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Thin monodromy in Sp(4)

Christopher Brav and Hugh Thomas

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ABSTRACT

We show that some hypergeometric monodromy groups in $\mathrm{Sp}(4, \mathbf{Z})$ split as free or amalgamated products and hence by cohomological considerations give examples of Zariski dense, non-arithmetic monodromy groups of real rank 2. In particular, we show that the monodromy group of the natural quotient of the Dwork family of quintic threefolds in $\mathbf{P}^4$ splits as $\mathbf{Z} * \mathbf{Z}/5\mathbf{Z}$. As a consequence, for a smooth quintic threefold X we show that the group of autoequivalences $D^b(X)$ generated by the spherical twist along $\mathcal{O}_X$ and by tensoring with $\mathcal{O}_X(1)$ is an Artin group of dihedral type.

1. Introduction

The question of the arithmeticity of Zariski dense monodromy groups of families of projective varieties was raised by Griffiths and Schmid [GS75]. Non-arithmetic or 'thin' examples were given by Deligne and Mostow [DM86] and Nori [Nor86] in products of some low-dimensional rank 1 groups, while examples in orthogonal groups of signature $(n, 1)$ will be provided by Fuchs, Meiri, and Sarnak [FMS13]. In all these examples, however, the Zariski closure of the monodromy group has real rank 1 or is a product of groups of rank 1, so it is natural to ask if there are examples of thin monodromy in higher rank. For more discussion of this problem, see Sarnak's notes on thin matrix groups [Sar12].

The simplest test cases are families over $\mathbf{P}^1 \backslash \{0, 1, \infty\}$ with monodromy having Zariski closure $\mathrm{Sp}(4, \mathbf{R})$ (which has real rank 2). Examples of such monodromy are provided by certain hypergeometric groups [BH89]. Very recently, some of these examples have been shown to be arithmetic by Singh and Venkataramana [SV12], and for others their methods are inconclusive.

In the present paper, we focus on the examples of hypergeometric groups in $\mathrm{Sp}(4, \mathbf{Z})$ having maximally unipotent monodromy at ∞, which have been extensively studied [CYY08, DM06, ES06]. In order to describe them more precisely, let us introduce some notation.

A triple $R, T, U \in \mathrm{GL}_n(\mathbf{C})$ with $R = TU$ is called *(irreducible) hypergeometric* if $\mathrm{rank}(T - I) = 1$ and R^{-1}, U have no common eigenvalues. This second condition ensures that the given representation of the *hypergeometric group* $\langle R, T, U \rangle \subset \mathrm{GL}_n(\mathbf{C})$ is irreducible. Hypergeometric groups are precisely the monodromy groups of generalised hypergeometric ordinary differential equations [BH89]. Such a group is uniquely determined, up to conjugacy, by the Jordan normal forms of R, T, and U, a property referred to as 'rigidity'.

We shall be interested in the cases in which $R, T, U \in \mathrm{Sp}(4, \mathbf{Z})$ and U has maximally unipotent monodromy (such arise as monodromy groups of families of Calabi–Yau threefolds with $h^{2,1} = 1$

Table 1. The 14 hypergeometric groups in $\mathrm{Sp}(4, \mathbf{Z})$ for which U is maximally unipotent.

(a_1, a_2, a_3, a_4)	d	k	Splitting	Additional relation
$(\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5})$	5	5	$\mathbf{Z} * \mathbf{Z}/5\mathbf{Z}$	None
$(\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8})$	2	4	$(\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}) *_{\mathbf{Z}/2\mathbf{Z}} \mathbf{Z}/8\mathbf{Z}$	None
$(\frac{1}{12}, \frac{5}{12}, \frac{7}{12}, \frac{11}{12})$	1	4	$(\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}) *_{\mathbf{Z}/2\mathbf{Z}} \mathbf{Z}/12\mathbf{Z}$	None
$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	16	8	$\mathbf{Z} * \mathbf{Z}$	None
$(\frac{1}{3}, \frac{1}{2}, \frac{1}{2}, \frac{2}{3})$	12	7	$\mathbf{Z} * \mathbf{Z}$	None
$(\frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{3}{4})$	8	6	$\mathbf{Z} * \mathbf{Z}$	None
$(\frac{1}{6}, \frac{1}{2}, \frac{1}{2}, \frac{5}{6})$	4	5	$\mathbf{Z} * \mathbf{Z}$	None
$(\frac{1}{4}, \frac{1}{3}, \frac{2}{3}, \frac{3}{4})$	2	3	No	$(R^6 T)^2 (R^6 T^{-1})^2$
$(\frac{1}{6}, \frac{1}{6}, \frac{5}{6}, \frac{5}{6})$	1	2	No	$(RT)^8$
$(\frac{1}{6}, \frac{1}{4}, \frac{3}{4}, \frac{5}{6})$	6	5	No	$(R^6 T)^2 (R^6 T^{-1})^2$
$(\frac{1}{6}, \frac{1}{3}, \frac{2}{3}, \frac{5}{6})$	3	4	No	$(R^3 T)^2 (R^3 T^{-1})^2$
$(\frac{1}{10}, \frac{3}{10}, \frac{7}{10}, \frac{9}{10})$	1	3	No	$(R^2 T)^{12}$
$(\frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4})$	4	4	?	?
$(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3})$	9	6	?	?

over $\mathbf{P}^1 \backslash \{0, 1, \infty\}$ with maximal degeneration at ∞). As shown by Doran and Morgan [DM06], there are precisely 14 such examples over $\mathbf{R}$, which are labeled by a quadruple of rational numbers (a_1, a_2, a_3, a_4) such that the eigenvalues of R are $\exp(2\pi i a_j)$. In all of these cases, the group is known to be Zariski dense in $\mathrm{Sp}(4, \mathbf{R})$ by a criterion of Beukers and Heckman [BH89]. Chen *et al.* [CYY08] show that with respect to an appropriate basis the 14 examples are contained in certain congruence subgroups of $\mathrm{Sp}(4, \mathbf{Z})$ and raise the question of whether they are arithmetic or thin.

The 14 possibilities are listed in Table 1.

THEOREM 1.1. *In seven of the 14 cases, the group splits as a free or an amalgamated product and contains a free subgroup of finite index. In particular, these examples give thin monodromy groups of real rank 2.*

More precisely, in the next section we show by a ping-pong argument that in seven of the cases, either there is a splitting $\langle T, R \rangle = \langle T \rangle * \langle R \rangle$ or $R^k = -I$ for some $k > 0$ and there is a splitting $\langle T, R \rangle = \langle R^k, T \rangle *_{\langle R^k \rangle} \langle R \rangle$. The precise form of the splitting is displayed in Table 1, from which it is immediate that either the group is free or the group generated by conjugates of T by powers of R is a free subgroup of finite index.

For thinness in these seven cases, note that the group $\mathrm{Sp}(4, \mathbf{Z})$ has cohomological dimension 2 over $\mathbf{Q}$ [LW85, Corollary 5.2.3]. But since a group of finite cohomological dimension has the same cohomological dimension as its finite-index subgroups [Swa69, Theorem 9.1], $\mathrm{Sp}(4, \mathbf{Z})$ cannot have free subgroups of finite index.

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Our ping-pong argument is uniform, works for seven of the 14 cases, and is inconclusive for the other seven. For five of these seven, we exhibit additional relations showing that the group does not split in the expected way. Note of course that among these seven other cases are three examples $((\frac{1}{6}, \frac{1}{6}, \frac{5}{6}, \frac{5}{6}), (\frac{1}{6}, \frac{1}{4}, \frac{3}{4}, \frac{5}{6})$, and $(\frac{1}{10}, \frac{3}{10}, \frac{7}{10}, \frac{9}{10}))$ of arithmetic groups from Singh and Venkataramana [SV12].

The monodromy groups appearing in the table have been well studied in the context of mirror symmetry and are expected to be mirror dual to certain groups of autoequivalences acting on the bounded derived category of coherent sheaves $D^b(X)$ on a Calabi–Yau threefold X. For a discussion of this, see van Enckevort and van Straten [ES06].

The case with parameter $(\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5})$ corresponds to the original and most famous example of mirror symmetry, the Dwork family of quintic threefolds and its mirror family, with the autoequivalences acting on $D^b(X)$, X a smooth quintic threefold. Let $\alpha = T_{\mathcal{O}_X}$ be the spherical twist functor along the structure sheaf $\mathcal{O}_X$ and $\beta = \mathcal{O}_X(1)\otimes?$ the functor of tensoring with $\mathcal{O}_X(1)$. We are interested in describing the subgroup of autoequivalences $\langle \alpha, \beta \rangle \subset \mathrm{Aut}(D^b(X))$.

THEOREM 1.2. *The subgroup $\langle \alpha, \beta \rangle \subset \mathrm{Aut}(D^b(X))$ is a two-generator Artin group with relation $(\alpha\beta)^5 = (\beta\alpha)^5$.*

To see this, first consider the action of $\langle \alpha, \beta \rangle$ on the even-degree cohomology $H^{\mathrm{ev}}(X, \mathbf{Q})$, for which it is easy to write down explicit matrices, which we denote by A and B [ES06]. Using this, one can check that A is a transvection, B is maximally unipotent, and $C = AB$ has the non-trivial fifth roots of unity for eigenvalues. A, B, C therefore give a hypergeometric triple and, by rigidity of hypergeometric triples, we can choose bases in which $A = T$, $B = U$.

Next, note that from the splitting $\langle A, B \rangle = \langle A, AB \rangle = \mathbf{Z} * \mathbf{Z}/5\mathbf{Z}$ we see that $\langle A, B \rangle$ is subject to the single relation

$$(AB)^5 = I.$$

The relation between α, β in $\mathrm{Aut}(D^b(X))$ turns out to be slightly more subtle. We have $(\alpha\beta)^5 \simeq [2]$, where $[2] \in \mathrm{Aut}(D^b(X))$ is the cohomological shift by two degrees [CK08, Kuz04]. In particular, $(\alpha\beta)^5$ is central. Since centrality gives $(\alpha\beta)^5 \simeq \beta(\alpha\beta)^5\beta^{-1}$, we have the relation

$$(\alpha\beta)^5 \simeq (\beta\alpha)^5.$$

Conversely, the relation $(\alpha\beta)^5 \simeq (\beta\alpha)^5$ implies centrality in a similar way.

It is then not hard to see that $(\alpha\beta)^5 \simeq (\beta\alpha)^5$ is the only relation between α, β. Indeed, consider the Artin group generated by x, y subject to the relation $(xy)^5 = (yx)^5$. Then we have a sequence of surjections $\langle x, y \rangle \twoheadrightarrow \langle \alpha, \beta \rangle \twoheadrightarrow \langle A, B \rangle$, where $x \mapsto \alpha, y \mapsto \beta$ and $\alpha \mapsto A, \beta \mapsto B$. Since the only relation among the A, B is $(AB)^5 = I$, the kernel of the surjection $\langle x, y \rangle \twoheadrightarrow \langle A, B \rangle$ is generated as a normal subgroup by $(xy)^5$, but since $(xy)^5$ is central, the kernel is in fact cyclic. Similarly, the kernel of the surjection $\langle \alpha, \beta \rangle \twoheadrightarrow \langle A, B \rangle$ is cyclic and generated by $(\alpha\beta)^5$. The surjection $\langle x, y \rangle \twoheadrightarrow \langle \alpha, \beta \rangle$ therefore induces an isomorphism between the kernels of $\langle x, y \rangle \twoheadrightarrow \langle A, B \rangle$ and of $\langle \alpha, \beta \rangle \twoheadrightarrow \langle A, B \rangle$ and hence is itself an isomorphism.

Table 1 gives the data for the 14 hypergeometric groups in $\mathrm{Sp}(4, \mathbf{Z})$ for which U is maximally unipotent. In the first column of the table are the parameters (a_1, a_2, a_3, a_4) giving the eigenvalues $\exp(2\pi i a_j)$ for R. With respect to a suitable basis [CYY08], there are integers d

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and k so that we have

$$U = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ d & d & 1 & 0 \\ 0 & -k & -1 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{Sp}(4, \mathbf{Z}).$$

The integers d, k appear in the second and third columns of the table. In the fourth column we describe a splitting of the group as a free or an amalgamated product in those cases in which our method is effective. In the final column we indicate cases in which there is a relation that prevents the group from splitting in the obvious way. We do not claim in these cases to give a complete set of relations, and indeed we omit relations of the form $R^m = I$.

2. Ping-pong

We begin with a quick proof via the ping-pong lemma of the fact that the SL(2) hypergeometric group having R with parameters $(1/3, 2/3)$ and U unipotent splits as $\mathbf{Z} * \mathbf{Z}/3\mathbf{Z}$. We include details for this example because we take a very similar approach in the Sp(4) case.

By rigidity, we can choose any convenient basis in which to write R, T, U. Let

$$U = \begin{pmatrix} 3 & 4 \\ -1 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, \quad R = TU = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

Now we recall the ping-pong lemma in a version convenient for us [LS77, Proposition III.12.4].

THEOREM 2.1. *Let a group G be generated by two subgroups G_1, G_2, whose intersection is H. Suppose that G acts on a set W, and suppose that there are disjoint non-empty subsets X, Y such that*

$$(G_1 - H)Y \subseteq X \quad and \quad (G_2 - H)X \subseteq Y,$$
$$HY \subseteq Y \quad and \quad HX \subseteq X.$$

Then $G = G_1 *_H G_2$.

We wish to apply the lemma with G_1 the subgroup generated by T, and G_2 the subgroup generated by R. We consider the given action of G on $\mathbf{R}^2$.

Divide $\mathbf{R}^2$ into 12 equally spaced cones (with respect to the angle measure obtained when we put the axes at $60°$). See Figure 1.

We define X to be the union of the interiors of the four cones which touch the horizontal axis, which are labeled in the diagram as $C^+, C^-, -C^+, -C^-$. We define Y to be the union of the interiors of the other eight cones.

We then must check that:

(i) X and Y are disjoint;

(ii) R and R^2 take X into Y;

(iii) T^i takes Y into X for $i \neq 0$.

Statements (i) and (ii) are completely trivial. We can break statement (iii) down into four even more trivial statements, as follows:

(iv) $T^{-1}C^+ \subseteq C^+$;

(v) $TC^- \subseteq C^-$;

(vi) $T^{-1}Y \subseteq \pm C^+$;
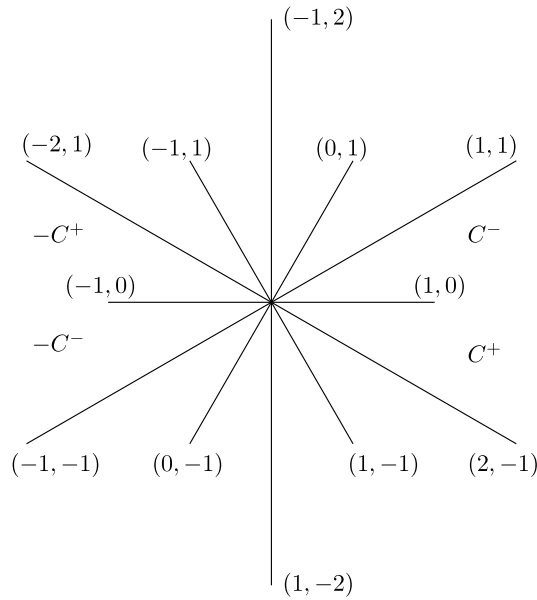
(vii) $TY \subseteq \pm C^-$.

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FIGURE 1. $\mathbf{R}^2$ divided into cones.

These four statements are easy to verify. For example, (iv) is equivalent to the statement that if we apply T^{-1} to an extreme ray of C^+ and re-express it in the basis of the extreme rays of C^+, the coefficients are non-negative. Statements (v)–(vii) can be rephrased similarly.

We now explain why (iv)–(vii) imply (iii). In order to show that T^i takes Y into X for i negative, (vi) tells us that T^{-1} takes Y into $\pm C^+$, and then (iv) tells us that applying further powers of T^{-1} will not take us outside $\pm C^+$. For i positive, the same approach is applied using (vii) and (v).

Examples in Sp(4)

We present our hypergeometric groups using the matrices

$$U = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ d & d & 1 & 0 \\ 0 & -k & -1 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$R = TU = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1-k & -1 & 1 \\ d & d & 1 & 0 \\ 0 & -k & -1 & 1 \end{pmatrix},$$

which preserve the standard symplectic form given by the matrix

$$J = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

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Consider the matrix

$$B = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ -d & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

which satisfies $B^2 = I$. We note that

$$BRB = R^{-1} \quad \text{and} \quad BT^{-1}B = T.$$

The fixed subspace of B is just the span of the second and third co-ordinate vectors. Let us write V for this subspace.

This subspace plays a similar role in our construction to the horizontal axis in the two-dimensional case discussed above, where the analogue of B is a reflection in the horizontal axis.

Remark 1. In the case where R is of finite order, V can be given a different description which is instructive, because it emphasises the analogy of V with the horizontal axis in the two-dimensional case.

The cases where R is finite order are exactly the cases where R has four distinct eigenvalues. In these cases, the eigenvalues are roots of unity and come in complex conjugate pairs, so R induces a decomposition of $\mathbf{R}^4$ into two-dimensional subspaces W_1 and W_2, on each of which R acts by a rotation in an appropriate basis.

Each of W_1, W_2 has a one-dimensional intersection with the fixed space of T. It turns out that V is the direct sum of these intersections, and B restricts to W_i, acting as a reflection fixing the intersection of W_i with the fixed space of T.

Returning to the case of general R, let

$$P = \log(T^{-1}R) = \log(U), \quad Q = \log(TR^{-1}).$$

Throughout this paper, whenever we write $\log(K)$, K is a matrix such that $K - I$ is nilpotent of degree at most 4. We therefore define $\log(K) = (K - I) - (K - I)^2/2 + (K - I)^3/3$.

If we consider the equation $v^T JPv = 0$, there are two non-zero solutions in V up to scalar multiples, which are $(0, 0, 1, 0)^T$ and

$$v = (0, 1, d/12 - k/2, 0)^T.$$

Now, define C^+ to be the open cone generated by $P^i v$ for $0 \leqslant i \leqslant 3$ (that is to say, linear combinations of these four vectors, with all coefficients strictly positive). Define $C^- = BC^+$, which can also be described as the open cone generated by $Q^i v$ for $0 \leqslant i \leqslant 3$. Note that $P^2 v = Q^2 v = (0, 0, d, 0)^T$, the other solution to the equation which we solved for v.

Finally, define

$$X = \pm C^+ \cup \pm C^-, \quad Y = \bigcup_{R^i \neq \pm I} R^i X.$$

We now show that X and Y give ping-pong tables in the first seven examples from Table 1.

Case $(\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5})$, $d = 5, k = 5$

The matrix R has order 5. $H = \{I\}$. G_1 is the group generated by T, and G_2 is the group of order 5 generated by R.

Write M for the matrix whose columns are the generating rays v, Pv, P^2v, P^3v of C^+. Similarly, write N for the matrix whose columns are the generating rays v, Qv, Q^2v, Q^3v of C^-.

We have

$$
M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -25/12 & 5/2 & 5 & 0 \\ 0 & -25/12 & 0 & -5 \end{pmatrix}, \quad
N = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 25/12 & 0 & 5 \\ -25/12 & -5/2 & 5 & 0 \\ 0 & 25/12 & 0 & 5 \end{pmatrix}.
$$

The conditions we need to check are:

(i) X and Y are disjoint;

(ii) $(G_2 - H)X \subseteq Y$;

(iii) $(G_1 - H)Y \subseteq X$.

In order to prove (i), it suffices to show that $R^j C^+$ and $R^j C^-$ are disjoint from $\pm C^+$ for $1 \leqslant j \leqslant 4$. (Disjointness from $\pm C^-$ then follows by the symmetry with respect to B.)

Consider, for example, showing that RC^+ is disjoint from $\pm C^+$. In order to do this, we expand $M^{-1}RM$:

$$
M^{-1}RM = \begin{pmatrix} -23/12 & -55/12 & -5 & -5 \\ 1 & 1 & 0 & 0 \\ -103/144 & -131/144 & -13/12 & -25/12 \\ 1/6 & 1/2 & 1 & 1 \end{pmatrix}.
$$

Note that the entries in two of the rows are all non-negative, while the entries in the other two rows are all non-positive. It follows that the same will be true for any positive linear combination of the columns, and thus for any point in RC^+ in the basis given by the columns of M. In particular, there is no intersection between RC^+ and $\pm C^+$.

The same argument works for the other cases of (i): it is always the case that two of the rows have non-negative entries and two of the rows have non-positive entries.

Condition (ii) is true by construction.

As in the two-dimensional case, we break condition (iii) down into four subclaims:

(iv) $T^{-1}C^+ \subseteq C^+$;

(v) $TC^- \subseteq C^-$;

(vi) $T^{-1}R^j(C^+ \cup C^-) \subseteq \pm C^+$ for $1 \leqslant j \leqslant 4$;

(vii) $TR^j(C^+ \cup C^-) \subseteq \pm C^-$ for $1 \leqslant j \leqslant 4$.

Statement (iii) follows from (iv)–(vii) in essentially the same way as in the two-dimensional case. An element of $(G_1 - I)Y$ is of the form $T^i R^j x$ for some $i \neq 0$, $1 \leqslant j \leqslant 4$, $x \in X$. If $i < 0$, then (vi) tells us that $T^{-1}R^j x \in \pm C^+$, and then (iv) applies to tell us that $T^i R^j x \in \pm C^+ \subseteq X$. Similarly, if $i > 0$, apply (vii) and (v).

To establish (iv), we first argue that $T^{-1}\overline{C^+} \subseteq \overline{C^+}$. To show this, it suffices by convexity to see that the generating rays of $T^{-1}\overline{C^+}$ are contained in $\overline{C^+}$. We evaluate

$$
M^{-1}T^{-1}M = \begin{pmatrix} 1 & 25/12 & 0 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 125/144 & 1 & 25/12 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
$$

We observe that the entries are all non-negative, which is exactly what we needed. Finally, since T^{-1} is an invertible linear transformation, it takes open sets to open sets, so the image of C^+ will be contained in C^+.

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We can show (v) either by a completely similar argument, checking that the entries of $N^{-1}TN$ are non-negative, or by deducing it from (iv) using the symmetry encoded by B.

The proof of (vi) works in exactly the same way: it reduces to checking that the entries of the following matrices are either all non-negative or all non-positive: $M^{-1}T^{-1}R^j M$ and $M^{-1}T^{-1}R^j N$ for $1 \leqslant j \leqslant 4$. Again (vii) is established by the same argument or by deducing it from (vi).

Ping-pong therefore establishes that the monodromy group is isomorphic to $\mathbf{Z} * \mathbf{Z}/5\mathbf{Z}$.

Case $(\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8})$, $d = 2$, $k = 4$

We have $R^4 = -I$. Now $H = \{I, R^4\}$. G_1 is generated by T and R^4 (forming a group isomorphic to $\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$). The subgroup G_2 is generated by R.

Since X and Y are symmetric with respect to negation, the conditions on H hold.

Condition (i) is proved exactly as before.

As before, condition (ii), that $(G_2 - H)X \subseteq Y$ is true by construction.

Condition (iii) follows as before from conditions (iv)–(vii). In conditions (vi) and (vii), note that now j runs from 1 to 3. The conditions are checked the same way as before.

Ping-pong therefore establishes that the monodromy group is isomorphic to $(\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}) *_{\mathbf{Z}/2\mathbf{Z}} \mathbf{Z}/8\mathbf{Z}$.

Case $(\frac{1}{12}, \frac{5}{12}, \frac{7}{12}, \frac{11}{12})$, $d = 1$, $k = 4$

We have $R^6 = -I$. As in the previous case, $H = \{I, -I\}$. G_1 is generated by T and R^6 (forming a group isomorphic to $\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$). The subgroup G_2 is generated by R.

Everything is checked exactly as in the previous case. Ping-pong establishes that the monodromy group is $(\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}) *_{\mathbf{Z}/2\mathbf{Z}} \mathbf{Z}/12\mathbf{Z}$.

Case $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, $d = 16$, $k = 8$

The order of R is infinite. $H = \{I\}$. G_1 is generated by T and G_2 by R.

The situation is very similar to the first case $d = k = 5$, except that, in order to perform the check of conditions (i), (vi), and (vii) as in that case, an infinite number of checks would be required. To establish (i) we must show that $R^j C^+$ and $R^j C^-$ are disjoint from C^+ for all $j \neq 0$. To establish (vi), we must show that the entries of $M^{-1}T^{-1}R^j M$ and $M^{-1}T^{-1}R^j N$ are all non-positive or all non-negative for each $j \neq 0$. As before, (vii) then follows by symmetry.

We note that R consists of a single Jordan block with eigenvalue -1. We therefore define $Z = \log(-R)$, which is nilpotent. Now

$$R^j = (-1)^j \exp(jZ) = (-1)^j (I + jZ + j^2 Z^2/2 + j^3 Z^3/6).$$

We find that $M^{-1}R^j M$ can therefore be expressed as

$$M^{-1}R^j M = (-1)^j (I + jA_1 + j^2 A_2 + j^3 A_3)$$

for certain matrices A_1, A_2, A_3. Since the entries in A_3 have all entries in two rows positive, and all entries in two rows negative, the cubic term will eventually dominate, leading us to conclude that $M^{-1}R^j M$ will have all entries in two rows positive and all entries in two rows negative, for j sufficiently large. In fact, it is easy to see that the cubic term dominates if $|j| \geqslant 6$; one then checks the cases $1 \leqslant |j| \leqslant 5$ individually.

To check condition (vi) we take a similar approach. We find that

$$M^{-1}T^{-1}R^j M = (-1)^j (M^{-1}T^{-1}M + jD_1 + j^2 D_2 + j^3 D_3)$$

for certain matrices D_1, D_2, D_3.

Since the entries of D_3 are all strictly negative, for $|j|$ sufficiently large the cubic term dominates, and the entries of $M^{-1}T^{-1}R^j M$ are all of the same sign. In fact, the absolute value of each entry of D_3 is at least as great as the corresponding entries in the other matrices, so the cubic term dominates starting with $|j| \geqslant 6$. We then check the cases $1 \leqslant |j| \leqslant 5$ individually. The same analysis is applied to $M^{-1}T^{-1}R^j N$.

Case $(\frac{1}{3}, \frac{1}{2}, \frac{1}{2}, \frac{2}{3})$, $d = 12, k = 7$

As in the previous case, the order of R is infinite. $H = \{I\}$. G_1 is generated by T and G_2 by R.

The analysis is very similar to the previous case. However, R has eigenvalues $\exp(\pm 2\pi i/3)$, and one two-dimensional Jordan block with eigenvalue -1. Therefore, we cannot simply take the logarithm of R. However, R^6 has all eigenvalues equal to 1, so we can define $Z = \log(R^6)$. Note that $Z^2 = 0$. Now

$$R^{6n+j} = R^j \exp(nZ) = R^j (I + nZ).$$

To check that $R^m C^+$ is disjoint from $\pm C^+$ for $m > 0$, we observe that for each $1 \leqslant j \leqslant 6$, $M^{-1}R^j M$ and $M^{-1}R^j ZM$ have the same two rows consisting of positive entries, and the same two rows consisting of negative entries. This suffices, since any positive m can be written as $6n+j$ for some $1 \leqslant j \leqslant 6$. For $m < 0$, we proceed similarly, writing $m = 6n + j$ with $-6 \leqslant j \leqslant -1$. To show that $R^k C^-$ is disjoint from $\pm C^+$, we proceed similarly.

To establish (vi), we consider

$$M^{-1}T^{-1}R^{6n+j}M = M^{-1}T^{-1}R^j M + n M^{-1}T^{-1}R^j ZM.$$

If $1 \leqslant j \leqslant 6$, all the entries of the two matrices on the right-hand side have the same sign. This establishes that $T^{-1}R^m C^+ \subseteq \pm C^+$ for any $m > 0$. For $m < 0$, we proceed similarly, writing $m = 6n + j$ with $-6 \leqslant j \leqslant -1$. Again, we check that $M^{-1}T^{-1}R^j M$ and $-M^{-1}T^{-1}R^j ZM$ have all entries with the same sign for $-6 \leqslant j \leqslant -1$. This establishes that $T^{-1}R^m C^+ \subseteq \pm C^+$ for $m < 0$. We then repeat the same two steps to show that $T^{-1}R^m C^- \subseteq \pm C^+$. This completes the proof of (vi). Then (vii) follows by symmetry.

Case $(\frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{3}{4})$, $d = 8, k = 6$

The matrix R has eigenvalues $\pm i$ together with one Jordan block of rank 2 with eigenvalue -1. We therefore proceed as in the previous case, but defining $Z = \log(R^4)$. The analysis goes through in exactly the same way.

Case $(\frac{1}{6}, \frac{1}{2}, \frac{1}{2}, \frac{5}{6})$, $d = 4, k = 5$

The matrix R has eigenvalues $\exp(\pm \pi i/3)$ together with a Jordan block of rank two with eigenvalue -1. We define $Z = \log(-R^3)$. The analysis goes through as in the previous two cases.

3. Logic behind the choice of ping-pong tables

The rough outline of the shapes of the ping-pong tables was inspired by the two-dimensional case. Given that, one wants to define cones C^+ and C^- (as in the previous section). C^+ should be stable under T^{-1} and U, while C^- should be stable under T and U^{-1}. The two cones will therefore lie on opposite sides of the fixed hyperplane of T.

It is not clear why C^+ and C^- should be chosen to have extreme rays in V, which is codimension one in the fixed hyperplane of T.

As we already remarked, the cone C^+ should be stable under U. An awkward feature of this condition is that if we, for some reason, decide that some vector x is in C^+, then it follows that the infinite set $Ux, U^2x, \ldots$ all also lie in C^+. The convex hull of this infinite set will typically have infinitely many extremal rays, making further analysis complicated.

Therefore, we instead chose to require that C^+ be stable under $P = \log(U)$. Since $U = \exp(P)$, stability under P implies stability under U, but since P is nilpotent, stability under P is easier to work with. Indeed, since $P^4 = 0$, for any x, the cone generated by $P^i x$ for $0 \leqslant i \leqslant 3$ will be stable under P.

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Christopher Brav   brav@maths.ox.ac.uk

Mathematical Institute, University of Oxford, 24–29 St Giles',
Oxford OX1 3LB, UK

Hugh Thomas   hthomas@unb.ca

Department of Mathematics and Statistics, University of New Brunswick,
Fredericton, New Brunswick, Canada E3B 5A3