical submanifolds and 2-type immersions in the Euclidean space. More specifically, we get:

THEOREM. *Let $x : M \longrightarrow S_0^m(1) \subset E^{m+1}$ be a pseudoumbilical 2-type immersion of a compact Riemannian surface M in the m-dimensional unit sphere centered in the origin of E^{m+1}, then the immersion has constant mean curvature and $m \geqq 5$.*

Received by the editors May 4, 1988 and, in revised form, November 30, 1988.
Research partially supported by a CAICYT Grant PR84-1242-C02-02 and by a grant of Junta de Andalucía. Spain.
1980 Mathematics Subject Classification: 53C40.

This work was done while the author was visiting Michigan State Univ. in Fall 1987. I would like to thank my colleagues there for their kindness. I am especially grateful to Prof. B-Y Chen for many valuable commentaries.

1. **Preliminaries.** Take $x : M^2 \longrightarrow S_0^m(1) \subset E^{m+1}$ an isometric immersion of a closed surface in the m-sphere, which without loss of generality we suppose it is the unit sphere centered in the origin of E^{m+1}. As $x(M^2)$ is included in $S_0^m(1)$ we say that the immersion is spherical. We denote by H the mean curvature vector of M^2 in E^{m+1} and by H' the mean curvature vector of M^2 in $S_0^m(1)$. Then $H = H' - x$ and so $\alpha^2 = 1 + (\alpha')^2$ where α and α' are the mean curvatures of M^2 in E^{m+1} and $S_0^m(1)$ respectively. Choose ξ as a unit normal vector parallel to $H', H' = \alpha'\xi$, and denote by A, D, and σ the Weingarten map, the normal connection, and the second fundmental form of M^2 in E^{m+1}, and A', D', σ' the same geometric elements of M^2 in $S_0^m(1)$. If one computes the Laplacian of H in terms of this kind of elements, one gets, [3],

$$\Delta H = (\Delta H)^T + \Delta^{D'} H' + \{|A_\xi|^2 + 2\} H' - 2\alpha^2 x \tag{1}$$

where $(\Delta H)^T$ is the tangent component of ΔH and $\Delta^{D'}$ represents the Laplacian associated to D'. The tangent component $(\Delta H)^T$, can be written [2], [4],

$$(\Delta H)^T = 2 \operatorname{Tr} A_{D'H'} + \nabla(\alpha)^2 \tag{2}$$

with $\operatorname{Tr} A_{D'H'} = \sum_{i=1}^{2} A_{D'_{E_i}H'} E_i$, $\{E_1, E_2\}$ being an orthonormal basis in the tangent bundle TM^2; and $\nabla(\alpha)^2$ is the gradient of α^2.

At this point we assume that the immersion (M^2, x) is of 2-type. This means that its position vector has the form $x = x_0 + x_p + x_q$, with x_p and x_q $(m+1)$-valuated eigenfunctions of Δ, the Laplacian of M^2, corresponding to the eigenvalues λ_p and λ_q respectively, and x_0 is a constant given by the center of mass of M^2. If x_0 coincides with the center of the sphere in which M^2 is included, we say that (M^2, x) is of mass-symmetric in $S_0^m(1)$. Using the well-known formula $\Delta x = -2H$ one obtains:

$$\Delta H = bH + c(x - x_0); \; b = \lambda_p + \lambda_q; \; c = \tfrac{1}{2}\lambda_p\lambda_q. \tag{3}$$

From formulas (1) and (3) we get

$$\langle x_0, x\rangle = 1/c\{n^2 + c - b\} \tag{4}$$

On the other hand, if X is a tangent vector field in M^2, one uses again (1) and (3) and now $\langle X, x_0\rangle = \langle X, (\Delta H)^T\rangle$. Therefore (2) and (4) give:

$$\operatorname{Tr} A_{D'H'} = (-1/2c)(2 + c)\nabla(\alpha)^2 \tag{5}$$

Hence:

https://doi.org/10.4153/CMB-1990-011-4 Published online by Cambridge University Press

LEMMA. *Let $x : M^2 \longrightarrow S_0^m(1)$ a 2-type spherical immersion of a closed surface in $S_0^m(1)$. Then it has constant mean curvature, if and only if,* $\mathrm{Tr}\, A_{D'H'} = 0$.

Now, we choose a system of isothermal coordinates $\{x_1, x_2\}$ covering M^2. The induced metric tensor g has the form $g = E(dx_1^2 + dx_2^2)$. We put $X_i = \partial/\partial X_i$, $i = 1, 2$, then by using Codazzi's equation one has

$$(6)\qquad \begin{aligned} D'_{X_2}\sigma'(X_1, X_1) - D'_{X_1}\sigma'(X_2, X_1) &= (X_2E)H' \\ D'_{X_2}\sigma'(X_1, X_2) - D'_{X_1}\sigma'(X_2, X_2) &= -(X_1E)H' \end{aligned}$$

As usual, we denote by $\partial z = \frac{1}{2}(X_1 - iX_2)$, $\partial \bar{z} = \frac{1}{2}(X_1 + iX_2)$, and then from (6), we obtain

$$(7)\qquad \begin{aligned} \partial\bar{z}(LnE)\sigma'(\partial z, \partial\bar{z}) &= D'_{\partial\bar{z}}\sigma'(\partial z, \partial\bar{z}) - D'_{\partial z}\sigma'(\partial\bar{z}, \partial\bar{z}) \\ \partial z(LnE)\sigma'(\partial z, \partial\bar{z}) &= D'_{\partial z}\sigma'(\partial z, \partial\bar{z}) - D'_{\partial\bar{z}}\sigma'(\partial z, \partial z) \end{aligned}$$

In this coordinate system, the mean curvature is $H' = 2E^{-1}\sigma'(\partial z, \partial\bar{z})$. Differentiating this formula and taking into account (7), we get

$$(8)\qquad \begin{aligned} \partial z(H') &= 2E^{-1}D'_{\partial\bar{z}}\sigma'(\partial z, \partial z) \\ \partial\bar{z}(H') &= 2E^{-1}D'_{\partial z}\sigma'(\partial\bar{z}, \partial\bar{z}) \end{aligned}$$

Next, we want to compute $\mathrm{Tr}\, A_{D'H'}$ in terms of isothermal coordinates. Since $\{E_i = X_i/\sqrt{E}\}$, $i = 1, 2$, is a local orthonormal basis

$$\begin{aligned} \mathrm{Tr}\, A_{D'H'} &= \sum_{i=1}^{2} A_{D'_{E_i}H'}E_i = \frac{2}{E}(A_{D'_{\partial z}H'}\partial\bar{z} + A_{D'_{\partial\bar{z}}H'}\partial z) \\ &= 4/E^2\{(\langle\sigma'(\partial\bar{z}, \partial\bar{z}), D'_{\partial z}H'\rangle + \langle\sigma'(\partial z, \partial\bar{z}), D'_{\partial\bar{z}}H'\rangle)\partial z \\ &\quad + \langle\sigma'(\partial\bar{z}, \partial z), D'_{\partial z}H'\rangle + \langle\sigma'(\partial z, \partial z), D'_{\partial\bar{z}}H'\rangle)\partial\bar{z} \end{aligned}$$

Hence, using (8), we finally obtain:

$$(9)\qquad \mathrm{Tr}\, A_{D'H'} = 4E^{-2}(\partial z\langle H', \sigma'(\partial\bar{z}, \partial\bar{z})\rangle\partial z + \partial\bar{z}\langle H', \sigma'(\partial z, \partial z)\rangle\partial\bar{z})$$

This formula holds for any spherical surface M^2 immersed in S_0^m. (Compare with lemma 1 of [8].)

2. **Proof of the theorem.** Suppose M^2 is pseudoumbilical. In this case

$$\langle H', \sigma'(\partial\bar{z}, \partial\bar{z})\rangle = \langle H', \sigma'(\partial z, \partial z)\rangle = 0$$

Thus, using (9), $\mathrm{Tr}\, A_{D'H'} = 0$. Therefore, from lemma 1, α and consequently α' are constant.

For the second part, we only need to know that a pseudoumbilical submanifold of dimension n in S^{n+2} is either minimal in S^{n+2} or a minimal hypersurface in a small $(n+1)$-sphere of S^{n+2}, [6]. But this kind of submanifolds are of 1-type by Takahashi's theorem. Then $m \geqq 5$.

REMARK. In [1] authors proved that there exist no 2-type mass-symmetric immersions of surfaces in S^4. As a consequence of our result, there exist no pseudoumbilical 2-type surfaces in S^4.

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