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RINGS OF FORMAL POWER SERIES

BY

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In this brief exposition we collect several results on rings of formal power series with coefficients from a field or a ring with some special properties. The results that are catalogued below are mostly algebraic in nature.

§1 initiates the discussion by giving a formal definition and listing some of the (elementary) properties. In §2, the preparation theorem of Weierstrass is stated for formal power series ring over a ring or a field, for convergent power series ring over a field which is complete with respect to a valuation and restricted power series ring over a ring with a linear topology. We study stability properties in the following section while §4 concerns itself with the structure of complete local rings. Algebraic geometric applications are dealt with in §5 while the closing section includes results which are not in any of the above categories.

All the rings that we consider are commutative and with identity element unless we state to the contrary.

§1. Let R be a ring and x be an indeterminate which commutes with every element of R. Then the set $R[[x]]$ consisting of all formal expressions of the form

$$\sum_{i=0}^{\infty} a_i x^i$$

where a_i are in R, under the two laws of composition

$$\sum_{i=0}^{\infty} a_i x^i + \sum_{i=0}^{\infty} b_i x^i = \sum_{i=0}^{\infty} (a_i + b_i) x^i$$

and

$$\left(\sum_{i=0}^{\infty} a_i x^i\right) \cdot \left(\sum_{i=0}^{\infty} b_i x^i\right) = \sum_{k=0}^{\infty} c_k x^k$$

where

$$c_k = \sum_{0<i,j<k} a_i b_j, \quad i+j = k$$

forms a ring and this ring is called the ring of formal power series over R.

The above can be generalized to more than one variable. We can look upon the ring $R[[x_1 x_2 \ldots x_n]]$ of formal power series in $x_1, x_2, \ldots, x_n$ over R as a graded ring in the following manner. First we define the total degree of a monomial

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$M_m = x_1^{m_1}. x_2^{m_2}. \ldots . x_n^{m_n}$ as the integer $m = m_1 + m_2 + \cdots + m_n$. Then we denote by $R^{(t)}$ the set of all formal finite linear combinations $\sum_t a_t M_t$ of monomials M_t of degree t with a_t in R. The additive group of $R[[x_1, x_2, \ldots, x_n]]$ is then isomorphic to the complete direct product of the additive groups $R^{(t)}$ for $t = 0, 1, 2, \ldots$. If we define the multiplication in this product by using the gradation as

$$R^{(p)} \cdot R^{(q)} = R^{(p+q)}$$

then we get

$$R[[x_1, x_2, \ldots, x_n]]$$

as a graded ring.

The following properties are easy to observe:

(1) $R[[x]]$ contains the ring of polynomials $R[x]$, and that this ring is dense in $R[[x]]$ under the topology given by the ideals $(x), (x^2), \ldots, (x^n), \ldots$. In fact $R[[x_1, x_2, \ldots, x_n]]$ is a topological ring under the ideal-adic topology induced by the ideal generated by $(x_1, x_2, \ldots, x_n)$. In case R is replaced by a field then we get a complete local ring.

(2) If R is a domain then so is $R[[x]]$ and $R[[x_1, x_2, \ldots, x_n]]$.

(3) If R is a field then $R[[x]]$ is a unique factorization domain.

(4) An element $f(x) = \sum_{i=0}^{\infty} a_i x^i$ in $R[[x]]$ is a unit if and only if $f(0) = a_0$ is a unit in R. Consequently every formal power series over a field with nonzero constant term is a unit.

(5) There is a one-to-one correspondence between the maximal ideals of R and the maximal ideals of $R[[x_1, x_2, \ldots, x_n]]$, the association being that the maximal ideal $\mathfrak{m}$ of R corresponds to

$$\mathfrak{m} \cdot R[[x_1, \ldots, x_n]] + \sum_{i=1}^{n} x_i R[[x_1, x_2, \ldots, x_n]].$$

(6) Let T be a nonsingular linear transformation of the variables $x_1, x_2, \ldots, x_n$ into $x_1', x_2', \ldots, x_n'$ with $B = (b_{ij})$ as the coefficient matrix of T where the entries of B are from a field K. Then T transforms an ideal of $K[[x_1, x_2, \ldots, x_n]]$ into an ideal of $K[[x_1', x_2', \ldots, x_n']]$. This result remains valid if K is replaced by a ring or the formal power series ring by convergent power series ring.

(7) If K is a field then any chain of prime ideals in $R = K[[x_1, x_2, \ldots, x_n]]$ is of length at most $n+1$. Moreover, any chain of prime ideals can be extended to a chain of $(n+1)$ prime ideals. We do not include the unit ideal R in the above.

For most of the above results the reader can refer to Nagata [19], Zariski and Samuel [38].

The following result which is concerned with the behavior of prime ideals in power series rings over a field under extensions of ground field is due to Chevalley [5].

(8) Let $K \subset L$ be two fields and P be a prime ideal in $K[[x_1, x_2, \ldots, x_n]]$ where

x_i are indeterminates. If $\wp$ is the ideal generated by $\mathfrak{p}$ in $L[[x_1, x_2, \ldots x_n]]$, then $\wp$ is not in general prime. In fact it is the intersection of primary ideals. If the dimension of $\wp$ in $K[[x_1, x_2, \ldots, x_n]]$ is r, then the dimension of the prime ideals associated with the primary components of $\wp$ in $L[[x_1, x_2, \ldots, x_n]]$ is also r. In order that $\wp$ be prime in $L[[x_1, x_2, \ldots, x_n]]$ we need to impose the condition that L is separably generated over K and K is algebraically closed in L.

§2. If $f(z)=\sum_{i=0}^{\infty} a_i z^i$ where a_i are complex numbers, is an analytic function of z in a neighborhood of the origin such that the origin is a zero of order n for $f(z)$, then for any function $g(z)=\sum_{i=0}^{\infty} b_i z^i$ which is also analytic in a neighborhood of the origin, there exists an analytic function $h(z)$ such that

$$h(z)\cdot f(z)-g(z) = \sum_{i=0}^{n-1} b_i z^i.$$

We shall now give a generalized version of the above for function of several variables with coefficients from an arbitrary ring in an algebraic language.

Let $f(x_1, x_2, \ldots, x_n)$ be a formal power series in n variables over a ring R such that $f(0, 0, \ldots, 0, x_n)$ is not identically zero and is expressible as $x_n^s\phi(x_n)$ where $s \geq 1$ and $\phi(x_n)$ is a unit in $R[[X_n]]$. Then for any element $g(x_1, x_2, \ldots, x_n)$ in $R[[x_1, x_2, \ldots, x_n]]$ there can be found power series $q(x_1, x_2, \ldots, x_n)$ and $r_i(x_1, x_2, \ldots, x_{n-1})$ in $R[[x_1, x_2, \ldots, x_{n-1}]]$ where $i=0, 1, \ldots, (s-1)$, such that

$$g(x_1, x_2, \ldots, x_n) = q(x_1, x_2, \ldots, x_n)\cdot f(x_1, x_2, \ldots, x_n) + \sum_{i=0}^{n-1} r_i(x_1, x_2, \ldots, x_{n-1})\cdot x_n^i.$$

In case R is the real field or the complex field or a field with a valuation, then we call a formal power series

$$\sum a_{r_1 r_2 \ldots r_n} x_1^{r_1} x_2^{r_2} \ldots x_n^{r_n} \quad \text{in} \quad R[[x_1, x_2, \ldots, x_n]]$$

a convergent power series with respect to the "absolute value" valuation in the first two cases and the given valuation in the third case (and we denote the valuation by v), if there exist positive real numbers $\lambda_1, \ldots, \lambda_n$ and M such that $v(a_{r_1 r_2 \ldots r_n})\cdot \lambda_1^{r_1} \ldots \lambda_n^{r_n} \leq M$ for every $r_1, r_2, \ldots, r_n$. The set of convergent power series forms a subring of the ring of formal power series and this convergent power series ring is usually denoted by $K\langle\langle x_1, x_2, \ldots, x_n\rangle\rangle$. By changing formal power series by convergent power series in the statement of the above preparation theorem we get the theorem for convergent power series.

If we take for $g(x_1, x_2, \ldots, x_n)$ the monomial $-x_n^s$, then we get in a unique way a power series $u(x_1, x_2, \ldots, x_n)$ which is a unit and power series $r_i(x_1, x_2, \ldots, x_{n-1})$ none of which is a unit, such that

$$f(x_1, x_2, \ldots, x_n) = u(x_1, x_2, \ldots, x_n)\cdot\left(x_n^s + \sum_{i=0}^{s-1} r_i(x_1, x_2, \ldots, x_{n-1})\cdot x_n^i\right).$$

The polynomial in x_n of degree s that occurs on the right-hand side of the above equation is called the distinguished pseudo-polynomial associated with $f(x_1, x_2, \ldots, x_n)$. The following properties of distinguished pseudo-polynomials are known in case of convergent power series (Ruckert [22]) and are easy to prove in general case as well.

(1) If $f(z) \in R[[x_1, \ldots, x_n, z]]$ with $f^*(z) \in R[[x_1, \ldots, x_n]][z]$ as its distinguished pseudo-polynomial, then there exists a power series $q(z)$ which is a unit such that $f(z) = f^*(z) \cdot q(z)$.

(2) If $f^*(z)$ and $g^*(z)$ are distinguished pseudo-polynomials then so is

$$f^*(z) \cdot g^*(z).$$

(3) If the distinguished pseudo-polynomial $f^*(z)$ of $f(z)$ in $R[[x_1, \ldots, x_n, z]]$ splits into two monic polynomials $f_1^*(z)$ and $f_2^*(z)$ such that all the coefficients of $f_i^*(z)$ $(i=1, 2)$ vanish at $x_1 = x_2 = \cdots = x_n = 0$ then $f(z)$ also decomposes (up to a unit factor) into $f_1(z) \cdot f_2(z)$. Further $f_1(z)$ and $f_2(z)$ have $f_1^*(z)$ and $f_2^*(z)$ as their distinguished pseudo-polynomials.

(4) In case R is an integral domain, with $f^*(z), g^*(z), f(z)$ and $g(z)$ as in (3) above and if $f^*(z) = g^*(z) \cdot g^*(z)$, then $q^*(z)$ is also a distinguished pseudo-polynomial.

We have the analogue of Weierstrass preparation theorem for yet another class of subrings of the ring of formal power series over a ring. Let R be a ring with a linear topology, that is a topology given by a neighborhood basis of 0 consisting of a family of ideals. (An ideal adic topology is one example of a linear topology.) A formal power series over such a ring is said to be a restricted power series if almost all coefficients belong to each neighborhood of zero.

Here is the analogue of the preparation theorem. Salmon [23].

Let R be a commutative ring with identity element and having a linear topology with respect to which it is separated and complete. Let $\mathfrak{m}$ be a closed ideal all of whose elements are topologically nil-potent. If $f(x)$ is a restricted power series whose image in $(R/\mathfrak{m})[x]$ is a monic polynomial of degree n, then for each restricted series $g(x)$ there exist uniquely a restricted power series $h(x)$ and a polynomial $r(x)$ of degree at most $(n-1)$ such that

$$g(x) = f(x) \cdot h(x) + r(x).$$

There are other versions of the preparation theorem like the differentiable preparation theorem for the germs of differential functions due to Malgrange [18] and the preparation theorem for the ring of formal algebraic series which is the Henselization of the localization of the ring of polynomials over a field localized at the ideal generated by the variables due to Lafon [16].

§3. In this section we list with due definitions those properties of the ring R that are inherited (or not) by the ring of formal power series in one or more variables over R.

(1) If R is a Noetherian ring (satisfying the ascending chain condition or basis condition or maximal condition for ideals) then so is the ring of formal power series and the proof of this is an adaptation of Hilbert's basis theorem (or can be derived as a corollary of the Weierstrass preparation theorem).

(2) If instead of R we take a field K for the coefficient domain then the ring of formal power series is a unique factorization domain and the proof of this can also be derived from the preparation theorem.

(3) If we take a unique factorization domain for the domain of coefficients then we cannot claim that $R[[x_1, x_2, \ldots, x_n]]$ is also a unique factorization domain. The following theorem due to Samuel [24] will substantiate the above statement.

Let R be a UFD, a, b, c, be three elements of R, i, j, k, be three integers. Assume that b is prime b and c are relatively prime, a^{i-1} is not in $Rb+Rc$ but a^i is in Rb^k+Rc^j and

$$i\cdot j\cdot k \geq i\cdot j+j\cdot k+k\cdot i.$$

Then $R[[x]]$ is not a unique factorization domain.

Using the above with a slightly more general b namely that b is a product of prime elements of R, Solomon [23], has shown that the ring of restricted power series also does not in general inherit the property of UFD.

However, if R is assumed to be a regular unique factorization domain, that is to say that the localization of R at each of its maximal ideals is a regular local ring, then $R[[x]]$ is also a regular unique factorization domain.

If we assume that R and all power series rings in any finite number of variables are unique factorization domains then $R[[x_1; i \in I]]$ is also a unique factorization domain and different proofs are given by Everett and Cashwell [2] and Deckard and Durst [7].

(4) If R is a Krull domain which means that (a) R is an integrally closed integral domain, (b) R is the intersection of discrete rank one valuation rings of the quotient field K of R, and (c) each nonzero element of R is a unit for almost all valuations induced by the valuations lying over R, then the power series ring over R is also a Krull domain. The proof is by induction on the number of variables. For $R[[x]]$ the proof is derived from the following facts. See Samuel [25] for details.

(i) The intersection R of a family of Krull rings contained in a field such that each nonzero element of R is a unit in almost all these Krull rings, is also a Krull ring.

(ii) The localization of a Krull ring is a Krull ring.

(iii) $R[[x]]=(\bigcap V_\alpha[[x]]_S) \cap K[[x]]$ where V_α are the essential valuation rings of R, S is the multiplicatively closed set generated by 1 and x and K is the field of quotients of R.

Gilmer and Heinzer [10] have extended the above result to the case when the number of variables is not countable.

In [12] Gilmer shows that if R is a Krull ring and P is a minimal prime ideal of

R, then there is a unique minimal prime ideal $\wp$ of $R[[x_\alpha]]$, which is the full ring of formal power series over R (Bourbaki [*Algebra*, Ch. IV, p. 66]), such that $\wp \cap R = P$.

(5) A ring R is said to be Henselian at a Maximal ideal $\mathfrak{m}$ if the couple $(R, \mathfrak{m})$ satisfies Hensel's lemma. Trivially any field is Henselian. Also the power series ring over a field is Henselian at the unique maximal ideal and this can be proved as a consequence of the preparation theorem. If R is a ring not necessarily local, which is Henselian at a maximal ideal M, then $R[[x]]$ is also Henselian at the maximal ideal $M\cdot R[[x]]+x\cdot R[[x]]$ that lies over M and this is true of any finite number of variables. For proof see [28]. We do not know if this is true if we have countably infinite number of variables.

Nagata has introduced the technique of associating a minimal Henselian local ring with every pair $(R, \mathfrak{p})$ where R is an integrally closed integral domain and $\mathfrak{p}$ is a prime ideal of R as follows. Let K be the field of quotients of R and $\bar{K}$ be its algebraic closure and K_s be the separable closure of K in $\bar{K}$. If we denote the integral closure of R in K_s by R_s and $\mathfrak{p}_s$ is a prime ideal of R that lies over $\mathfrak{p}$ then the localization $\tilde{R}$ of the decomposition ring $\hat{R}$ of $G(\mathfrak{p}_s)$, where $G(\mathfrak{p}_s)$ is the group of automorphisms of K_s over K that leave $\mathfrak{p}_s$ fixed, with respect to $\mathfrak{p}=\mathfrak{p}_s \cap \hat{R}$ is Henselian and this is called the Henselization of R at $\mathfrak{p}$.

Clearly $\tilde{R}[[x]]$ is Henselian at the maximal ideal lying over $\mathfrak{p}$. We believed to have proved that $\tilde{R}[[x]]$ is also the Henselization of $R[[x]]$ at the prime ideal $\mathfrak{p}[[x]]+xR[[x]]$ that lies over $\mathfrak{p}$. But this is false because $R[[x]]$ has transcendency over the Henselization of $R[[x]]$ at $\mathfrak{p}\cdot R[[x]]+x\cdot R[[x]]$.(2)

The convergent power series ring in a finite number of variables over a field K which is complete with respect to a multiplicative valuation v (that is $v(a+b) \leq v(a)+v(b)$, for any a, b in K, where $v(x)$ is a nonnegative real number) is Henselian at its unique maximal ideal.

(6) Let $(R, \mathfrak{A})$ be a Zariski ring that is to say that R is a Noetherian ring with a linear topology induced by the ideal and its powers such that every ideal of R is closed under this topology. Then $R[[x]]$ is also a Zariski ring with respect to the ideal $\mathfrak{A}\cdot R[[x]]+x\cdot R[[x]]$. By induction we can extend the above result to any finite number of variables.

(7) Let R be a local ring with a distinct system of parameters, the number of which is equal to the dimension of the ring R. Such a ring is called a Macaulay ring. The polynomial ring over a Macaulay ring is a Macaulay ring and the completion of a Macaulay ring is also Macaulay. From these it is easily seen that $R[[x]]$ is also Macaulay if R is so. For details see Nagata [19], Zariski and Samuel [38].

(8) An integral domain R is called a Japanese ring, if for each finite extension K' of the field of quotients K of R, the integral closure R' of R is an R-module of

(2) The author thanks Prof. M. Nagata for pointing out the error and the reason mentioned above.

finite type. Grothendieck [*Eléments de Géometrie Algébrique*, Ch. IV, 0; 23.1.1] has shown that if R is an integrally closed Noetherian Japanese ring then the ring of formal power series over R in finite number of indeterminates is also a Japanese ring.

(9) A commutative ring R with identity is said to be coherent if each direct product of flat R-modules is flat. A necessary and sufficient condition for R to be coherent is that the annihilator of each element of R is an ideal of finite type and the intersection of two ideals of finite type is again of finite type. Clearly every Noetherian ring R is coherent and this property of coherence is inherited by $R[x]$ and $R[[x]]$ if R is assumed to be Noetherian. However, coherence is not stable under polynomial or power series extensions. R may be coherent without $R[x]$ being so and $R[x]$ may be coherent without $R[[x]]$ being so [for details see Soublin [31], [32], [33]).

(10) A commutative ring is said to be rigid if the second cohomology group of R with coefficients in itself vanishes. This property of rigidity is inherited by the power series ring. For details see Gerstenhaber [8].

(11) Separable generation and relative algebraic closure are also inherited by fields of formal power series. More explicitly, if $K \subset L$ are two fields such that K is algebraically closed in L then $K((x_1, x_2, \ldots, x_n))$ is also algebraically closed in $L((x, x, \ldots, x))$ and if L is separable extension of a purely transcendental extension of K (that is L is separably generated over K) then this property carries over to $L((x_1 x_2 \ldots x_n))$ over $K((x_1 x_2 \ldots x_n))$. The proof is quite straightforward. See Chevalley [5].

(12) Recently Gilmer [J. of Algebra **11** (1969), 488–502] has shown that if $K \subset L$, K and L are fields then $L[[x_1, \ldots, x_n]]$ is integral over $K[[x_1, \ldots, x_n]]$ in case L is separable algebraic over K if and only if L is finite over K. If L is purely inseparable over K, then the above is valid if and only if L is of finite exponent over K. While if L is inseparable over K then for the integrality of $L[[x_1, \ldots, x_n]]$ over $K[[x_1, \ldots, x_n]]$ it is necessary and sufficient that the separable degree of L over K is finite and L is of finite exponent over K. He has also given conditions for an element $f(x_1, \ldots, x_n)$ of $L[[x_1, \ldots, x_n]]$ to be integral over $K[[x_1, \ldots, x_n]]$. The conditions are the same as above except that L is replaced by L_1 where L_1 is the subfield of L generated by the coefficients of $f(x_1, \ldots, x_n)$ over K.

But there are several properties that are not inherited by power series rings. We list some of these.

(1) If R is a Dedekind domain then $R[[x]]$ need not be so.

(2) If R is Artinian, that is R satisfies the minimum condition for ideals, then $R[[x]]$ is not Artinian.

(3) If R is a Jacobson ring (i.e. for each ideal $\mathfrak{A}$ of R, Rad $\mathfrak{A}=\{x$ in R such that x^n belongs to $\mathfrak{A}$ for some integer $n\}$, is equal to Raj $\mathfrak{A}=$the intersection of all maximal ideals containing $\mathfrak{A}$), then no power series ring can be Jacobson. For instance any field K is Jacobson (trivial) while $K[[x]]$ is not.

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(4) If R is a valuation ring, $R[[x]]$ is not so.

(5) R may be integrally closed without $R[[x]]$ being so as the following example due to J. Ohm [20] shows: Let R be the integral closure of $K[d, a_0, a_1, \ldots, b_0, b_1, \ldots]$ where $d, a_0, a_1, \ldots, b_0, b_1, \ldots$ are indeterminates over a field K whose characteristic is different from 2 and the b_i are defined as below:

$$b_0 = a_0 \cdot d^{-1},$$

$$b_n = (1/d^2) \sum_{i=0}^{n} a_i a_{n-i}, \quad n \geq 1.$$

If we set

$$w = (1/d) \sum_{i=0}^{\infty} a_i x^i$$

then

$$w^2 = b_0^2 + \sum_{i=1}^{\infty} b_i x^i$$

is an element of $R[[x]]$ which shows that w is integral over $R[[x]]$. Moreover w belongs to the quotient field of $R[[x]]$. But w is not an element of $R[[x]]$.

(6) If R is a domain then the field of quotients of L of $R[[x]]$ may be properly contained in the field of formal power series over the quotient field K of R. Gilmer [9] has shown that the necessary and sufficient conditions for $K((x)) = L$ is any sequence of nonzero principal ideals of R has a nonzero intersection (that is contains a nonzero element) and $K[[x]]$ is the localization of $R[[x]]$ by a set on nonzero elements of R.

§4. Property 8 of §1 stated that the ring of formal power series over a field is a complete local ring. We now ask: Is every complete local ring a ring of formal power series over a field? This section deals with the answer to the above question. The results and the rough sketch of the proof are due to Cohen [6] and the last theorem dealing with Eisenstein extension is due to Nagata [19]. Other proofs using homological methods are available.

If R is a complete local ring with M as its maximal ideal, then there exists a subring K called the coefficient ring of R enjoying the following properties: if the characteristic of the quotient field R/M is $p \neq 0$, then the ideal $M \cap K$ is generated by p-fold identity. K is a complete local ring and R/M is isomorphic to $K/(M \cap K)$. If $\{x_\lambda\}$ forms a basis for M, then R is isomorphic to $K[[x_\lambda]]$.

There are three possible cases. The two equicharacteristic cases (that is when the characteristics of R and R/M are equal) and one unequal characteristic case when the characteristic of R is zero and that of R/M is different from zero. In the case when R and R/M are of characteristic zero, we use Zorn's lemma to get a maximal subfield K of R that is isomorphic to R/M. if $u_1, u_2, u_3, \ldots, u_n$ is a system of parameters of R, then $S = K[[u_1, u_2, \ldots, u_n]]$ which is the completion of $K[u_1, \ldots, u_n]$ under the topology induced by the maximal ideal $(u_1, \ldots, u_n)$ is a regular local ring and R is a finite S-module.

If the characteristic of R and R/M are both equal to $p \neq 0$, then we have two cases arising according as R/M is a perfect or an imperfect field. In case R/M is perfect, there is a unique field K in R formed by the multiplicative representatives of R/M in R where by a multiplicative representative of α in R/M we mean an element a in R such that $a \equiv \alpha \pmod{M}$ and a has p^r-th root in R for every positive integer r. In case $L = R/M$ is not perfect, we have a set of elements $\Gamma = \{\gamma_i \text{ in } R/M\}$ with

(i) $(L^p(\gamma_1, \ldots, \gamma_r) : L^p) = p^r$ for any r distinct elements of Γ.
(ii) $L = L^p(\Gamma)$.

We call such a set a p-basis for L. The existence of such a p-basis can be established. Let c_r in R be a preimage of γ_r. Then we extend R to a complete local ring S which is unramified with respect to R and having a perfect residue class field. Here by unramified we mean that if there exists a minimal basis for the maximal ideal M of R, then it is also a minimal basis for the maximal ideal $\mathfrak{M}$ of S, and for every positive integer r, $M^r = R \cap \mathfrak{M}^r$.

Once we prove the existence of the multiplicative representatives, then as in the perfect residue class field case we can show that $R = K[[x_1, \ldots, x_n]]$ or R is a finite module over $K[[x_1, x_2, \ldots, x_n]]$.

In the unequal characteristic case R contains a complete, discrete unramified valuation ring of characteristic zero called a V-ring, with a residue class field of finite characteristic such that if the maximal ideal M of R has a minimal basis of n-elements, then R is homomorphic to a ring of formal power series in n variables (or $(n-1)$ in case p is not in M^2) over a V-ring whose residue class field is R/M. Here again we distinguish between perfect and imperfect residue class fields. We use the fact that to any given field k of finite characteristic p there can be found a field K of characteristic zero, with a valuation v with respect to which K is complete and having k as its quotient field.

The following result is due to Nagata [19] for which we need the definition of an Eisenstein extension of a ring and this we formulate first.

By an Eisenstein polynomial over a local ring R with the unique maximal ideal M we mean a monic polynomial in one indeterminate, all of whose coefficients belong to M and the constant term does not belong to M^2. If $f(x)$ is an Eisenstein polynomial over R, then $R[x]/f(x)$ is called an Eisenstein extension of R.

Every complete regular local ring (R, M) is an Eisenstein extension of a complete unramified regular local ring (R_0, M_0) where (R_0, M_0) is necessarily the power series ring in a finite number of analytically independent elements over a coefficient ring of R_0. Furthermore, Eisenstein extension of a regular local ring is again a regular local ring.

§5. The properties of formal power series find application in the study of local properties of algebraic curves and surfaces like singularities, multiplicities, etc.

Let us consider the projective plane S with a definite coordinate system over an algebraically closed field K of characteristic zero and $f(X, Y, Z)$ be an irreducible homogeneous polynomial of degree $n>0$ over K. Then the set of points $P(=(x, y, z))$ of S which satisfy the equation $f(X, Y, Z)=0$ is called an irreducible algebraic curve. It is not hard to check that the irreducible algebraic curve is independent of the choice of the coordinate system. For studying the local properties, we consider the set of all parametrizations of the algebraic curve, which consists of all points $(\bar{X}, \bar{Y}, \bar{Z})$ where $\bar{X}$, etc., are elements of the field of formal power series over K in one variable t (say). If $(\bar{X}, \bar{Y}, \bar{Z})$ is any parametrization of the algebraic curve and $\bar{X}(0)=x$, $\bar{Y}(0)=y$, and $\bar{Z}(0)=z$, then (x, y, z) is said to be the center of the parametrization. We say two parametrizations are equivalent, if we can pass from one to the other by a substitution of power series having no constant term and in this situation the centres of these parametrizations are the same. An equivalence class of irreducible parametrizations is called a place and the common center of these parametrizations is called the center of the place. It is quite easy to observe that the center of any place of an algebraic curve C is a point of C while each point of C is the center of at least one place of C. The proof of the last statement involves the algebraic closure of the field of fractional power series which can be viewed as the inductive limit of $K((X^{1/n}))$—the field of formal power series in $X^{1/n}$ over K—as $n\to\infty$. Using the notions of places and parametrization we can give a meaningful definition of the number of intersection of two algebraic curves of orders m and n and can show that if two curves have no common components then they have exactly $m\cdot n$ intersections (for details see Walker [24]). We define the multiplicity of two (affine) algebraic curves f and g having a common intersection P as the sum of the orders of $f(g)$ at the places of $g(f)$ with centers at P.

For defining multiplicity for higher dimensional algebraic varieties we need the structure theory of complete local domains discussed in the previous section. First, we define the multiplicity of a complete local domain that contains a field as follows: Let D be a complete local domain, and K be any basic field of D and $x_1, \ldots, x_r$ be a system of parameters. If R denotes the ring of formal power series $K[[x_1, \ldots, x_n]]$ and e is the integer such that

$$[D\colon R] = [D/M\colon K]\cdot e,$$

where M is the unique maximal ideal of D, then e is the multiplicity of D with respect to the system of parameters $x_1, \ldots, x_r$. In case D is a regular local domain we get $e=1$ from the structure theorem for local rings.

If we have a complete local ring, we call it equidimensional if all the prime divisors of the zero ideal in D have the same dimension as D itself. If $\mathfrak{q}_1 \cap \mathfrak{q}_2 \cap\cdots\cap \mathfrak{q}_h=(0)$ is an irredundant representation of (0) as the intersection of primary ideals and $P_1, \ldots, P_m$ are the associated prime ideals then we define the multiplicity of D as follows: Let l_t be the length of $\mathfrak{q}_t (1\le t\le h)$ and $x_1, \ldots, x_n$ be a system of parameters for D. We denote by $x_{1,t}, \ldots, x_{n,t}$ the residue classes of $x_1, \ldots, x_n$ module P_t. Then $\{x_{i,t}\}$ will form a system of parameters for D/P_t and for this we have the notion of multiplicity defined above (as D/P_t is a domain). If e_t is the multiplicity of D/P_t with respect to $x_{1,t}, \ldots, x_{n,t}$ as parameters then

$$e = \sum_{t=1}^{h} e_t l_t$$

is called the multiplicity of D with respect to $x_1, \ldots, x_n$ as parameters (see [5]).

Let P be a point on an algebraic variety V in the affine space of dimension n over a field K and $(a_1, \ldots, a_n)$ be the coordinates of P. Consider the ring

$$[K[x_1, \ldots, x_n]/(x_1-a_1, \ldots, x_n-a_n)]$$

and its completion $K[[x_1-a_1, x_2-a_2, \ldots, x_n-a_n]]$. The couple composed of the n-dimensional local space $E^n(\bar{x})$ associated with the power series ring $K[[\bar{x}_1, \ldots, \bar{x}_n]]$ and the isomorphism j_p defined by $j_p(x_i)=\bar{x}_i+a_i$ is called the local space attached to the point P of the affine space A_n. If M is the ideal of V and $\overline{M}_1, \ldots, \overline{M}_g$ are the prime ideal of $j_p(M)\cdot K[[\bar{x}_1, \ldots, \bar{x}_n]]$, then the algebroid varieties in the local space attached to P corresponding to these ideals $\overline{M}_1, \ldots, \overline{M}_g$ are called the sheets of V at P. We have already seen that the dimension of each $\overline{M}_i$ is the same as the dimension of M. Thus each point of V has a certain number of sheets of same dimension which is equal to the dimension of V lying over it. In case P is a simple point there is only one sheet of V at P. We observe that the sheets at a point of a variety correspond to places on a curve.

Other notions of multiplicity are due to Samuel and Nagata. For the study of algebroid curves and varieties, the ring of formal power series over a field plays the role that the ring of polynomials play for the algebraic varieties.

By an algebroid curve C over an algebraically closed field K we mean a local ring of the form $k[[x, y]]/f(x, y)$ where x and y are indeterminates and $f(x, y)$ is a nonunit of $k[[x, y]]$ and is devoid of multiple factors. $f(x, y)=0$ is said to be the equation of the algebroid curve. If $F(x, y)$ is irreducible then C is said to be irreducible algebroid curve. Zariski [37] has recently initiated a study of the singularities of algebroid plane curves.

§6. A field K is said to be a $C_i(d)$ if every form in K in n-variables of degree d where $n > d^i$ has a nontrivial zero in K. Here n, d are integers and i is a real number. K is said to be a C_i if it is a $C_i(d)$ for all d. The infimum of the real numbers i for which K is a C_i is called the diophantine dimension (d.d. for short) of K. It can be shown that the d.d. is an integer. Clearly the d.d. of an algebraically closed field is 0. Fields whose d.d. is 1 are called quasi-algebraically closed and they are studied in [17]. There it is also shown that if the d.d. of a field is n, then the d.d. of $K((t))$—the field of formal power series over K in one variable is $(n+1)$. Consequently we see that d.d. of a field of formal power series is ≥ 1. Since no quasi-algebraically closed field has any finite division algebras over it, $k((t))$ has no finite division algebras over it when k is algebraically closed.

In [17] Lang raised the problem. If a valuated field K which is complete under a discrete valuation has a C_i-residue field $\bar{K}$, then is $\bar{K}$ a C_{i+1}? In case the characteristic of $\bar{K}$ is zero the answer is yes, since $K((t))$—the field of formal power series satisfies the requirements. Lang himself answered the question in the affirmative when $\bar{K}$ is algebraically closed. Recently Terjanian [34] has disproved Lang's conjecture by showing that a p-adic field Q_p is not C_2.

Results involving diophantine problems, but with logical bias may be found in a series of articles by Ax and Kochin [1].

We call a nondiscrete locally compact field K a p-field, where p is a prime, if $\mathrm{mod}_k\,(p\cdot 1_k)<1$ where $\mathrm{mod}_k\,(p\cdot 1_k)$ is the module of the automorphism p of K and this is the constant factor by which the unique Haar measure of the locally compact group K is transformed by the automorphism. A commutative p-field of finite characteristic is isomorphic to a field of formal power series in one indeterminate over a finite field (see Weil [36]). These are local fields and their diophantine dimension is 2. However, the diophantine dimension does not characterize commutative p-fields of finite characteristic.

Let R be a complete local Noetherian domain and G be a finite group of R-automorphisms of $R[[x]]$. Then there exists a formal power series $f(x)$ in $R[[x]]$ such that

$$R[[x]]^G = \{g(x) \in R[[x]] \mid \sigma(g(x)) = g(x) \text{ for all } \sigma \in G\}$$

is $R[[f]]$ (Samuel [26]). This leads to a problem of investigating the relationship between G and f. Thus for instance, if G is finite cyclic will f be a power of an irreducible power series?

The question of the automorphisms of a power series ring on a ring R that leave the elements of R fixed has also been considered. In case R is a field K then substitution by any power series $\sum a_i x^i$ for x gives rise to an automorphism if and only if $a_0=0$ and $a_1\neq 0$. In case of arbitrary commutative ring we need some notion of convergence in the ring of coefficients. O'Malley [21] has given a characterization of all the R-automorphisms of $R[[x]]$ under some restrictions on R. The following are his results. Let $f=\sum_{i=0}^{\infty} a_i x^i$, $a_i \in R$. Let a_0 be not a unit in R and R be complete under the (a_0)-adic topology. Then we can associate with f in a unique manner an R-endomorphism ψ_f of $R[[x]]$ such that $\psi_f(x)=f(x)$. Thus ψ_f is an R-automorphism (and indeed a substitution map) if and only if a_1 is a unit. If σ is an R-automorphism of $R[[x]]$ with $\sigma(x)=f(x)=\sum a_i x^i$ with $\bigcap_n (a_0^n)=(0)$, then R is complete under the (a_0)-adic topology and $\sigma=\psi_f$. The question that if σ is an R-automorphism of $R[[x]]$ with $\sigma(x)=f(x)$ must $\bigcap_n (f(0)^n)=(0)$ was answered by Gilmer [13] in the affirmative if R is Noetherian, or if $f(0)$ is regular and in the negative in the general case. Thus, in short O'Malley has given a complete characterization of all the R-automorphisms of $R[[x]]$ in case R is Noetherian or, an integral domain, or a ring in which $\bigcap_n (a_0^n)=(0)$ for any a_0 in the Jacobson radical. In [30] some of the results are extended to restricted power series ring.

Let R be a Krull domain and $C(R)$ denote its ideal class group. Then it is known [21], [25] that there is an injection from $C(R)$ to $C(R[[x_1,\ldots,x_n]])$, and that mapping is surjective if only R is a regular unique factorization domain. It is also known that the mapping is bijective even if R is a regular Noetherian domain.

Recently Koch [Math. Nachrichten **35** (1967), 323–327] has investigated the structure of the Galois group of the maximal separable extension of a power series field in one variable over a field of characteristic p. We quote his result here. Let k_0 be a finite field with $q=p^f$ elements and $k=k_0((t))$. Denote by K the maximal separable extension of k (the separable closure of k in its algebraic closure), and by K_0 the maximal tamely ramified extension of k. If $G(K \mid K_0)$ denotes the Galois group of the first entry over the second then

(a) $G(K \mid k)$ is the semidirect product of $G(K_0 \mid k)$ with $G(K \mid K_0)$

(b) $G(K_0 \mid k)$ is a profinite group with two generators σ, τ with a single defining relation $\sigma^{-1}\tau\sigma=\tau^p$ and

(c) $G(K \mid K_0)$ is as $G(K_0 \mid k)$ operator group, a free operator, p-group with countably many generators.

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