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SL_n, Orthogonality Relations and Transfer

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Abstract. Let π be a square integrable representation of $G' = \mathrm{SL}_n(D)$, with D a central division algebra of finite dimension over a local field F *of non-zero characteristic*. We prove that, on the elliptic set, the character of π equals the complex conjugate of the orbital integral of one of the pseudocoefficients of π. We prove also the orthogonality relations for characters of square integrable representations of G'. We prove the stable transfer of orbital integrals between $\mathrm{SL}_n(F)$ and its inner forms.

1 Introduction

Let F be a local field of non-zero characteristic and D a central division algebra of finite dimension over F. Let G' be the group $\mathrm{SL}_n(D)$. If π is a square integrable representation of G', we show that the well-known (in zero characteristic, [Cl]) relation between the character of π and the orbital integral of one of its pseudocoefficients holds for G'. Since Lemaire [Le2] proved the local integrability of characters for representations of $\mathrm{SL}_n(D)$, the orthogonality relations for characters follows.

The idea of the proof is the same we used in [Ba1] to prove the same result for $\mathrm{GL}_n(F)$. It uses basically two ingredients: the close fields theory *à la* Kazhdan [Ka] for $\mathrm{SL}_n(D)$ and a result about the lifting of orbital integrals on $\mathrm{GL}_n(F)$ by Lemaire. Here we show that our construction [Ba2] of the close fields theory for $\mathrm{GL}_n(D)$ easily implies the construction of the close fields theory for $\mathrm{SL}_n(D)$, and the result of Lemaire implies an analogous result for $\mathrm{SL}_n(D)$. First we work under the conditions $D = F$ and the characteristic of F does not divide n. We remove these two conditions later.

In the end we remark that our formula relating orbital integrals on SL_n with orbital integrals on GL_n implies the transfer of stable orbital integrals (see [LL, Sh] for SL_2) in all characteristics.

2 $\mathrm{GL}_n(D)$ and Hecke algebras

Let F be a non-archimedean local field, o its ring of integers and i the maximal ideal of o. Let q be the cardinal of the residual field o/i. Let D be a central division algebra of dimension d^2 over F. Let O be the ring of integers of D and I the maximal ideal of O. Let π be a uniformizer for D. Set $G = \mathrm{GL}_n(D)$. Set $K_0 = \mathrm{GL}_n(O)$ and, for all $j \in \mathbb{N}^*$, $K_j = 1 + M_n(I^{dj})$. Let H (or $H(G)$, if more than one group are involved) be the convolution algebra of locally constant functions on G with compact support. For each j, let H_j be the sub-algebra of H formed by the K_j bi-invariant functions. H_j will be called the *Hecke algebra of level* j. Let Z be the center of G. The way of

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defining a characteristic polynomial for elements in G may be found in [Pi, 16.1]. If $g \in G$, g is called *regular semisimple* if its characteristic polynomial has distinct roots in an algebraic closure of F. It is called *elliptic* if in addition its characteristic polynomial is irreducible (some authors then call it "regular elliptic").

Recall the Cartan decomposition. Let $\mathcal{A}$ be the set of matrices $(a_{i,j})_{1\leq i,j\leq n}$ such that $a_{i,j} = \delta_{i,j}\pi^{a_i}$ where $\delta_{i,j}$ is the Kronecker symbol and $a_1 \leq a_2 \leq \cdots \leq a_n$. Then we have

$$G = \coprod_{A\in\mathcal{A}} K_0 A K_0.$$

So the characteristic functions of the sets K_0AK_0 form a basis of H_0 when A lies in $\mathcal{A}$. If $j \in \mathbb{N}$, then K_j is a normal subgroup of K_0. The kernel of the natural projection from K_0 onto $\mathrm{GL}_n(O/I^j)$ is K_j, so there is a canonical isomorphism $K_0/K_j \simeq \mathrm{GL}_n(O/I^j)$. Hence we will identify these two groups. In particular we write:

$$K_0 = \coprod_{B\in \mathrm{GL}_n(O/I^j)} K_j B = \coprod_{B\in \mathrm{GL}_n(O/I^j)} BK_j.$$

Now set $T_j = \mathrm{GL}_n(O/I^j) \times \mathrm{GL}_n(O/I^j)$. The Cartan decomposition may then be written:

$$G = \coprod_{A\in\mathcal{A}} \bigcup_{(B,C)\in T_j} K_j BAC^{-1} K_j.$$

It is not a disjoint union. However, two sets in the union are either equal or disjoint. Let X_A be the subgroup of $\mathrm{GL}_n(O) \times \mathrm{GL}_n(O)$ made of couples (B, C) such that $BAC^{-1} = A$. Let $X_{A,j}$ be the image of X_A in T_j. Then we have $K_jBAC^{-1}K_j = K_jbAc^{-1}K_j$ if and only if $(b^{-1}B, c^{-1}C) \in X_{A,j}$. So, the set $K_jBAC^{-1}K_j$ is well defined for $(B, C) \in T_j/X_{A,j}$, and we have

$$G = \coprod_{A\in\mathcal{A}} \coprod_{(B,C)\in T_j/X_{A,j}} K_j BAC^{-1} K_j.$$

So the set of characteristic functions of sets $K_jBAC^{-1}K_j$ is a basis of H_j when A lies in $\mathcal{A}$, and for every such A, (B, C) lies in $T_j/X_{A,j}$. See [Ba2] for details.

3 $\mathrm{GL}_n(D)$ and Close Fields

Now suppose L is another non-archimedian local field. All the objects we described before are defined for L too, and in the following they will take an index F or L to specify the field to which they are attached. Suppose that there is an isomorphism $\lambda_j : o_F/i_F^j \simeq o_L/i_L^j$ for some positive integer j. We say then that the fields F and L are *j-close*. If D_L is the central division algebra of dimension d^2 over L with the same Hasse invariant as D_F, then λ_j induces an isomorphism $O_F/I_F^{dj} \simeq O_L/I_L^{dj}$, which we still denote by λ_j. Fix a uniformizer π_L of D_L such that the image by λ_j of the class of π_L is the class of π_F. The set $\mathcal{A}_L$ is defined with respect to this choice, and we get a natural bijection, still denoted λ_j, from $\mathcal{A}_F$ onto $\mathcal{A}_L$. It is clear that the isomorphism $\lambda_j : O_F/I_F^{dj} \simeq O_L/I_L^{dj}$ induces an isomorphism $\lambda_j : T_{j,F} \simeq T_{j,L}$. One may prove

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that the restriction of this isomorphism induces, for every $A \in \mathcal{A}_F$, an isomorphism between the subgroups $X_{A,j,F}$ and $X_{\lambda_j(A),j,L}$. So we get a natural bijection between the basis of $H_{j,F}$ and $H_{j,L}$ which defines an isomorphism λ_j between these two vector spaces.

One may show that if $l \leq j$, λ_j induces an isomorphism between o_F/i_F^l and o_L/i_L^l, then the fields F and L are also l-close. If we use this isomorphism and the same choice of uniformizer for D_F and D_L, then the isomorphism $\lambda_l : H_{l,F} \simeq H_{l,L}$ obtained is induced by the restriction of the isomorphism $\lambda_j : H_{j,F} \simeq H_{j,L}$. If K is a compact subset of G_F bi-invariant by $K_{j,F}$, its characteristic function is an element of $H_{j,F}$, and the image by λ_j of this function in $H_{j,L}$ is the characteristic function of an open compact set denoted $\lambda_j(K)$. Fix a Haar measure on G_F (resp., G_L) such that the volume of the subgroup $K_{0,F}$ (resp., $K_{0,L}$) is 1. Then the volume of $\lambda_j(K)$ equals the volume of K. All these results are proved in [Ba2].

4 $\mathrm{SL}_n(D)$ and Hecke Algebras

We forget L for a moment and we turn back to our F, D and the construction of the beginning. Let G' be the subgroup $\mathrm{SL}_n(D)$ of G. For all positive integers j, set $K'_j = K_j \cap G'$. The K'_j make up a basis of open compact neighborhood of 1 in G'. Let H' (or $H'(G')$ when more than one group are involved) be the algebra of convolution of locally constant function on G' with compact support. Let H'_j be the Hecke algebra of level j of G' made by K'_j-bi-invariant functions on G' which have compact support. Set $\mathcal{A}' = \mathcal{A} \cap G'$. The kernel of the natural projection from K'_0 onto $\mathrm{SL}_n(O/I^j)$ is K'_j, so there is a canonical isomorphism $K'_0/K'_j \simeq \mathrm{SL}_n(O/I^j)$, and we will identify these two groups. Now let T'_j be the subgroup $\mathrm{SL}_n(O/I^j) \times \mathrm{SL}_n(O/I^j)$ of T_j. For each $A \in \mathcal{A}'$, set $X'_{A,j} = X_{A,j} \cap T'_j$. Let $Z' = Z \cap G'$ be the center of G'.

Proposition 4.1 *For every $(B,C) \in T'_j/X'_{A,j}$, $K'_jBAC^{-1}K'_j$ is well defined and we have*

$$G' = \coprod_{A \in \mathcal{A}'} \coprod_{(B,C) \in T'_j/X'_{A,j}} K'_jBAC^{-1}K'_j.$$

Proof We use the Cartan decomposition

$$G' = \coprod_{A \in \mathcal{A}'} K'_0AK'_0.$$

As

$$K'_0 = \coprod_{B \in K'_0/K'_j} K'_jB = \coprod_{B \in K'_0/K'_j} BK'_j$$

and $K'_0/K'_j \simeq \mathrm{SL}_n(O/I^j)$, we have

$$G' = \coprod_{A \in \mathcal{A}'} \bigcup_{(B,C) \in T'_j} K'_jBAC^{-1}K'_j.$$

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Now suppose that $K'_jBAC^{-1}K'_j = K'_jbAc^{-1}K'_j$ for some (B, C) and (b, c) in T'_j. If we consider (B, C) and (b, c) as elements of T_j, then in G we must have $K_jBAC^{-1}K_j = K_jbAc^{-1}K_j$, because these two sets have non-void intersection. So we know that $(b^{-1}B, c^{-1}C) \in X_{A,j}$. As $(b^{-1}B, c^{-1}C)$ is an element of T'_j, we must then have $(b^{-1}B, c^{-1}C) \in X'_{A,j}$. The converse is also true: if $(b^{-1}B, c^{-1}C) \in X'_{A,j}$, then

$$K'_jBAC^{-1}K'_j = K'_jbAc^{-1}K'_j.$$

(It suffices to consider a representative of $(b^{-1}B, c^{-1}C)$ in X_A.) ■

Choose a Haar measure on G' such that the volume of K'_0 is 1.

Lemma 4.2 *If $A \in G$, then for every $j \in \mathbb{N}$, we have*

$$\operatorname{card}(K'_j/(AK'_jA^{-1} \cap K'_j)) = \operatorname{card}(K_j/(AK_jA^{-1} \cap K_j)).$$

As $G = K_0\mathcal{A}K_0$ and K_0 normalizes K_j and K'_j, it suffices to prove the lemma for $A \in \mathcal{A}$. Write

$$K_j = \coprod_{i=1}^{l} k_i(AK_jA^{-1} \cap K_j).$$

First, suppose that $D = F$. If $A \in \mathcal{A}$, then the diagonal matrix with 1 on the first $n - 1$ positions and $\det(k_i)^{-1}$ on the last is always in $AK_jA^{-1} \cap K_j$, so we may and will assume that $k_i \in G'$ for all i. Then

$$K'_j = K_j \cap G' = \coprod_{i=1}^{l} \big(k_i(AK_jA^{-1} \cap K_j) \cap G'\big) = \coprod_{i=1}^{l} \big(k_i(AK_jA^{-1} \cap K_j \cap G')\big)$$

because $k_i \in G'$. But G' is a normal subgroup of G, so

$$AK_jA^{-1} \cap K_j \cap G' = (A(K_j \cap G')A^{-1}) \cap (K_j \cap G') = AK'_jA^{-1} \cap K'_j,$$

and we have proved that

$$K'_j = \coprod_{i=1}^{l} k_i(AK'_jA^{-1} \cap K'_j),$$

hence the equality for cardinals.

Suppose now that $D \neq F$. We want to do the same and to find a diagonal matrix in K_j whose determinant is $\det(k_i)^{-1}$. As it is diagonal, it will be in $AK_jA^{-1} \cap K_j$, and the proof will be the same after. First, with elementary operations on lines of k_i^{-1}, we obtain by a standard algorithm a triangular matrix in K_j with the same determinant $\det(k_i)^{-1}$. Now, if in this triangular matrix we keep only the diagonal and put zero for all the other entries, we obtain a diagonal matrix with the same determinant (one has to apply [We, Corollary 2, p. 169] on reduced norms). ■

Lemma 4.3 *Let $j \geq 1$ and $A \in \mathcal{A}$, and let $a_1 \leq a_2 \leq \cdots \leq a_n$ be the powers of the uniformizer on the diagonal of A. Then*

$$\mathrm{vol}(K_j' A K_j') = q^{d \sum_{1\leq i<i'\leq n} a_{i'} - a_i} \mathrm{vol}(K_j').$$

Proof Using the last lemma, it follows from [Ba2, proof of Lemma 2.10]. ∎

Remark The volumes of K_0 and K_0' are one and, for $j \geq 1$, $K_0/K_j \simeq \mathrm{GL}_n(O/I^{dj})$ and $K_0'/K_j' \simeq \mathrm{SL}_n(O/I^{dj})$. The determinant is a surjective map $\mathrm{GL}_n(O/I^{dj})$ to $\mathrm{GL}_1(O/I^{dj})$ with kernel $\mathrm{SL}_n(O/I^{dj})$. So we have

$$\mathrm{vol}(K_j) = \mathrm{card}(\mathrm{GL}_1(O/I^{dj}))\,\mathrm{vol}(K_j') = (q^d - 1)q^{d^2 j - d}\,\mathrm{vol}(K_j').$$

Proposition 4.4 *For every $a \in G$, the automorphism $f_a\colon x \mapsto axa^{-1}$ of G' is measure preserving.*

Proof Let us show that $\mathrm{vol}(aK_0'a^{-1}) = 1$. Applying Lemma 4.2 to a and a^{-1} we get

$$\mathrm{card}(K_0'/aK_0'a^{-1} \cap K_0') = \mathrm{card}(K_0/aK_0a^{-1} \cap K_0),$$
$$\mathrm{card}(K_0'/a^{-1}K_0'a \cap K_0') = \mathrm{card}(K_0/a^{-1}K_0a \cap K_0).$$

On the other hand, $\mathrm{card}(K_0/aK_0a^{-1} \cap K_0) = \mathrm{card}(K_0/a^{-1}K_0a \cap K_0)$, because the (finite) cardinals are quotients of volumes, and conjugation with a^{-1} in G (to pass from $aK_0a^{-1} \cap K_0$ to $a^{-1}K_0a \cap K_0$) is measure-preserving with respect to a Haar measure. We also have $\mathrm{card}(K_0'/a^{-1}K_0'a \cap K_0') = \mathrm{card}(aK_0'a^{-1}/aK_0'a^{-1} \cap K_0')$, because conjugation with a induces an isomorphism between these two groups. The result follows. ∎

If $g \in G'$, set $h(g) = (\mathrm{vol}(K_j')^{-1})1_{K_j' g K_j'}$.

Lemma 4.5

(i) *If $A, A' \in \mathcal{A}'$, then $h(A) * h(A') = h(AA')$.*
(ii) *If $(B, C) \in T_0'$, then $h(B) * h(A) * h(C) = h(BAC)$.*

The proof is exactly like that for [Ba2, Lemma 2.11].

We remark that for every function $f \in H_j$, the restriction of f to G' belongs to H_j'. This restriction commutes with the inclusions $H_j \subset H_i$ and $H_j' \subset H_i'$ for $i \geq j$. Conversely, every function $f' \in H_j'$ can be lifted in a standard way to a function $f \in H_j$, using the natural inclusion of the standard basis of H_j' into the standard basis of H_j. But this operation no longer commutes with the inclusions between Hecke algebras. The restriction and the lifting will be used more than once in the following.

5 $\mathrm{SL}_n(D)$ and Close Fields

Let us consider again the situation of the two j-close fields, F and L, and all the other constructions from Section 3. Embody in the situation the groups $G'_F(=\mathrm{SL}_n(D_F))$ and $G'_L(=\mathrm{SL}_n(D_L))$. The bijection $\lambda_j\colon \mathcal{A}_F \to \mathcal{A}_L$ induces a bijection $\lambda'_j\colon \mathcal{A}'_F \to \mathcal{A}'_L$, and the isomorphism $\lambda_j\colon T_{j,F} \to T_{j,L}$ induces an isomorphism $\lambda'_j\colon T'_{j,F} \to T'_{j,L}$. As a consequence, the isomorphism $\lambda_j\colon H_{j,A,F} \to H_{j,\lambda_j(A),L}$ induces an isomorphism $\lambda'_j\colon H'_{j,A,F} \to H'_{j,\lambda_j(A),L}$. (This last result in the case of GL_n [Ba2, Lemma 2.7] needed some painful calculations in the first part of [Ba2], and to avoid recalling all the notations, we choose to get it here by this embedding of G' in G). We then obtain an isomorphism λ'_j of vector spaces from $H'_{j,F}$ to $H'_{j,L}$. We recall that if m is an integer greater than j, if F and L are m-close, then F and L are also j-close.

Theorem 5.1 *There exists an integer $m \geq j$ such that if F and L are m-close, then the isomorphism λ'_j is an isomorphism of (Hecke) algebras.*

We need a lemma before we can prove the theorem.

Lemma 5.2 *Let $\mathcal{C}$ be a finite subset of $\mathcal{A}'_F$, and set*

$$G'_F(\mathcal{C}) = \bigcap_{A\in\mathcal{C}} K'_{0,F} A K'_{0,F}.$$

Then

(i) *There exist $m \geq j$ depending on $\mathcal{C}$ such that, for all $g \in G'_F(\mathcal{C})$, we have*

$$gK'_{m,F}g^{-1} \subset K'_{j,F}.$$

(ii) *If L is m-close to F, then for all $f_1, f_2 \in H'_{j,F}$ supported on $G'_F(\mathcal{C})$ we have $\lambda'_j(f_1 * f_2) = \lambda'_j(f_1) * \lambda'_j(f_2)$.*

Proof This lemma is analogous to $G' = \mathrm{SL}_n$ of [Ba2, Lemma 2.14] for the group $G = \mathrm{GL}_n$. The point (i) here follows obviously "by intersection with G'" from the point (a) there. The point (ii) is then proven exactly like the point (b) of [Ba2, Lemma 2.14]. ■

Proof of Theorem 5.1 It goes exactly like the proof of [Ba2, Theorem 2.13]. ■

6 Hecke Algebras and Representations

We forget the close fields for a moment and turn back to notations in Section 4. Let (π, V) be an irreducible smooth representation of G'. If K is a subgroup of G', let V^K be the subspace of vectors which are fixed under $\pi(k)$ for all $k \in K$. If K is open, V^K has finite dimension. The *level* of π is the lowest integer l such that $V^{K_l} \neq 0$. If $f \in H'_j$, we set $\pi(f) = \int_{G'} f(g)\pi(g)\,dg$. The image of $\pi(f)$ is then included in $V^{K'_j}$. In particular, if j is less than l, then $\pi(f) = 0$. If j is greater than or equal to l, then $\pi(f)$ induces an endomorphism of $V^{K'_j}$. It is also clear that the trace of $\pi(f)$ equals the trace of this endomorphism. The space $V^{K'_j}$ is an H'_j-module with the external low: $f * v = \pi(f)v$ for all $f \in H'_j$ and all $v \in V^{K'_j}$. To any irreducible smooth representation π of level less than or equal to j we associate this way an H'_j-module. This construction gives a bijection from the set of equivalence classes of irreducible smooth representations of G' with level less than or equal to j and the set of isomorphism classes of irreducible non-degenerated H'_j-modules, (see [Be] for example).

7 Close Fields and Representations

Let F, L, j and m be as in Theorem 5.1; in view of what has been said in the last section, λ'_j induces a bijection between the set of equivalence classes of irreducible smooth representations of G'_F with level less than or equal to j and the set of equivalence classes of irreducible smooth representations of G'_L with level less than or equal to j. As the maps λ'_i for $i \leq j$ are compatible with the inclusions' relations between Hecke algebras, we see that λ'_j is level preserving. Also, if $f \in H'_{j,F}$ and π is an irreducible smooth representation of level less than or equal to j of G'_F, we have obviously $\operatorname{tr}\pi(f) = \operatorname{tr}\lambda'_j(\pi)(\lambda'_j(f))$.

Proposition 7.1 *Let π be an irreducible smooth representation of G'_F of level less than or equal to j. Then π is square integrable if and only if $\lambda'_j(\pi)$ is. Thus, π is tempered if and only if $\lambda'_j(\pi)$ is.*

Proof For square integrable representations, the proof is the same as for [Ba2, Theorem 2.17]. Now, the tempered representations of G'_F are its irreducible unitary representations π such that for all $\epsilon > 0$, there exists a non-trivial coefficient of π belonging to $L^{2+\epsilon}(G'_F)$, and the same for G'_L. The same proof as for square integrable representations shows that λ'_j sends tempered representations to tempered representations. ■

Corollary 7.2 *If π is a square integrable representation of G'_F of level less than or equal to j and f is a pseudocoefficient of π, then $\lambda'_j(f)$ is a pseudocoefficient of $\lambda'_j(\pi)$.*

Proof The corollary is an easy consequence of the above proposition. See [Ba1, Lemma 4.2], (as well as [Ba1, §2] for a definition and a survey of pseudocoefficients in all characteristic). ■

8 Orbital Integrals

Let F be a non-archimedian local field as in Section 1. Here $D = F$. Recall that we fixed Haar measures dg and dg' on G and G' such that $\operatorname{vol}(K_0, dg) = 1$ and $\operatorname{vol}(K'_0, dg') = 1$. If γ is a regular semisimple element of G and $Z_G(\gamma)$ is the stabilizer of γ in G, we put a Haar measure on $Z_G(\gamma)$ such that the volume of the subgroup of its points over O is one. On $G/Z_G(\gamma)$ we put the quotient measure. The same if $\gamma \in G'$ and we consider its commutator $Z_{G'}(\gamma) = Z_G(\gamma) \cap G'$ in G'. The orbital

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integrals $\Phi(f,\cdot)$ of functions $f \in H$ or $f \in H'$ at the point γ will be calculated with respect to these choices of measures.

Let us fix the following notations: if A is a subset of F, $A^{[n]}$ is the set of all n-th powers of elements of A in F. If A is a subset of G, then $\det(A)$ is the image in F of A under the determinant map. If A and B are subsets of G, then AB is the set of all products ab with $a \in A$ and $b \in B$.

From here to the end of the section we suppose that the characteristic of F is either zero or prime with n.

Lemma 8.1 *We have* $1 + I^{2n} \subset O^{*[n]}$.

Proof If the characteristic of F is zero, the lemma is an obvious consequence of [BS, Exercise 2, p. 46] (which is an easy application of [BS, Theorem 3, p.42]). In our opinion, there is a mistake in the statement of the exercise, and one has to replace $2^{\delta} + 1$ by $2\delta + 1$, which is stronger and comes straight from the standard proof. The δ in the exercise is the greatest power of p dividing n. In particular, we have $\delta < n$, so $2\delta + 1 < 2n$, so the exercise implies our statement. The same proof works in non-zero characteristic if p is prime to n. ■

Let S be a set of representatives of $O^*/1 + I^{2n}$ in O^*. Choose a subset S' of S which is a system of representatives of $O^*/O^{*[n]}$ (always possible, thanks to Lemma 8.1).

Let $X_{G'}$ be the set of diagonal matrices in G with 1 in the first $n-1$ places and an element of S' in the last one. Let X be the set of diagonal matrices in G with 1 in the first $n-1$ places and an element of $\{1, \pi, \pi^2, \dots, \pi^{n-1}\}$ in the last one.

It is clear that $F^{*[n]} = \det(Z)$, $O^{*[n]} = \det(Z(O))$ and

$$F/F^{*[n]} = \coprod_{i=0}^{n-1} \pi^i O^*/O^{*[n]}.$$

Using the natural inclusion of G' in G, we realize $x(G'/Z')$ as a subset of G/Z for all $x \in G$. It is easy to check that $G(O)/Z(O) = \coprod_{x \in X_{G'}} x(G'(O)/Z'(O))$ and $G/Z = \coprod_{x \in X_{G'}X} x(G'/Z')$.

Remark If $j \geq 2n$, the natural inclusion $K'_j/Z(K'_j) \to K_j/Z(K_j)$ is a bijection (where $Z(K'_j)$ is the center of K'_j and $Z(K_j)$ is the center of K_j).

Let $\gamma \in G'$. As $Z \subset Z_G(\gamma)$, $\det(Z) \subset \det(Z_G(\gamma))$. So we may (and will) choose a subset S'_γ of S' which forms a system of representatives for $O^*/\det(Z_G(\gamma)(O))$ in O^*. We denote X_γ the corresponding subset of $X_{G'}$. The valuation map sends the set $\det(Z_G(\gamma))$ into a subgroup W of $\mathbb{Z}$ containing $n\mathbb{Z}$. Consider a system of representatives J_γ of $\mathbb{Z}/W$ in the set $\{0, 1, 2, \dots, n-1\}$. We have

$$F^*/\det(Z_{G'}(\gamma)) = \coprod_{j \in J_\gamma} \pi^j O^*/\det(Z_G(\gamma)(O)).$$

So, $\coprod_{j\in J_\gamma} S'_\gamma \pi^j$ is a system of representatives of $F^*/\det(Z_G(\gamma))$ in F^*. Let Y_γ be the set of diagonal matrices in G with 1 in the first $n-1$ places and π^j, with $j \in J_\gamma$, in the last one.

One may show that

$$G(O)/Z_G(\gamma)(O) = \coprod_{x\in X_\gamma} x(G'(O)/Z_{G'}(\gamma)(O)),$$

$$G/Z_G(\gamma) = \coprod_{x\in X_\gamma Y_\gamma} x(G'/Z_{G'}(\gamma)).$$

Let x_γ be the cardinal of X_γ. The first relation shows that with our choice of measures, the measure we put on $G'/Z_{G'}(\gamma)$ is x_γ times the restricted measure from $G/Z_G(\gamma)$.

One may also verify that if δ is conjugated to γ in G, there exist exactly one element $x \in X_\gamma Y_\gamma$ such that δ is conjugated to $x\gamma x^{-1}$ in G'.

Let us look at this construction from another point of view. We say that U is a *system adapted to* γ if for each $\delta \in G$ conjugated to γ in G, there exist exactly one element $x \in U$ such that δ is conjugated to $x\gamma x^{-1}$ in G'. Then we have

$$G/Z_G(\gamma) = \coprod_{x\in U} x(G'/Z_{G'}(\gamma)).$$

We just proved that $X_\gamma Y_\gamma$ is a system adapted to γ. But what is remarkable from our discussion is that knowing just the set $\det(Z_{G'}(\gamma))$, we may construct a system adapted to γ and we know x_γ (which is the quotient of two cardinals: those of $F^*/\det(Z_{G'}(\gamma))$ and of $\mathbb{Z}$ by its subgroup of valuations of elements in $\det(Z_{G'}(\gamma))$). So, our previous construction allows us to construct a particular such system depending only on $O^*/1+I^{2n}$ and on the first n powers of π.

Now let U be a system adapted to γ. If we denote by $\mathcal{O}_G(\gamma)$ (resp., $\mathcal{O}_{G'}(\gamma)$) the orbit in G (resp., in G') of γ, then:

$$\mathcal{O}_G(\gamma) = \coprod_{x\in U} \mathcal{O}_{G'}(x\gamma x^{-1}).$$

If $d\bar{g}$ (resp., $d\bar{g}'$) is the measure fixed on $G/Z_G(\gamma)$ (resp., $G'/Z_{G'}(\gamma)$), then we have that for every $f \in H(G)$,

$$\begin{aligned}\Phi(f,\gamma) &= \int_{G/Z_G(\gamma)} f(g\gamma g^{-1})d\bar{g} = \sum_{x\in U}\int_{G'/Z_{G'}(\gamma)} f(xg\gamma g^{-1}x^{-1})d\bar{g}\\ &= \sum_{x\in U}\int_{G'/Z_{G'}(\gamma)} f((xgx^{-1})(x\gamma x^{-1})(xg^{-1}x^{-1}))d\bar{g}\\ &= \sum_{x\in U}\int_{G'/Z_{G'}(\gamma)} f(g(x\gamma x^{-1})g^{-1})d\bar{g},\end{aligned}$$

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the last equality coming from Proposition 4.4. So, if $f' \in H'$ is the restriction of f to G', we obtain, as $d\bar{g} = \frac{1}{x_\gamma} d\bar{g}'$,

$$\Phi(f,\gamma) = \frac{1}{x_\gamma} \sum_{x \in U} \Phi(f', x\gamma x^{-1}). \tag{8.1}$$

Recall that γ is regular semisimple. Let V_γ be an open and compact neighborhood of γ in G' containing only elements of G' which are conjugated under G' to a regular element in the torus $Z_{G'}(\gamma)$. Such a neighborhood always exists by the submersion theorem of Harish-Chandra. Using the same theorem for G, we may and will assume that an element in V_γ is conjugated in G to exactly one element of $Z_{G'}(\gamma)$. Then $Z_G(t)$ is conjugated to $Z_G(\gamma)$ in G' for all $t \in V_\gamma$. In particular, $\det(Z_G(t)) = \det(Z_G(\gamma))$ which shows that the system $X_\gamma Y_\gamma$ is adapted to t, too, and $x_t = x_\gamma$. So the formula

$$\Phi(f,t) = \frac{1}{x_\gamma} \sum_{x \in X_\gamma Y_\gamma} \Phi(f', xtx^{-1})$$

works in the whole neighborhood V_γ (actually the formula works for every regular element of $\mathrm{Ad}_G Z_G(\gamma) \cap G'$). For some precise choices when constructing X_t, we even have $X_t = X_\gamma$.

For each $x \in X_\gamma Y_\gamma$, set $V_{x\gamma x^{-1}} = xV_\gamma x^{-1}$ (it is an open and compact neighborhood of $x\gamma x^{-1}$ in G'). If $A \subset G$, let $\mathrm{Ad}_{G'}(A)$ stand for the set of all conjugates of elements in A by elements of G'. The sets $\mathrm{Ad}_{G'}(V_{x\gamma x^{-1}})$, $x \in U$, are disjoint because of the choice of V_γ. They are all open and closed also. The fact that they are open is obvious (union of open sets). Then the fact that they are closed would be a consequence of their union being closed. But their union is $\mathrm{Ad}_G(V_\gamma)$, and this is closed: if P is the (continuous) map *characteristic polynomial* from G to F^n, then $P(V_\gamma)$ is compact because V_γ is, hence the reciprocal image $P^{-1}(P(V_\gamma)) = \mathrm{Ad}_G(V_\gamma)$ is closed.

Now let $f' \in H'$. We may write $f' = f'_0 + \sum_{x \in X_\gamma Y_\gamma} f'_x$, where the support of f'_0 does not intersect any $\mathrm{Ad}_{G'}(V_{x\gamma x^{-1}})$, and the support of each f'_x is included in $\mathrm{Ad}_{G'}(V_{x\gamma x^{-1}})$. The orbital integral of f'_0 vanishes on all $xV_\gamma x^{-1}$. The orbital integral of f'_{x_0} vanishes on all $xV_\gamma x^{-1}$ with $x \in X_\gamma Y_\gamma \backslash \{x_0\}$. If $f'_1 \in H'_j$, we just lift it to a function $f_1 \in H_j$, and using the relation between orbital integrals we get

$$\Phi(f',t) = x_\gamma \Phi(f_1,t) \tag{8.2}$$

for all $t \in V_\gamma$. In particular, if $\Phi(f_1, \cdot\,)$ is constant in a neighborhood V of γ in G, then $\Phi(f', \cdot\,)$ is constant on $V_\gamma \cap V$.

9 Orbital Integrals and Close Fields

We will deal again with two different fields F and L, and the subscript F or L will indicate the one to which the object is attached. The field F is fixed. If γ is an elliptic element of G'_F, then we fix X_γ as in the previous section, and, if L is a field m close to F with $m \geq 2n$, we define $\lambda_m(X_\gamma)$ in the following way: We take the image of

X_γ in $O_F^*/1+I_F^m = (O_F/I_F^m)^*$ defined by its last coefficient on the diagonal. Then we take the image of this set under the ring isomorphism $\lambda_m\colon O_F/I_F^m \to O_L/I_L^m$. We then consider a system S_L of representatives of this set in O_L^* and finally we let $\lambda_m(X_\gamma)$ be the set of diagonal matrices in G_L with 1 in the first $n-1$ places of the diagonal and an element of S_L in the last. The set Y_γ is defined only in terms of powers of the uniformizer π_F of F, so there is a canonical way of defining the corresponding set $\lambda_m(Y_\gamma)$ using the uniformizer π_L of L. It is also clear how we define $\lambda_m(x)$ for each x in $X_\gamma Y_\gamma$. Actually, $X_\gamma \subset K_{0,F}$, and $Y_\gamma \subset \mathcal{A}_F$, so every $x \in X_\gamma Y_\gamma$ is an element of type BAC^{-1} (with $C=1$) like those used in the standard decomposition of G_F. Hence, for all $m \geq 2n$, if L is m-close to F we automatically have $\lambda_m(x) \in \lambda_m(K_m x K_m)$, so for this particular adapted system we have defined a point-wise lifting always compatible with the general lifting of open compact sets.

Theorem 9.1 (Lemaire) *Let γ be an elliptic element of G_F. Let j be a positive integer. Then there exist l and m such that*

(i) *for every $f \in H_{j,F}$, $\Phi(f,\cdot)$ is constant on $K_{l,F}\gamma K_{l,F}$, equal to $\Phi(f,\gamma)$,*
(ii) *m is greater than j and l and for every field L which is m-close to F, for every $f \in H_{j,F}$, $\Phi(\lambda_j(f),\cdot)$ is constant on $\lambda_l(K_{l,F}\gamma K_{l,F})$, equal to $\Phi(f,\gamma)$.*

Proof [Le1, p.1054]. ■

Lemma 9.2 *Let $\gamma \in G_F$ be an elliptic element and let j be a positive integer. There exist l and m such that if L and F are m close, then for all $\gamma' \in \lambda_l(K_{l,F}\gamma K_{l,F})$ we have $K_{j,L}Z_{G_L}(\gamma')K_{j,L} = \lambda_l(K_{j,F}Z_{G_F}(\gamma)K_{j,F})$.*

Proof It is shown in the first paragraphs of [Le1, proof of (i), p. 1043]. ■

Let $\gamma \in G_F'$ be an elliptic element. Apply the last lemma for a $j \geq 2n$. Then we have the following.

Proposition 9.3 *If L and F are m-close, then for all $\gamma' \in \lambda_l(K_{l,F}'\gamma K_{l,F}')$, the system $\lambda_l(X_\gamma)\lambda_l(Y_\gamma)$ is adapted to γ' and $x_{\gamma'} = x_\gamma$.*

Proof We have seen that $1+I_L^{2n} = \det(K_{2n,L}) \subset \det(Z_{G_L}(\gamma'))$ and $1+I_F^{2n} = \det(K_{2n}) \subset \det(Z_{G_F}(\gamma))$. So $\det(Z_{G_L}(\gamma')) = \det(K_{j,F}Z_{G_L}(\gamma')K_{j,F})$ and

$$\det(K_{j,F}Z_{G_F}(\gamma)K_{j,F}) = \det(Z_{G_F}(\gamma)).$$

Now, by the previous lemma we get $K_{j,L}Z_{G_L}(\gamma')K_{j,L} = \lambda_l(K_{j,F}Z_{G_F}(\gamma)K_{j,F})$. But, if V is a $K_{j,F}$ bi-invariant set, then $\det(V)$ is invariant by $1+I_F^j$, and $\det(V) \subset \mathrm{GL}_1(F)$ correspond to $\det(\lambda_j(V)) \subset \mathrm{GL}_1(L)$ by the close fields theory for GL_1 (it suffices to verify this on standard sets $K_jBAC^{-1}K_j$, and this is obvious). ■

Let γ be an elliptic element of G_F'.

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Theorem 9.4 *Let $f' \in H$. There exist p and m such that*

(i) *$\Phi(f', \cdot)$ is constant on $K'_p \gamma K'_p$, equal to $\Phi(f', \gamma)$;*
(ii) *for every field L which is m-close to F, $\Phi(\lambda_m(f'), \cdot)$ is constant on $\lambda_m(K'_p \gamma K'_p)$, equal to $\Phi(f', \gamma)$.*

We begin with a lemma studying the behavior of the lifting under conjugation. It implies, for example, that if two open compact sets A and B are conjugated, the same is true for their lifting to a field close enough. It also implies that if no element of A is conjugated with an element of B, the same is true for their lifting to a field close enough.

Lemma 9.5 *Let H_1, H_2 be open compact subsets of G_F and $g \in G_F$ such that*

$$gH_1g^{-1} \subset H_2.$$

If H_1 and H_2 are bi-invariant under some $K_{j,F}$, then $K_{j,F}gK_{j,F}H_1K_{j,F}g^{-1}K_{j,F} \subset H_2$. Moreover, there exist $m > j$ such that if L is m-close to F, then

$$\lambda_m(K_{j,F}gK_{j,F})\lambda_m(H_1)\lambda_m(K_{j,F}g^{-1}K_{j,F}) \subset \lambda_m(H_2).$$

Proof As $gH_1g^{-1} \subset H_2$ and H_1 and H_2 are bi-invariant under $K_{j,F}$, we obviously have $K_{j,F}gK_{j,F}H_1K_{j,F}g^{-1}K_{j,F} \subset H_2$. For the second assertion, it suffices to show that $\lambda_m(K_{j,F}xK_{j,F}yK_{j,F}) = \lambda_m(K_{j,F}xK_{j,F})\lambda_m(K_{j,F}yK_{j,F})$ for all $x, y \in G_F$. But $K_{j,F}xK_{j,F}yK_{j,F}$ is the support of the function obtained by the convolution product of characteristic functions $1_{K_{j,F}xK_{j,F}}$ and $1_{K_{j,F}yK_{j,F}}$. So, when m is big enough for the linear isomorphism between $H_{j,F}$ and $H_{j,L}$ to be an algebra isomorphism (Theorem 5.1), we also have our relation. ∎

Proof of Theorem 9.4 The proof of the theorem is now straightforward. Thanks to Proposition 9.3 and Lemma 9.5, if L is m-close to F, m big enough, then the construction for L at the end of the last section is parallel to that for F (just pick a γ_L in $\lambda_m(V_\gamma)$ and use Lemma 9.5 to show (for m big enough) that for all $x \in X_\gamma Y_\gamma$, $\lambda_m(V_{x\gamma x^{-1}}) = V_{\lambda_m(x)\gamma_L\lambda_m(x)^{-1}})$. To conclude (i) of our theorem, just use Theorem 9.1(ii) and relation (8.2). ∎

10 The Orthogonality Relations for Characters

If $^-$ denotes complex conjugation, we have the following.

Theorem 10.1 *Let F be a local field of non-zero characteristic p. Let n be a positive integer such that p does not divide n. Then if π is a square integrable representation of $G'_F = \mathrm{SL}_n(F)$, if f'_π is a pseudocoefficient of π, we have*

(i) *$\chi_\pi(g) = \overline{\Phi(f'_\pi, g)}$ if g is an elliptic element of G'_F;*
(ii) *$\Phi(f'_\pi, g) = 0$ if g is a regular semisimple element of G'_F which is not elliptic.*

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Proof The proof of (i) is then the same as for [Ba1, Theorem 4.3]. Point (ii) is true in every characteristic and for every connected reductive algebraic group (see for example [Ba1, Lemme 2.4]). ∎

Corollary 10.2 *The orthogonality relations for characters hold on G'_F.*

Proof The proof is the same as in [DKV, 4.4.h], as Lemaire showed the local integrability of characters for SL_n in non-zero characteristic [Le2]. ∎

11 Removing Condition $p \nmid n$

What happens if F is of non-zero characteristic p, and p divides n? First of all, Theorem 9.1 is absolutely independent of that. Otherwise, the decomposition of G/Z as cosets of G'/Z' is no longer finite, because $F^{*[n]}$ no longer contains an open neighborhood of 1. But, if a field E is an extension of F, then the norm map from E^* to F^* contains an open neighborhood of 1, say $1 + I^{p_{E^*}}$ [We, Proposition 5, p. 143]. So, if γ is an elliptic element of G'_F, then we may still consider a system of representatives of $O_F^*/1+I_F^{p_{Z_{G_F}(\gamma)}} = (O_F/I_F^{p_{Z_{G_F}(\gamma)}})^*$ in O_F^*, and it will be a *finite* set containing a system of representatives for $O_F^*/\det(Z_{G_F}(\gamma))$. The diagonal matrix with 1 on the first $n-1$ positions and an element of this system of representatives on the last will be our X_γ, adapted to γ. More generally, if γ is any regular semisimple element of G_F, $Z_{G_F}(\gamma)$ is isomorphic to the group of invertible elements of a product of finite extensions of F, and this isomorphism sends the determinant to the product of reduced norms, so $\det(Z_{G_F}(\gamma))$ still contains an open subgroup of O_L^* and the whole construction goes the same. All the other fields L involved when applying the close fields theory to G_F and G'_F are of zero characteristic, so for them the construction of X_δ involves $O_L^*/1+I_L^{2n} = (O_L/I_L^{2n})^*$ independently of the field L or the element δ. So we just have to replace the condition $m = 2n$ by $m = \max(2n, p_{Z_{G_F}(\gamma)})$ in the discussion of how to lift adapted systems. All the proofs go then the same. We remark that Proposition 9.3 implies *afterwords* that even in this case of bad characteristic, we still have $x_\gamma \leq 2qn^2$ independently of the regular semisimple element γ, where q is the cardinal of the residual field.

12 Removing Condition $D = F$

Let d^2 be the dimension of D over F. If γ is a regular semisimple element of $\mathrm{GL}_n(D)$, if δ is an element of $\mathrm{GL}_{dn}(F)$, we say that δ *corresponds* to γ if the characteristic polynomial of δ is equal to that of γ. We then write $\delta \leftrightarrow \gamma$. Such δ always exist and are regular semisimple. If γ is elliptic, then such δ are always elliptic. If $f \in H(\mathrm{GL}_n(D_F))$, one may find a function $e \in H(\mathrm{GL}_{nd}(F))$ such that the orbital integral of e verifies:

(i) $\Phi(e, \delta) = \Phi(f, \gamma)$ for all elliptic $\gamma \in \mathrm{GL}_n(D)$ and all $\delta \leftrightarrow \gamma$;

(ii) $\Phi(e, \delta) = 0$ for every regular semisimple element $\delta \in \mathrm{GL}_{nd}(F)$ which does not correspond to any regular semisimple element of $\mathrm{GL}_n(D)$.

This result is proved in [DKV] for F of characteristic zero and in [Ba3] for F of non-zero characteristic. We will call it *orbital integrals transfer* over F.

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Now, if $\gamma \in \mathrm{GL}_n(D)$ is regular semisimple and $\delta \in \mathrm{GL}_{nd}(F)$ corresponds to γ, then $Z_{\mathrm{GL}_n(D)}(\gamma)$ is isomorphic to $Z_{\mathrm{GL}_{nd}(F)}(\delta)$ by an isomorphism preserving the determinant. So $S'_\gamma = S'_\delta$, and the theory of the set X_γ and adapted systems to γ is the same as for δ: for any $x \in S'_\gamma$ choose an element $y_x \in D$ such that the reduced norm of y is x, and then X_γ is the set of diagonal matrices in $\mathrm{GL}_n(D)$ with 1 in the first $n-1$ positions and y_x in the last. Not only is it possible to find such a $y_x \in O_D^*$, but one may choose it in O_E^* where E is the unramified extension of dimension d of F contained in D [We, Proposition 3, p. 141], so that all y_x commute with each other. The construction of Y_γ in $\mathrm{GL}_n(D)$ is the same as in $\mathrm{GL}_n(F)$, with the uniformizer of D instead of that of F. We may suppose the uniformizer of F used for these constructions is the power d of the uniformizer of D, so that the reduced norm of the uniformizer of D is the uniformizer of F. We then have an obvious bijection form $X_\delta Y_\delta$ onto $X_\gamma Y_\gamma$ which preserves the determinant (thanks to [We, Corollary 2, p. 169]).

We prove an analog of Theorem 9.1 for $\mathrm{GL}_n(D)$. This version of Lemaire's theorem that we prove below is weaker, but we need Lemaire's result only for any *fixed* function, as we used it only for a finite number of functions in the proof of our main theorem.

Theorem 12.1 *Let γ be a regular semisimple element of $G = \mathrm{GL}_n(D_F)$. Let $f \in H(\mathrm{GL}_n(D_F))$. Then there exist l and m such that:*

(i) *$\Phi(f, \cdot)$ is constant on $K_{l,F}\gamma K_{l,F}$, equal to $\Phi(f,\gamma)$,*
(ii) *m is greater than l and for every field L which is m-close to F, $\Phi(\lambda_l(f), \cdot)$ is constant on $\lambda_l(K_{l,F}\gamma K_{l,F})$, equal to $\Phi(f,\gamma)$.*

As f is fixed, the real problem is (ii). We get it by transferring integral orbitals to $\mathrm{GL}_{dn}(F)$, and using Theorem 9.1. So we will deal with four groups: $\mathrm{GL}_n(D_F)$, $\mathrm{GL}_{nd}(F)$, $\mathrm{GL}_n(D_L)$ and $\mathrm{GL}_{nd}(L)$, where L is a non-archimedian local field of zero characteristic m-close to F for some m. Let $M \in \mathrm{GL}_{nd}(F)$ be the companion matrix of the characteristic polynomial of γ. Then M corresponds to γ.

We will need the following lemma.

Lemma 12.2 *Let U_1 and U_2 be neighborhoods of γ and M, respectively. Then there exist open compact neighborhoods V_1 of γ and V_2 of M and an integer m such that,*

(i) *$V_1 \subset U_1$ and $V_2 \subset U_2$.*
(ii) *for all field L m-close to F, $\lambda_m(V_1)$ ($\subset \mathrm{GL}_n(D_L)$) and $\lambda_m(V_2)$ ($\subset \mathrm{GL}_{nd}(L)$) are well defined (*i.e.*, V_1 and V_2 are $K_{m,F}$ bi-invariant) and for all $g \in \lambda_m(V_1)$ there exist $h \in \lambda_m(V_2)$ corresponding to g.*

Proof This is a direct consequence of [Ba2, Propositions 4.5, 4.10]. The reader may verify it by formal logic, without knowing what "polynômes proches" means. ■

Proof of Theorem 12.1 Now we have proved that given j, if m is big enough and L is m close to F, then the orbital integrals transfer over F and over L commute with the map λ_j for functions [Ba3]. So our proposition follows from Lemma 12.2 and Theorem 9.1 applied after transferring f.

The analog of Proposition 9.3 in the case $D \neq F$ is also true. If the V_2 of Lemma 12.2 is included in the $K_{l,F}\gamma K_{l,F}$ of Proposition 9.3, and if we apply the proposition and the lemma, we find that the proposition is true for $\mathrm{GL}_n(D)$. One has just to replace the neighborhood $K_{l,F}\gamma K_{l,F}$ of γ with the V_1 of the lemma.

Last but not least is the fact that the characters of irreducible smooth representations of $\mathrm{SL}_n(D)$ are locally integrable in non-zero characteristic. This result may be found in [Le2]. The proof of the orthogonality relations for $\mathrm{SL}_n(D)$ is now exactly the same as the proof for $\mathrm{GL}_n(F)$. ■

13 Stable Transfer

Let F be a non-archimedian local field of any characteristic and D a central division algebra of dimension d^2 over F. If γ is a regular semisimple element of $\mathrm{SL}_n(D)$ or $\mathrm{SL}_{nd}(F)$, fix U_γ a system adapted to γ. We note that the set of regular semisimple classes in $\mathrm{SL}_{nd}(F)$ is parametrized via the characteristic polynomial by the set of all polynomials P of degree n with coefficients in F such that the first and the last coefficients of P are equal to 1, while the set of regular semisimple classes in $\mathrm{SL}_n(D)$ is parametrized by the set of all polynomials P of degree n with coefficients in F such that the first and the last coefficients of P are equal to 1 and the decomposition of P as a product of irreducible polynomials over F involves only polynomials of degrees divisible by d.

We have the following theorem of stable transfer of orbital integrals for SL_n.

Theorem 13.1

(i) *Let $f \in H(\mathrm{SL}_n(D))$. There exists $h \in H(\mathrm{SL}_{nd}(F))$ such that:*

(a) *for all regular semisimple element $\gamma \in \mathrm{SL}_n(D)$, $\delta \in \mathrm{SL}_{nd}(F)$ such that $\delta \leftrightarrow \gamma$,*

$$\sum_{x \in U_\gamma} \Phi(f, x\gamma x^{-1}) = \sum_{x \in U_\delta} \Phi(h, x\delta x^{-1}),$$

(b) *for all regular semisimple elements $\delta \in \mathrm{SL}_{nd}(F)$ which do not correspond to any regular semisimple element of $\mathrm{SL}_n(D)$,*

$$\sum_{x \in U_\delta} \Phi(h, x\delta x^{-1}) = 0.$$

(ii) *Let $h \in H(\mathrm{SL}_{nd}(F))$ verify* (i)(b). *Then there exists $f \in H(\mathrm{SL}_{nd}(F))$ such that for all regular semisimple elements $\gamma \in \mathrm{SL}_n(D)$, for all regular semisimple elements $\delta \in \mathrm{SL}_{nd}(F)$ such that the characteristic polynomials of γ and δ are equal, we have*

$$\sum_{x \in U_\gamma} \Phi(f, x\gamma x^{-1}) = \sum_{x \in U_\delta} \Phi(h, x\delta x^{-1}).$$

Proof In the previous section we explained the transfer of orbital integrals for GL_n ([DKV] for the zero characteristic case and [Ba3] for the non-zero characteristic case). Transferring f to h may be done by lifting f to a function on $\mathrm{GL}_n(D)$, transferring this function to $\mathrm{GL}_{nd}(F)$ and then taking the restriction to $\mathrm{SL}_{nd}(F)$ to be h. Then h verifies (a) and (b) thanks to (8.1) (as we already pointed out, $x_\delta = x_\gamma$). ■

There is a natural question to ask for (i): Could we find h such that all of its orbital integrals would be zero at regular semisimple points of $\mathrm{SL}_{nd}(F)$ that do not correspond to any regular semisimple element of $\mathrm{SL}_n(D)$ (*i.e.*, each of the terms in the sum of the (b) of our theorem is zero)? We do not know the answer to this question.

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