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PSEUDOREGULAR RADICAL CLASSES

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We consider radical classes specified by an associating polynomial in two variables which have a similar form to the polynomial defining the class of quasiregular rings. In particular, the relationships of these classes to the classes of semiprime, nil and quasiregular rings are explored.

Throughout, all rings are associative. The ring of integers will be denoted by $\mathbf{Z}$. We denote by $\circ$ the circle composition operation on any ring R, defined by $a \circ b = a + b - ab$ for all $a, b \in R$.

Let $\mathbf{Z}_0[x,y,z]$ be the free ring (that is, the ring of integer polynomials without constant term) on the generators x, y, z; similarly, let $\mathbf{Z}_0[x,y]$ be the free ring on the generators x, y. We view the latter as a subring of the former. An element f of $\mathbf{Z}_0[x,y]$ is said to be *associating* if there exists $g \in \mathbf{Z}_0[x,y,z]$ such that $f\big(f(x,y),z\big) = f\big(x, g(x,y,z)\big)$. This notion is defined in much greater generality in [**2**], for instance by allowing more than one "existential" variable y, or by allowing the expression f to be drawn from a set so that the nil radical can be dealt with (amongst others).

For any $f \in \mathbf{Z}_0[x,y]$, let $\mathcal{R}_f$ be the class of rings in $\mathcal{V}$ defined as follows: R is in $\mathcal{R}_f$ providing that, for every $r \in R$ there exists $s \in R$ such that $f(r,s) = 0$. By [**2**, Theorem 4], $\mathcal{R}_f$ is a radical class providing f is associating.

The classes of quasiregular and von Neumann regular rings have the form $\mathcal{R}_f$ for some f: in the former case, we may let $f = x + y + xy$ and $g = y + z + yz$; in the latter, $f = x - xyx$ and $g = y + z - 2xyz + xyzyx$. Indeed these examples may be generalised as follows: if $p, q \in \mathbf{Z}[x]$, then the element $x - p(x)yq(x)$ of $\mathbf{Z}_0[x,y]$ is associating, as is implicitly shown in [**5**], and the resulting radical class is the class of $(p;q)$-*regular* rings. This family also includes Divinsky's D-regular radical class.

1. PSEUDOREGULARITY

In this paper we consider the special case in which $f(x,y) = p(x) + q(x)y$, where we allow $q(x) \in \mathbf{Z}[x]$; thus we assume $f(x,y)$ is (right) linear in y. (There is obviously a dual theory for $f(x,y) = p(x) + yq(x)$.) We use the notation $\mathcal{R}_{p,q}$ as an alternative to $\mathcal{R}_f$.

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We briefly put this choice in context. It is easy to show there are no associating $f(x,y)$ of the form $f(x,y) = p_0(x)+p_1(x)y+p_2(x)y^2+\cdots$ except those for which $p_i(x) = 0$ for all $i > 1$. On the other hand, if we assume commutativity, the family $f_m(x,y) = x \circ y \circ y \circ \cdots \circ y$ (m times) for any square-free m are associating and give rise to the distinct radical classes $\mathcal{J}_m = \mathcal{R}_{f_m}$ as in [**3**]. (There are other similar associating f defined in terms of $\circ$, but all give rise to one of these classes.) We know of no other commutative cases. Returning to the non-commutative case, there may perhaps be cases in which y occurs non-linearly in one or more monomials in $f(x,y)$, although we conjecture there are none. Thus it is quite possible that our apparently restrictive assumption on the form of $f(x,y)$ is not very restrictive at all and only rules out of consideration cases where y makes a single appearance in any one monomial.

It will be convenient to say that $(p,q) \in \mathbf{Z}[x] \times \mathbf{Z}[x]$ is an *$\mathcal{R}$-pair* if $p(0) = 0$ and $f(x,y) = p(x) + q(x)y$ is associating. Whether or not $f(x,y)$ is associating, we call the condition $f(x,y) = 0$ the *$(p \mid q)$-pseudoregularity* condition. (Generally, any pair of polynomials written in the form (p,q) will be assumed to satisfy $p(0) = 0$.)

It turns out that even in the general non-commutative case, whether $\big(p(x), q(x)\big)$ is an $\mathcal{R}$-pair depends on single variable, and hence commutative, conditions on p, q.

Theorem 1.1. *The pair $\big(p(x), q(x)\big)$ is an $\mathcal{R}$-pair if and only if*

1. $p\big(p(x)\big) - p(x) \in \big(q(x)\big)$ *and*
2. $q\big(p(x)\big) \in \big(q(x)\big)$,

where $\big(q(x)\big)$ is the principal ideal generated by $q(x)$ in $\mathbf{Z}[x]$.

Proof: First suppose that $f(x,y) = p(x)+q(x)y$ is associating. Then in $\mathbf{Z}_0[x,y,z]$, $f\big(f(x,y),z\big) = f(x,g)$ for some g, so

$$p\big(p(x)+q(x)y\big) + q\big(p(x)+q(x)y\big)z = p(x) + q(x)g(x,y,z).$$

Letting $g(x,y,z) = s(x,y) + t(x,y)z$, we have that

$$p\big(p(x)+q(x)y\big) + q\big(p(x)+q(x)y\big)z = p(x) + q(x)s(x,y) + q(x)t(x,y)z.$$

Hence $p\big(p(x)+q(x)y\big) = p(x) + q(x)s(x,y)$ and $q\big(p(x)+q(x)y\big) = q(x)t(x,y)$ in $\mathbf{Z}[x,y]$. Letting $y = 0$ gives the identities $p\big(p(x)\big) = p(x) + q(x)s(x,0)$ and $q\big(p(x)\big) = q(x)t(x,0)$, holding in $\mathbf{Z}[x]$, and so $p\big(p(x)\big) - p(x) \in \big(q(x)\big)$ and $q\big(p(x)\big) \in \big(q(x)\big)$ in $\mathbf{Z}[x]$.

Conversely, if $p\big(p(x)\big) - p(x) \in \big(q(x)\big)$ and $q\big(p(x)\big) \in \big(q(x)\big)$ in $\mathbf{Z}[x]$, then we can write $p\big(p(x)\big) = p(x) + r(x)q(x)$ and $q\big(p(x)\big) = s(x)q(x)$ for some $r, s \in \mathbf{Z}[x]$. Hence

$$\begin{aligned} f\big(f(x,y),z\big) &= p\big(p(x)+q(x)y\big) + q\big(p(x)+q(x)y\big)z \\ &= p\big(p(x)\big) + q(x)h(x,y) + q\big(p(x)\big)z + q(x)k(x,y)z, \end{aligned}$$

for some $h \in \mathbf{Z}_0[x,y]$, $k \in \mathbf{Z}[x,y]$, as can be seen by expanding. Hence

$$\begin{aligned} f\bigl(f(x,y),z\bigr) &= p(x) + q(x)r(x) + q(x)h(x,y) + q(x)s(x)z + q(x)k(x,y)z \\ &= p(x) + q(x)\bigl(r(x) + h(x,y) + s(x)z + k(x,y)z\bigr) \\ &= p(x) + q(x)g(x,y,z), \end{aligned}$$

where $g(x,y,z) = r(x) + h(x,y) + s(x)z + k(x,y)z$, and so $f(x,y)$ is associating. ∎

Further to the earlier comments, there is no obvious way to generalise the form $f(x,y) = p(x) + q(x)y$ and still get simple conditions on $p(x), q(x)$ as above, whether by allowing an $r(x)$ term on the right of the y or even by introducing new existential variables to supplement y. Thus, there is a certain inevitability about our choice of form for $f(x,y)$. On the other hand, the conditions are by no means necessary. For instance, the class of Boolean rings $\mathcal{B}$ is $\mathcal{R}_{x^2+x,0}$, and hence is a $(p \mid q)$-pseudoregular radical class, although surely $(x^2+x,0)$ is not an $\mathcal{R}$-pair; nor does it seem likely that $\mathcal{B}$ is a $(p \mid q)$-pseudoregular class for any $\mathcal{R}$-pair (p,q).

An important special case is $f(x,y) = x + q(x)y$, a radical class for any $q(x) \in \mathbf{Z}[x]$, as is immediate from the above theorem. These are $(q;1)$-regular classes. (The discovery of the general $(p;q)$-regularity condition is attributed to McKnight in [**5**].) For instance, $f(x,y) = x + y - xy = x + (1-x)y$ has this form, as does $f(x,y) = x + xy$. It is of considerable interest to determine which of our classes are always equal to such classes; that is, which $(p \mid q)$-pseudoregular radical classes are also $(x \mid q)$-pseudoregular (possibly for a different q). At present we conjecture that if $\bigl|q(0)\bigr| = 1$, then $\mathcal{R}_{p,q}$ is such a case.

Proposition 1.2. *The class $\mathcal{R}_{p,q}$ contains the $(q;1)$-regular class $\mathcal{R}_{x,q}$.*

Proof: Write $p(x) = xh(x)$, $h(x) \in \mathbf{Z}[x]$. If R is such that for all $a \in R$ there exists $b \in R$ for which $a + q(a)b = 0$, then $a = -q(a)b$, so $p(a) = ah(a) = -q(a)bh\bigl(-q(a)b\bigr)$, so $p(a) + q(a)c = 0$, where $c = bh\bigl(-q(a)b\bigr) \in R$, and so $R \in \mathcal{R}_{p,q}$. ∎

There are a number of easily identified circumstances in which the conditions of Theorem 1.1 are satisfied. Here is one.

Proposition 1.3. *For any $q(x), s(x) \in \mathbf{Z}[x]$ such that $q(0) = 0$, $\bigl(s(x)q(x), q(x)\bigr)$ is an $\mathcal{R}$-pair.*

Proof: If $p(x)$ has the stated form, then, modulo $q(x)$,

$$p\bigl(p(x)\bigr) - p(x) = s\bigl(s(x)q(x)\bigr)q\bigl(s(x)q(x)\bigr) - s(x)q(x) \equiv s(0)q(0) = 0,$$

and $q\bigl(p(x)\bigr) = q\bigl(s(x)q(x)\bigr) \equiv q(0) = 0$, as required. ∎

The D-regular radical class defined by $f(x,y) = x + xy$ has this form, with $s(x) = 1$. If $s(x) \in \mathbf{Z}_0[x]$ in the above proposition, then one may let $b = -s(a)$ and then $p(a) + q(a)b = 0$ for all a in any ring R, so $\mathcal{R}_{p,q}$ is the class of all rings.

Not all factors $q(x)$ of $p(p(x)) - p(x)$ will divide $q(p(x))$. For instance, if $p(x) = x^2 + 2x$ then $p(p(x)) - p(x) = x(x+1)^2(x+2)$. Now $q(x) = x, x+1$ both satisfy (2) in Theorem 1.1 (as well as (1) of course), although $q(x) = x+2$ does not. Nonetheless, $q(x) = x(x+2) = p(x)$ does, by Proposition 1.3. In general we have the following, which may be well-known and in a more general form, but we include its proof anyway.

Lemma 1.4. *Let $p, q, h \in \mathbf{Z}[x]$. Then letting $d = \gcd(p,q)$ and $r = \operatorname{lcm}(p,q)$, we have that $\gcd(p(h), q(h)) = d(h)$, and $\operatorname{lcm}(p(h), q(h)) = r(h)$.*

Proof: Now we have that $p = dp', q = dq'$ for some p', q'. Then $p(h) = d(h)p'(h)$ and $q(h) = d(h)q'(h)$, and so $d(h)$ is a common divisor of $p(h), q(h)$. Also, $d = up + vq$ for some u, v, so $d(h) = u(h)p(h) + v(h)q(h)$ and any common divisor of $p(h), q(h)$ is a divisor of $d(h)$. This proves $d(h) = \gcd(p(h), q(h))$. However, we also have that $r = \operatorname{lcm}(p,q) = pq/\gcd(p,q) = pq/d$, and so $\operatorname{lcm}(p(h), q(h)) = p(h)q(h)/\gcd(p(h), q(h)) = p(h)q(h)/d(h) = r(h)$. ∎

Theorem 1.5. *Suppose (p, q_1) and (p, q_2) are both $\mathcal{R}$-pairs. Then so are $(p, \gcd(q_1, q_2))$ and $(p, \operatorname{lcm}(q_1, q_2))$. If $q_1(x)q_2(x)$ divides $p(p(x)) - p(x)$, then (p, q_1q_2) is an $\mathcal{R}$-pair.*

Proof: If $q_1(x), q_2(x)$ both divide $p(p(x)) - p(x)$, so do $q = \gcd(q_1, q_2)$ and $r = \operatorname{lcm}(q_1, q_2)$. Moreover, $\operatorname{lcm}(q_1(p), q_2(p)) = q(p)$. Also, $q_1(x)$ divides $q_1(p(x))$ and $q_2(x)$ divides $q_2(p(x))$, so certainly $q = \gcd(q1, q2)$ divides $\gcd(q_1(p), q_2(p)) = q(p)$, and $r = \operatorname{lcm}(q_1, q_2)$ divides $\operatorname{lcm}\Big(q_1(p(x)), q_2(p(x))\Big) = r(p(x))$. Hence both (p,q) and (p,r) are $\mathcal{R}$-pairs as required.

Now because $q_i(x)$ divides $q_i(p(x))$, $i = 1, 2$, we have that $q(p(x)) = q_1(p(x))q_2(p(x))$ is divisible by $q_1(x)q_2(x)$, so if also $q_1(x)q_2(x)$ divides $p(p(x)) - p(x)$, then (p, q_1q_2) is an $\mathcal{R}$-pair. ∎

Suppose $q(x) = r(p(x))$ divides $p(p(x)) - p(x)$. Then $q(p(x)) = r\Big(p(p(x))\Big) \equiv r(p(x)) = q(x) \equiv 0$ (modulo $q(x)$), so $\Big(p(x), r(p(x))\Big)$ is an $\mathcal{R}$-pair. There are certain factors of $p(p(x)) - p(x)$ which have this form and hence always form $\mathcal{R}$-pairs with $p(x)$.

In general, for $p(x)$ satisfying $p(0) = 0$, we have that $p(x) = xh(x)$ where $h(x) = p(x)/x \in \mathbf{Z}[x]$, so

$$\begin{aligned} p(p(x)) - p(x) &= xh(x)h(xh(x)) - xh(x) \\ &= xh(x)\Big(h(xh(x)) - 1\Big) \\ &= p(x)\Big(h(p(x)) - 1\Big), \end{aligned}$$

of which three obvious factors are $p(x)$ itself, $h(p(x)) - 1$, and their product $p(p(x)) - p(x)$ itself. Thus $q(x) = r(p(x))$ where $r(x) = x$, $r(x) = h(x) - 1$, or $r(x) = p(x) - x$, is a factor of $p(p(x)) - p(x)$. Summing up, we have

Proposition 1.6. *Given $p(x)$ with $p(0) = 0$, the following are $\mathcal{R}$-pairs:*

1. $(p(x), p(x))$,
2. $\big(p(x), p(p(x)) - p(x)\big)$, *and*
3. $\big(p(x), p(p(x))/p(x) - 1\big)$.

2. Semiprime classes

A radical class is *semiprime* if it contains the prime (Baer) radical class. Semiprime $(p;q)$-regular classes turn out to be of special importance, and are characterised by the property that $|p(0)| = |q(0)| = 1$. In fact, the semiprime $(p;q)$-regular radical class is equal to the semiprime $(pq;1)$-regular class by [**4**, Corollary 16], in turn equal to the $(x \mid pq)$-pseudoregular class.

Thus an $(x \mid q)$-pseudoregular class (which is always radical, being a $(q;1)$-regular radical class) is semiprime if and only if $|q(0)| = 1$. On the other hand, there are pseudoregular classes $\mathcal{R}_{p,q}$ with $|q(0)| = 1$ which are not even radical classes. For instance, $(p,q) = (2x, 1-x)$ is easily seen not to be an $\mathcal{R}$-pair, but does not even define a radical class. To see this, first note that no ring with identity of characteristic other than 2 is in $\mathcal{R}_{p,q}$, so in particular $R = \mathbf{Z}_4 \notin \mathcal{R}_{p,q}$. However, the ideal $I = \{0,2\} \cong \mathbf{Z}_2^0 \in \mathcal{R}_{p,q}$ (where M^0 denotes the zeroring on the Abelian group M), and $R/I \cong \mathbf{Z}_2 \in \mathcal{R}_{p,q}$. Hence $\mathcal{R}_{p,q}$ is not closed under extensions. It is possible that the third class of examples in Proposition 1.6 will provide examples of $(p \mid q)$-pseudoregular radical classes for which $|q(0)| = 1$ yet which is not a $(p;q)$-regular class for any p, q. For instance, putting $p(x) = x^2$ gives the radical class $\mathcal{R}_{x^2, x^3-1}$, which may or may not be a $(p;q)$-regular class for any p, q.

A radical class is semiprime if and only if it contains all zerorings ([**5**, Lemma 3]). Quite generally we have

Theorem 2.1. *$\mathcal{R}_{p,q}$ contains all zerorings if and only if the coefficient of x in $p(x)$ is a multiple of $q(0)$.*

Proof: Let $p(x) = a_1x + a_2x^2 + \cdots$ and $q(x) = b_0 + b_1x + \cdots$. Now suppose $q(0) = b_0 \neq 0$. Let R be the zeroring on $\mathbf{Z}_{b_0}$, the additive cyclic group of integers modulo b_0. If $\mathcal{R}_{p,q}$ contains all zerorings, then for all $r \in R$ there exists $s \in S$ for which $a_1r + b_0s = 0$, that is, $a_1r = 0$ for all $r \in R$, and so b_0 divides a_1. Conversely, if $a_1 = b_0c$ for some integer c, then for any zeroring R, if $r \in R$, then $p(r) = a_1r = cb_0r = b_0(cr) = crq(r)$, and so $R \in \mathcal{R}_{p,q}$.

Now suppose $b_0 = 0$. Then for any zeroring R and any $r, s \in R$, $p(r) + q(r)s = a_1r$, so if $a_1 = 0$ then $\mathcal{R}_{p,q}$ contains all zerorings. Conversely, if $\mathcal{R}_{p,q}$ contains all zerorings, then letting R be the zeroring on $\mathbf{Z}$, we see that $R \in \mathcal{R}_{p,q}$ implies $a_1 = 0$. ∎

Corollary 2.2. *If (p,q) is an $\mathcal{R}$-pair, then $\mathcal{R}_{p,q}$ is semiprime if and only if the coefficient of x in $p(x)$ is a multiple of $q(0)$.*

From the earlier comments, this actually generalises the corresponding fact for $(p;q)$-regular classes.

All semiprime $(p;q)$-regular classes contain $\mathcal{J}$, the Jacobson radical. Here is an example to show that in general a semiprime pseudoregular radical class need not. Let $f(x,y) = x^2 + (x^4 - x^2)y$, so that $p(x) = x^2$ and $q(x) = x^4 - x^2$; the conditions of Theorem 1.1 are satisfied as is easily checked, so $\mathcal{R}_f$ is a radical class. It is easy to see that all zerorings are in $\mathcal{R}_f$, which is therefore semiprime, so if it were $(p;q)$-regular for some p, q, then it would contain $\mathcal{J}$ (by [**4**, Theorem 3]). However, $R = \mathbf{Z}_0[X]/(X^3)$ is nil and not in $\mathcal{R}_f$, since (writing $\overline{X}$ for the image of X in R), there would need to be a $P(\overline{X}) \in R$ such that $\overline{X}^2 - \overline{X}^2 P(\overline{X}) = 0$, which there is not.

For $(p;q)$-regularity, semiprimeness is equivalent to $|p(0)q(0)| = 1$. If this latter condition is satisfied, then from [**1**, Corollary 7] and Proposition 1.2, $\mathcal{J} \subseteq \mathcal{R}_{x,q} \subseteq \mathcal{R}_{p,q}$. Hence we have

Proposition 2.3. *If $|q(0)| = 1$ then all quasiregular rings are in $\mathcal{R}_{p,q}$.*

3. Equivalent pairs

We say (p,q) and (p',q') are *equivalent* if $\mathcal{R}_{p,q} = \mathcal{R}_{p',q'}$. In general we have the following

Proposition 3.1. *If (p,q) is an $\mathcal{R}$-pair then so is $(p+sq,q)$ for any s for which $s(0)q(0) = 0$. If $s(0) = 0$ then (p,q) and $(p+sq,q)$ are equivalent.*

Proof: If (p,q) is an $\mathcal{R}$-pair, then letting $p' = p + sq$, we see that $p'(0) = 0$, and that modulo $q(x)$, $p'(x) \equiv p(x)$, so

$$\begin{aligned} p'\big(p'(x)\big) - p'(x) &\equiv p'\big(p(x)\big) - p(x) \\ &= p\big(p(x)\big) + s\big(p(x)\big)q\big(p(x)\big) - p(x) \\ &\equiv p\big(p(x)\big) - p(x) \\ &\equiv 0, \end{aligned}$$

and $q\big(p'(x)\big) \equiv q\big(p(x)\big) \equiv 0$, so (p',q) is an $\mathcal{R}$-pair.

Now note that $p'(x) + q(x)y = p(x) + q(x)\big(s(x) + y\big)$, so if $s(0) = 0$ then evidently $\mathcal{R}_{p,q}$ is equivalent to $\mathcal{R}_{p',q}$. ∎

We say (p,q) and (p',q') are *strongly equivalent* if $q = q'$ and $p - p' \in \big(xq(x)\big)$, the ideal generated by $xq(x)$ in $\mathbf{Z}[x]$. Notation: $(p,q) \Leftrightarrow (p',q')$. From the last proposition, strong equivalence of (p,q) and (p',q') implies their equivalence, and each or neither is an $\mathcal{R}$-pair. Equivalence does not imply strong equivalence: for instance, all $(p,1)$ are equivalent, since all define the (radical) class of all rings.

Proposition 3.2. *If $q(x)$ has leading coefficient of magnitude 1, then (p,q) is strongly equivalent to (r,q), where $r(x)$ is the remainder of $p(x)$ on division by $xq(x)$.*

PROOF: We can divide $p(x)$ by $xq(x)$ since the latter has leading coefficient of size 1, in such a way that the remainder $r(x)$, congruent to $p(x)$ modulo $xq(x)$, has degree at most that of $q(x)$. Moreover (r,q) is strongly equivalent to (p,q) by Proposition 3.1. ▯

4. Some special cases

THEOREM 4.1. *For $n > 0$, the expression $f(x,y) = x^n + (x^m - x^l)y$ is associating if and only if $m, l \leqslant n^2$ and $m - l$ divides $n^2 - n$.*

PROOF: We want $q(x) = x^m - x^l$ to divide $q(p(x)) = x^{nm} - x^{nl}$, which it surely does as is easily seen from working modulo $q(x)$, but also to divide $p(p(x)) - p(x) = x^{n^2} - x^n$. For this to occur it must be the case that $m-l$ can divide n^2-n but also that $m, l \leqslant n^2$. ▯

For example, from this we obtain that $f(x,y) = x^{n+1} + x^n y - x^{n+1}y = x^n(x \circ y)$ is associating. $\mathcal{R}_f$ defines a class which obviously contains $\mathcal{J}$. However, $x^{n+1} + (x^n - x^{n+1})y = x^{n+1} + (1-x)z$ where $z = x^n y$, so $\mathcal{R}_{x^{n+1}+(x^n-x^{n+1})y} \subseteq \mathcal{R}_{x^{n+1}+(1-x)y}$; but $x^{n+1} + (1-x)y$ defines the same class as $x^{n+1} + (1-x)(x + x^2 + \cdots + x^n) + (1-x)y = x + (1-x)y = x \circ y$, so $\mathcal{R}_{x^n(x\circ y)} = \mathcal{J}$, for all $n \geqslant 0$. A great many radical $\mathcal{R}_{p,q}$ turn out to be $\mathcal{J}$ as is evident from results in [4]: if $|q(0)| = 1$ then this is the case exactly when $q(x)$ has a factor of the form $(ax+1)$, where $a \neq 0$ and for each prime divisor m of a, there is an integer n such that m divides $p(n)$.

THEOREM 4.2. *Suppose (p,q) is an $\mathcal{R}$-pair. If $q(1) = 0$ and $p(1) = 1$ then $\mathcal{R}_{p,q} \subseteq \mathcal{J}$.*

PROOF: Now if $q(1) = 0$ then $q(x) = (x-1)h(x)$ for some $h(x)$. If in the ring R, $p(a) + (a-1)h(a)b = 0$, then $p(a) + (a-1)c = 0$ for $c = h(a)b \in R$, so $\mathcal{R}_{p,q} \subseteq \mathcal{R}_{p,x-1}$. Now if (p,q) is an $\mathcal{R}$-pair, then $p(p(x)) - p(x)$ is divisible by $q(x)$ and hence certainly by $x-1$, and moreover $p(x) - 1$ is divisible by $x - 1$ exactly when $p(1) = 1$, so that in this case, $(p(x), x-1)$ is an $\mathcal{R}$-pair. But as can be seen by computing $p(x)$ modulo $x(x-1) = x^2 - x$, $(p(x), x-1) \Leftrightarrow (p(1)x, x-1) = (x, x-1)$, which defines $\mathcal{J}$. ▯

Letting $p(x) = q(x) = x$ gives the D-regular radical, which is surely not in $\mathcal{J}$ since it contains all rings with identity, while $p(x) = q(x) = x^2 - x$ gives a still smaller radical class, although one which still contains all rings with identity and hence is not in $\mathcal{J}$ either. These provide examples to show that both conditions in the above theorem statement are necessary.

Combining Theorem 4.2 with Proposition 2.3 gives

COROLLARY 4.3. *If (p,q) is an $\mathcal{R}$-pair with $p(1) = 1$, $q(1) = 0$ and $|q(0)| = 1$ then $\mathcal{R}_{p,q} = \mathcal{J}$.*

The requirement that $|q(0)| = 1$ in this corollary cannot be dropped, as the fact that $\mathcal{J} \neq \mathcal{R}_{x^2, x^4 - x^2}$ shows.

COROLLARY 4.4. *If $p(1) = 1$, then $\big(p(x), 1 - p(x)\big)$ is an $\mathcal{R}$-pair, that is $p(x) \circ y$ is associating, and moreover $\mathcal{R}_{p,1-p} = \mathcal{J}$.*

This result is rather surprising; it implies for instance that for a ring to be quasiregular, it is sufficient for all n-th powers to be quasiregular, for any single $n > 0$. On the other hand, the following result holds quite generally, and follows easily from Proposition 3.1.

PROPOSITION 4.5. *For any $t(x) \in \mathbf{Z}[X]$, the class of rings defined by $f(x,y) = x\big(t(x) \circ y\big)$ is a radical class.*

Next we consider which pseudoregular classes contain all nil rings.

THEOREM 4.6. *Suppose $p(x) = a_s x^s + a_{s+1}x^{s+1} + \cdots$ and $q(x) = b_t x^t + b_{t+1}x^{t+1} + \cdots$ with $a_s, b_t \neq 0$. If the class $\mathcal{R}_{p,q}$ contains all nil rings then $s > t$, and if also b_t divides all other b_j, then b_t divides all a_i.*

Conversely, if $s > t$ and b_t divides all other a_i as well as all b_j, then $\mathcal{R}_{p,q}$ contains all nil rings.

PROOF: Suppose first that $\mathcal{R}_{p,q}$ contains all nil rings. Suppose $s \leqslant t$. Let $R = \mathbf{Z}_0[X]/(X^{t+1})$. Then R is nil and so for the image $\overline{X}$ of X in R, there is some $h \in R$ for which $a_s\overline{X}^s + a_{s+1}\overline{X}^{s+1} + \cdots + a_t\overline{X}^t = (b_t\overline{X}^t)h$, which is zero since h is a multiple of $\overline{X}$; hence $a_s = a_{s+1} = \cdots = a_t = 0$, a contradiction, so $s > t$. Hence we can write $p(x) = a_{t+1}x^{t+1} + a_{t+2}x^{t+2} + \cdots$, where all $a_i = 0$ for $i < s$ (if any such $i \geqslant t+1$ exist).

Now assume all b_j are divisible by b_t. Let $R = \mathbf{Z}_{b_t}^0[X]/(X^m)$, the ring of polynomials over $\mathbf{Z}_{b_t}$ with zero constant term factored by the ideal (X^m), where m exceeds the maximal degrees of x in either $p(x)$ or $q(x)$. Then R is nil, so, again letting $\overline{X}$ be the image of X in R, there is $h \in R$ for which $a_{t+1}\overline{X}^{t+1} + a_{t+2}\overline{X}^{t+2} + \cdots = 0$ in R, since all b_j are divisible by b_t. Hence $a_{t+1}, a_{t+2}, \ldots$ are divisible by b_t also.

Now suppose conversely that b_t divides all b_j and all a_i. Let $b_i' = b_i/b_t$ and $a_i' = a_i/b_t$ for all $i > t$. Then let $r(x) = b_{t+1}'x + b_{t+2}'x^2 + \cdots$ and $w(x) = a_s'x_{s-t+1} + a_{s+1}'x_{s-t+2} + \cdots$; then $p(x) = b_t x^t w(x)$ and $q(x) = b_t x^t\big(1 - r(x)\big)$. Suppose R is nil, with $a \in R$ such that $a^n = 0$. Let $b = -w(a)\big(1 + r(a) + r(a)^2 + \cdots + r(a)^{n-1}\big)$, noting that $r(a)^n = 0$. Then

$$
\begin{aligned}
&p(a) + q(a)b \\
&\quad = b_t a^t w(a) - b_t a^t\big(1 - r(a)\big)w(a)\big(1 + r(a) + r(a)^2 + \cdots + r(a)^{n-1}\big) \\
&\quad = b_t a^t w(a)\Big(1 - \big(1 - r(a)\big)\big(1 + r(a) + r(a)^2 + \cdots + r(a)^{n-1}\big)\Big) \\
&\quad = b_t a^t w(a)\Big(1 - \big(1 + r(a) + r(a)^2 + \cdots + r(a)^{n-1} - r(a) - r(a)^2 - \cdots - r(a)^n\big)\Big) \\
&\quad = b_t a^t w(a)(1 - 1) \\
&\quad = 0.
\end{aligned}
$$

Hence $R \in \mathcal{R}_{p,q}$. ∎

This has immediate application to cases where (p,q) is an $\mathcal{R}$-pair, in terms of radical classes containing the nil radical class. It would be nice to tighten the above result to if and only if status, dropping the need to assume b_t divides all other b_j in the first part of the statement. There is also the question of the uniqueness of the element b given a in the nil ring R, satisfying $p(a)+q(a)b=0$, a question of interest for pseudoregular rings generally. Both of these issues may require the assumption that (p,q) is an $\mathcal{R}$-pair in order to make progress.

It is easy to give sufficient conditions for hereditariness of $\mathcal{R}_f$. These are the relevant version of conditions given in [**5**], which they generalise in the $(q;1)$-regular case.

Proposition 4.7. *$\mathcal{R}_f$ is hereditary providing either of the following holds:*

1. $|q(0)|=1$; *or*
2. $q(x)=p(x)s(x)$, $s(0)=0$.

Proof: If $|q(0)|=1$ then $q(x)=1+xs(x)$ or $-1+xs(x)$ for some $s(x)\in\mathbf{Z}[x]$. Hence in the first case, $f(x,y)=p(x)+q(x)y=p(x)+y+xs(x)y$, so if $R\in\mathcal{R}_f$, with I an ideal of R, and if $a\in I$ then $p(a)+b+as(a)b=0$ for some $b\in R$, so $b=-p(a)-as(a)b\in I$. Similarly with the second case.

Suppose $q(x)=p(x)s(x)$, $s(0)=0$. Let $R\in\mathcal{R}_f$, $f(x)=p(x)+q(x)y$, with I an ideal of R. If $a\in I$ then there exists $b\in R$ for which $0=p(a)+q(a)b=p(a)+p(a)s(a)b=p(a)+s(a)p(a)b=p(a)+s(a)\big(-q(a)b\big)b=p(a)+q(a)\big(-s(a)b^2\big)=p(a)+q(a)c$, where $c=-s(a)b^2\in I$. Hence $I\in\mathcal{R}_f$ and so $\mathcal{R}_f$ is hereditary. $\square$

It is not known if a converse for this result can be obtained.

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