Can. J. Math., Vol. XXXIII, No. 1, 1981, pp. 210–228

VARIATIONAL METHODS FOR ONE AND SEVERAL PARAMETER NON-LINEAR EIGENVALUE PROBLEMS

PAUL BINDING

1. Introduction. We shall consider a multiparameter eigenvalue problem of the form

$$W_n(\lambda)x_n = 0 \neq x_n$$

$$(1.1) \qquad W_n(\lambda) = T_n + V_n(\lambda) \quad n = 1, 2, \ldots, k,$$

where $\lambda \in \mathbf{R}^k$ while T_n and $V_n(\lambda)$ are self-adjoint linear operators on a Hilbert space H_n. If $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbf{R}^k$ and $x = (x_1, \ldots, x_k) \in \bigoplus_{n=1}^k H_n$ satisfy (1.1) then we call λ an *eigenvalue*, x an *eigenvector* and (λ, x) an *eigenpair*. While our main thrust is towards the general case of several parameters λ_n, the method ultimately involves reduction to a sequence of one parameter problems. Our chief contributions are (i) to generalise the conditions under which this reduction is possible, and (ii) to develop methods for the one parameter problem particularly suited to the multiparameter application. For example, we give rather general results on the magnitude and direction of the movement of non-linear eigenvalues under perturbation.

Until Section 8, we restrict ourselves to the case where the T_n have compact resolvents and the $V_n(\lambda)$ are bounded, for each n and λ. This includes, for example, second order linear ordinary differential equations (de) on intervals $[a_n, b_n]$ with self-adjoint boundary conditions. The case where the $V_n(\lambda)$ are linear in λ has received attention from many authors, starting explicitly for $k = 2$ with Klein's investigation of Lamé's equation [**11**]. Not long afterwards, Bôcher [**6**] showed, under certain restrictions on the de, that for any non-negative integer multi-index $i = (i_1, \ldots, i_k)$ there exists a unique eigenpair (λ, x) so that x_n has i_n zeros in $[a_n, b_n]$. There are also various results on monotonic and continuous dependence of both the eigenvalues and the zeros (or, more generally, of the focal points) of the eigenvector. Such results go back to Sturm for the case $k = 1$ and may be found for the general multiparameter case in [**4**].

Non-linear one parameter problems have an extensive literature; see for example [**17**] and the references there. Their generalisation to several

Received May 18, 1979 and in revised form May 27, 1980. This research was supported by a grant from the NSERC of Canada, and was written up at the Control Theory Centre, University of Warwick.

https://doi.org/10.4153/CJM-1981-018-4 Published online by Cambridge University Press

parameters is quite recent; see for example [**7**], [**8**], and [**2**]. The last two references deal with problems non-linear in λ and both make use of degree theory to obtain existence of eigenpairs. Here we use hypotheses related to those of [**2**] but our methods differ from those of all the cited works. Instead we have non-linearised the approach of [**3**] to produce tools yielding uniqueness and comparison results for which degree theory is unsuited.

In Section 2, we set up our notation and we describe the variational approach by means of a converse (Theorem 2.1) to an existence theorem. Our basic monotonic dependence results are deduced from very weak hypotheses in Section 3. We discuss existence, uniqueness, comparison and dependence results for $k = 1$ in Sections 4 and 5. In Section 6 the assumptions are reformulated so as to generalise more readily to several parameters. Section 7 contains the main result for $k > 1$. The existence part is contained in [**2**, Corollary 1], but the method used here automatically gives uniqueness, as well as new continuous and Lipschitz dependence results. Finally in Section 8 we consider some connections between the work here and that in the literature. In particular we briefly consider some alternative settings more appropriate for differential operators $V_n(\lambda)$ and for integral equations.

2. Preliminaries. We consider (1.1) as posed in the first paragraph of Section 1. Throughout, the symbol u will denote a k-tuple $(u_1, \ldots, u_k) \in \bigoplus_{n=1}^k H_n$ with $\|u_n\| = 1$ for each n. For each u, we write $v(\lambda, u)$ and $w(\lambda, u)$ for the vectors in $\mathbf{R}^k$ with nth entries

$$v_n(\lambda, u) = (u_n, V_n(\lambda)u_n) \quad \text{and} \quad w_n(\lambda, u) = (u_n, W_n(\lambda)u_n),$$

where defined.

It follows from [**3**, Corollary to Lemma 1] that each $W_n(\lambda)$ has compact resolvent. We shall also assume that each T_n is bounded below, i.e., that each (u_n, T_nu_n) is bounded below. This assumption is not essential, but holds for many physical applications and permits us to introduce

$$\text{(2.1)} \qquad \rho_n^{i_n}(\lambda) = \max\{\min\{w_n(\lambda, u): u_n \in \mathscr{D}(T_n), (u_n, y_j) = 0\}: \\ y_j \in H_n, 1 \leqq j \leqq i_n\}.$$

By virtue of the minimax principle, $\rho_n^{i_n}(\lambda)$ is the i_nth eigenvalue of $W_n(\lambda)$, counted from $i_n = 0$ according to multiplicity. Recall that the maximinisation is unnecessary if we choose the y_j as the first i_n eigenvectors of $W_n(\lambda)$; i.e., the eigenvectors corresponding to $\rho_n^0(\lambda), \ldots, \rho_n^{i_n-1}(\lambda)$; we shall refer to this fact as Rayleigh's principle.

(2.1) yields the following characterization of eigenpairs; cf. [**3**, Theorem 4(ii)].

THEOREM 2.1. *If* (λ, x) *satisfies* (1.1) *then there exist integers* $i_n \geqq 0$ *so* *that* $\rho^i(\lambda) = (\rho_1{}^{i_1}(\lambda), \ldots, \rho_k{}^{i_k}(\lambda)) = 0$ *and* $u_n = x_n/\|x_n\|$ *is a minimiser in* (2.1).

Henceforth i will denote a non-negative integer multi-index. Further, we shall label an eigenpair characterized by Theorem 2.1 with superfix i, so in particular $\rho^i(\lambda^i) = 0$. It is worth pointing out that λ^i and x^i need not be unique functions of i, and one of our tasks is to produce uniqueness conditions.

Our comparison results will concern (1.1) and a similar system distinguished by primes. The auxiliary constructions $v(\lambda, u)$, λ^i, etc., will also be primed as necessary, e.g. $v'(\lambda, u)$, λ'^i.

Finally, the bounded linear operators on either $\mathbf{R}^k$ or H_n will be given the uniform operator norm, and the self-adjoint operators will be given the partial order

$$A \geqq B \Leftrightarrow A - B \text{ is non-negative definite.}$$

3. Basic comparison theory. For Sections 3–6, unless otherwise stated, we assume $k = 1$ and accordingly we shall suppress subscripts. Our first comparison result is basic to much of what follows. We shall employ the following notation:

$$\alpha(\lambda, \mu) = \sup_u\{v(\lambda, u) - v(\mu, u)\}. \tag{3.1}$$

THEOREM 3.1. (i) $\rho^i(\lambda) - \rho^i(\mu) \leqq \alpha(\lambda, \mu)$.

(ii) *If* $\mathscr{D}(T) = \mathscr{D}(T')$ *then* $\rho^i(\lambda) - \rho'^i(\mu) \leqq \sup\{w(\lambda, u) - w'(\mu, u): u \in D(T)\}$.

Proof. (i) Let $u = \bar{u}$ minimise $w(\mu, u)$ subject to $u \in \mathscr{D}(T)$ and u orthogonal to the first i eigenvectors of $W(\lambda)$. Then Rayleigh's principle gives

$$\rho^i(\lambda) \leqq w(\lambda, \bar{u})$$

while

$$\rho^i(\mu) \geqq w(\mu, \bar{u})$$

follows from the minimax principle (2.1). Thus

$$\rho^i(\lambda) - \rho^i(\mu) \leqq v(\lambda, \bar{u}) - v(\mu, \bar{u}) \leqq \alpha(\lambda, \mu). \tag{3.2}$$

(ii) The argument is similar, with $u = \hat{u}$ minimising $w'(\mu, u)$ subject to $u \in \mathscr{D}(T)$ and orthogonal to the first i eigenvectors of $W(\lambda)$, yielding

$$\rho^i(\lambda) - \rho'^i(\mu) \leqq w(\lambda, \hat{u}) - w'(\mu, \hat{u}). \tag{3.3}$$

Instead of bounding differences in ρ^i by differences in v, we can deduce complementary bounds, using the λ^i introduced after Theorem 2.1. In what follows, λ^i denotes an arbitrary element of the set $(\rho^i)^{-1}(0)$, which we tacitly assume to be nonempty. Also, we shall denote λ'^j by ν throughout. A statement like "$\xi(\lambda^i) - \xi(\nu) \leqq \pi$" (cf. Corollary 3.2 below) is therefore shorthand for "if $(\rho^i)^{-1}(0) \cap (\rho'^j)^{-1}(0) \neq \emptyset$ then $\sup\{\xi(\lambda) - \xi(\mu): \rho^i(\lambda) = 0 = \rho'^j(\mu)\} \leqq \pi$".

COROLLARY 3.2. *For any non-negative integers i and j,*
(i) $v(\nu, u) - v(\lambda^i, u) \leqq \sup_\lambda\{\rho^i(\lambda) - \rho'^j(\lambda)\}$ *for some* $u \in (T)$
(ii) $v'(\nu, u') - v'(\lambda^i, u') \leqq \sup_\lambda\{\rho^i(\lambda) - \rho'^j(\lambda)\}$ *for some* $u' \in (T')$.

Proof. From (3.2) with $\lambda = \lambda^i$ and $\mu = \nu$, we have

$$(3.4) \qquad v(\nu, \bar{u}) - v(\lambda^i, \bar{u}) \leqq \rho^i(\nu) - \rho^i(\lambda^i) = \rho^i(\nu) - \rho'^j(\nu)$$

since $\rho^i(\lambda^i)$ and $\rho'^j(\nu)$ are both zero. This establishes (i), and the proof of (ii) is analogous.

We return now to general k. Let $\sigma \in \mathbf{R}^k$ be one of the 2^k vectors such that $\sigma_n = \pm 1$ for each n. We define

$$a \leqq_\sigma b \text{ if } \sigma_n a_n \leqq \sigma_n b_n \quad \text{for each } n.$$

Our basic multiparameter comparison result is then as follows.

COROLLARY 3.3. *Suppose that for each i there is j so that*

$$(3.5) \qquad \rho^i(\lambda) \leqq_\sigma \rho'^j(\lambda) \text{ for each } \lambda.$$

Then

$$v(\nu, u) \leqq_\sigma v(\lambda^i, u) \quad \text{for some } u_n \in \mathscr{D}(T_n),\ 1 \leqq n \leqq k,$$

and

$$v'(\nu, u') \leqq_\sigma v'(\lambda^i, u') \quad \text{for some } u_n' \in \mathscr{D}(T_n'),\ 1 \leqq n \leqq k.$$

In particular, none of the operators $\sigma_n(V_n(\nu) - V_n(\lambda^i))$, $\sigma_n(V_n'(\nu) - V_n'(\lambda^i))$ can be positive definite.

This follows directly from Corollary 3.2 applied to each W_n and W_n' in turn, at least when $\sigma_n = 1$. (When $\sigma_n = -1$, we exchange W_n' and W_n and apply (i) and (ii) of Corollary 3.2 in the reverse order.) Further, a strict inequality in any component of (3.5) leads to corresponding inequalities in the conclusions, as is easily seen.

We emphasize the weak hypotheses used here. Dependence on λ enters only via (3.5), while the usual multiparameter "definiteness condition" (cf. (8.1) below) is not needed at all. Note that the conclusions of Corollary 3.3 depend only on the V_n and V_n'. This is particularly useful in the de context, where the $V_n(\lambda)$ are simply multiplication operators. Various applications of Corollaries 3.3 and 3.4 below, to second order linear de are given in [**4**], and they may be carried over with appropriate modifications to the non-linear case here.

It is worth pointing out that Corollary 3.3 may be rewritten in terms of the sets

(3.6) $\quad C_\sigma(\lambda) = \{\mu \in \mathbf{R}^k : v(\mu, u) \leqq_\sigma v(\lambda, u) \text{ for some } u\}$

and the similarly defined sets $C_\sigma'(\lambda)$. In terms of these constructions we have the following.

COROLLARY 3.4. *Under the hypotheses of Corollary* 3.3, $\nu \in C_\sigma(\lambda^i) \cap C_\sigma'(\lambda^i)$.

Two cases where the $C_\sigma(\lambda)$ are easily constructed are as follows: First, if $V_n(\lambda)$ is independent of λ_m for $m \neq n$ and is monotonic in λ_n, then $C_\sigma(\lambda) - \lambda$ is one of the co-ordinate orthants. Second, if each $V_n(\lambda)$ is linear in λ then $C_\sigma(\lambda) - \lambda = C_\sigma(0)$. In both cases, then, Corollary 3.4 provides bounds on the movement of any eigenvalue, based on a single calculation.

4. One parameter monotonicity conditions. In Section 3 we related differences in ρ^i to differences in v. Now we shall use "monotonicity" conditions to replace the differences in v by differences in λ^i. In this way we shall produce a uniqueness and perturbation theory for (1.1) in the case $k = 1$.

The first monotonicity condition is as follows. We write $v(\ \ , u)$: $\lambda \to v(\lambda, u)$.

Assumption I. For each u, $v(\ \ , u)$ is strictly decreasing.

Our next result shows the consequences of I for Corollary 3.2. We write

(4.1) $\quad \nu = \lambda'^j, \delta = \rho^i(\nu)$

and we set

$$\operatorname{sgn} \lambda = 1, 0 \text{ or } -1 \text{ when } \lambda > 0, = 0 \text{ or } < 0,$$

respectively.

THEOREM 4.1. *Assuming* I, $\operatorname{sgn}(\lambda^i - \nu) = \operatorname{sgn} \delta$.

Proof. If $\delta < 0$ then from (3.4) we obtain $v(\nu, \bar{u}) < v(\lambda^i, \bar{u})$ since $\rho^i(\lambda^i) = 0$, so I yields $\lambda^i < \nu$.

If $\delta > 0$ then we replace (3.4) by

(4.2) $\quad \nu(\lambda^i, u) - v(\nu, u) \leqq -\delta$

which is obtained from (3.2) by setting $\lambda = \nu$ and $\mu = \lambda^i$, and by choosing a suitable u to replace $\bar{u}$. (4.2) gives $v(\lambda^i, u) < v(\nu, u)$, and I now yields $\lambda^i > \nu$.

Finally, if $\delta = 0$ then (3.4) gives $\lambda^i \not> \nu$. On the other hand, $\lambda^i \not< \nu$ follows from (4.2), so we obtain $\lambda^i = \nu$ as required.

https://doi.org/10.4153/CJM-1981-018-4 Published online by Cambridge University Press

If the word "strictly" is omitted from I, then we still obtain $\delta(\lambda^i - \nu) \geqq 0$, as the reader will easily verify. The method above also establishes uniqueness of λ^i, in the sense that $(\rho^i)^{-1}(0)$ is at most a singleton. Indeed, if $\rho^i(\lambda) = \rho^i(\mu) = 0$ with $\lambda > \mu$, then (3.2) and I yield $\lambda \leqq \mu$, a contradiction. Of course, x^i need not be unique, since dim ker $W(\lambda^i)$ can be any positive integer. The condition

$$\rho^{i-1}(\lambda) < \rho^i(\lambda) < \rho^{i+1}(\lambda) \quad \text{for all } \lambda$$

would guarantee uniqueness for x^i.

Our next task is to give "uniform" versions of Assumption I and Theorem 4.1, and we shall employ (3.1) for this.

Assumption I_u. $\alpha(\lambda, \mu) < 0$ whenever $\lambda > \mu$.

LEMMA 4.2. *Assuming* I_u, $\alpha(\lambda, \mu)$ *is strictly decreasing [increasing] in* $\lambda[\mu]$ *for each fixed* $\mu[\lambda]$. *Further, there exist increasing functions* β_μ *and* γ_λ *so that*

$$\beta_\mu(0) = \gamma_\lambda(0) = 0,\ \lambda - \mu = \beta_\mu(-\alpha(\lambda, \mu)) \quad \textit{and} \quad \mu - \lambda = \gamma_\lambda(\alpha(\lambda, \mu)).$$

Proof. The triangle inequalities

$$\alpha(\lambda, \mu) - \alpha(\lambda', \mu) \leqq \alpha(\lambda, \lambda') \quad \text{and} \quad \alpha(\lambda, \mu) - \alpha(\lambda, \mu') \leqq \alpha(\mu', \mu)$$

follow easily from (3.1). These establish the monotonicity assertions about $\alpha(\lambda, \mu)$. Thus

$$\text{(4.3)} \qquad \varphi_\mu: \lambda \rightarrow -\alpha(\lambda + \mu, \mu) \quad \text{and} \quad \psi_\lambda: \mu \rightarrow \alpha(\lambda, \lambda + \mu)$$

are both strictly increasing functions. We now pick $\beta_\mu = \varphi_\mu{}^{-1}$ and $\gamma_\lambda = \psi_\lambda{}^{-1}$.

Theorem 4.1 gives a direction for the movement of eigenvalues under perturbation. We can now deduce bounds for that movement; ν and δ are as defined in (4.1).

COROLLARY 4.3. *Assuming* I_μ, $\gamma_\nu(\delta) \leqq \lambda^i - \nu \leqq \beta_\nu(\delta)$.

Proof. We consider only the case $\delta \neq 0$, $\delta = 0$ having already been treated in Theorem 4.1. From (3.4) we have

$$\text{(4.4)} \qquad -\alpha(\lambda^i, \nu) \leqq \delta,$$

and using Lemma 4.2, we can apply the increasing function β_ν to give

$$\lambda^i - \nu \leqq \beta_\nu(\delta).$$

Likewise we may deduce

$$\text{(4.5)} \qquad \alpha(\nu, \lambda^i) \geqq \delta$$

https://doi.org/10.4153/CJM-1981-018-4 Published online by Cambridge University Press

from (4.2). Applying γ_ν, we obtain

$$\lambda^i - \nu \geqq \gamma_\nu(\delta)$$

as required.

We point out that if we premultiply (4.4) and (4.5) by -1 first, then

$$\gamma_\lambda{}^i(-\delta) \leqq \nu - \lambda^i \leqq \beta_\lambda{}^i(-\delta) \tag{4.6}$$

results. Of course, further bounds can be obtained in terms of β_ν', etc., if the roles of λ^i and ν are interchanged.

Various parametric dependence results now follow easily. We suppose that W is also a function of a parameter ϵ from a set E, but that $(W(\lambda, \epsilon))$ is independent of ϵ; the latter condition can be relaxed to some extent but we shall not pursue this. Here and below we shall interpret the unprimed system (1.1) at ϵ, and the primed system at $\epsilon' \in E$.

First let E be partially ordered. If $W(\lambda, \epsilon)$ increases with ϵ, then Theorem 3.1 (ii) shows that $\rho^i(\lambda, \epsilon)$ increases with ϵ, so $\lambda^i(\epsilon)$ increases with ϵ, by virtue of Theorem 4.1.

Next, let E be a topological space.

Definition 4.4. W is *continuous in* ϵ if for each λ and each $\epsilon, \epsilon' \in E$, $\omega_\lambda(\epsilon, \epsilon') = \|W(\lambda) - W'(\lambda)\|$ is finite and tends to zero as $\epsilon' \to \epsilon$.

COROLLARY 4.5. *If W is continuous in ϵ, and if I_u holds uniformly in ϵ, i.e.,*

$$\sup_\epsilon \alpha(\lambda, \mu, \epsilon) < 0 \textit{ if } \lambda > \mu,$$

then λ^i is continuous in ϵ for each i.

Proof. We fix i and suppress the superscript. Setting $i = j$ in Corollary 4.3 with $\lambda = \lambda(\epsilon)$ and $\nu = \lambda(\epsilon')$ we have

$$|\lambda - \nu| \leqq \max\{|\beta_\nu(\delta)|, |\gamma_\nu(\delta)|\}, \tag{4.7}$$

where $\delta = \rho(\nu) - \rho'(\nu)$ comes from (4.1). Thus

$$|\delta| \leqq \omega_\nu(\epsilon, \epsilon')$$

follows from Theorem 3.1 (ii).

It suffices, then, to show that β_ν and γ_ν are continuous at 0, uniformly in ϵ. But this follows because the strict monotonicity of the functions φ_ν and ψ_ν (4.3) is uniform in ϵ.

Now let E possess a metric d. We define W to be *Lipschitz* in ϵ if

$$\omega_\lambda(\epsilon, \epsilon')/d(\epsilon, \epsilon') \leqq \zeta_\lambda \text{ for all } \epsilon \neq \epsilon'.$$

Corollary 4.5 easily extends to show that λ^i is Lipschitz in ϵ if one strengthens I_u to

https://doi.org/10.4153/CJM-1981-018-4 Published online by Cambridge University Press

Assumption I_L. There exists $\theta > 0$ so that $\alpha(\lambda, \mu, \epsilon) < \theta(\mu - \lambda)$ whenever $\lambda > \mu$.

Finally we point out that (4.6) may be used instead of Corollary 4.3. This leads to analogous results where I_u [or I_L] need not be uniform in ϵ, but the continuity of W in ϵ [or the Lipschitz constant ζ_λ] must be uniform in λ.

5. One parameter existence theory. So far we have a uniqueness and monotonic dependence theory for (1.1) from Assumption I via Theorem 4.1, and a continuous dependence theory from Assumption I_u via Corollary 4.5. Easy examples, given at the end of this section, show, however, that we still may not have existence of eigenpairs. To that end we introduce further assumptions.

Assumption II. For some u^*, $v(\lambda, u^*)v(-\lambda, u^*) \to -\infty$ as $|\lambda| \to \infty$.

Assumption III_u. $|\nu(\lambda, u)| \to \infty$ as $|\lambda| \to \infty$, uniformly in u.

LEMMA 5.1. *Assuming* II *and* III_u, $v(\lambda, u)v(-\lambda, u) \to -\infty$ *as* $|\lambda| \to \infty$, *for each* u.

Proof. Suppose, for some u, that $v(\lambda, u^*)v(\lambda, u) \to -\infty$ as $\lambda \to +\infty$ (say). From continuity of $v(\lambda, \ )$ it follows that there is $u(\lambda)$ satisfying $v(\lambda, u(\lambda)) = 0$. This, however, contradicts III_u.

Thus $v(\ , u)$ has the same sign as $\lambda \to \pm\infty$ for each u, and II now completes the argument.

In line with our earlier analysis where the $v(\ , u)$ were decreasing, we shall assume for simplicity that $v(\lambda, u) \to \mp\infty$ as $\lambda \to \pm\infty$. The alternative case is completely analogous.

Assumption IV_u. $v(\lambda, u)$ is continuous in λ, uniformly in u.

Equivalently, $V(\lambda)$ is norm continuous in λ. We are now ready for our existence results.

THEOREM 5.2. *Assuming* II, III_u *and* IV_u, (1.1) *has at least one eigenpair* (λ^i, x^i) *for each* i.

Proof. If suffices to show that ρ^i is continuous and that $\rho^i(\lambda) \to \mp\infty$ as $\lambda \to \pm\infty$. These conclusions all follow from (3.2), by virtue of our remarks following Lemma 5.1.

Indeed

$$|\rho^i(\lambda) - \rho^i(\mu)| \leqq \sup_u |v(\lambda, u) - v(\mu, u)|$$

demonstrates continuity of ρ^i. Further, fixing μ in (3.2), we obtain

$$\rho^i(\lambda) \leqq v(\lambda, \bar{u}) + \rho^i(\mu) + \|V(\mu)\|$$

and the right side tends to $-\infty$ as $\lambda \to +\infty$. Finally,

$$\rho^i(\mu) \geqq v(\mu, \bar{u}) + \rho^i(\lambda) - \|V(\lambda)\|$$

also follows from (3.2) and, with λ fixed, the right side tends to $+\infty$ as $\mu \to -\infty$.

The examples below show that uniqueness cannot be guaranteed. We therefore combine the above work with that of Section 4.

COROLLARY 5.3. *Assuming* I, III_u *and* IV_u, (1.1) *has exactly one eigenvalue* λ^i *for each* i, *and any monotonic parametric dependence in* W *is reflected in* λ^i. *In particular*, λ^i *increases with* i.

Proof. If II fails then $v(\lambda, u^*)v(-\lambda, u^*) \to +\infty$ as $\lambda \to +\infty$ by virtue of III_u. This patently contradicts I. Thus II holds, and we may apply Theorems 4.1 and 5.2.

Of course, the λ^i are difficult to compute this way, since knowledge of each ρ^i function requires a continuum of maximinimisations of the form (2.1). The situation can be alleviated by means of the following construction.

COROLLARY 5.4, *Under the assumptions of Corollary* 5.3, *for each* $u \in (T)$ *there exists a unique* $\lambda(u)$ *satisfying* $w(\lambda(u), u) = 0$, *and furthermore*

$$\lambda^i = \max\{\min\{\lambda(u) : u \in \mathscr{D}(T), u \perp y_j\} : y_j \in H, 1 \leqq j \leqq i\}.$$

Proof. Under the stated conditions, existence of $\lambda(u)$ is trivial. Fix $y_1, \ldots, y_i \in H$, let O_y be their orthogonal complement, and define

$$H_y = O_y \cap \mathscr{D}(T).$$

Suppose for the moment that

$$\lambda^i(y) = \min\{\lambda(u) : u \in H_y\}$$

exists. Now (2.1) shows that

$$w(\lambda^i, u) \leqq \rho^i(\lambda^i) = 0 \tag{5.1}$$

for some $u \in H_y$. I and the definition of $\lambda(u)$ give $\lambda(u) \leqq \lambda^i$, so

$$\lambda^i(y) \leqq \lambda(u) \leqq \lambda^i \tag{5.2}$$

follows by definition of $\lambda^i(y)$. On the other hand, if we choose the y_j as the first i eigenvectors of $W(\lambda^i)$, then Rayleigh's principle gives

$$w(\lambda^i, u) \geqq 0 \tag{5.3}$$

for each $u \in H_y$. Thus $\lambda^i(y) \geqq \lambda^i$, which combined with (5.2) yields the desired result.

It remains to establish the existence of $\lambda^i(y)$, for given $y_1, \ldots, y_i$. Let P_y denote the orthogonal projection onto O_y, and define $V_y(\lambda)$ and $W_y(\lambda)$ as the restrictions of $P_y V(\lambda) P_y$ and $P_y W(\lambda) P_y$ to O_y. Since $(u, V_y(\lambda)u) = v(\lambda, u)$ for $u \in O_y$, $W_y(\lambda)$ satisfies the same conditions as $W(\lambda)$, so we may repeat the analysis for W_y instead. This leads to an eigenvalue $\lambda_y{}^i$ and eigenvector $u_y{}^i$ such that

$$(5.4) \qquad w_y(\lambda_y{}^i, u_y{}^i) = 0 \leqq w_y(\lambda_y{}^i, u)$$

for all $u \in H_y$; note that $H_y \subseteq \mathscr{D}(W_y(\lambda))$ for any λ. Since $P_y u = u$ for all $u \in H_y$, (5.4) yields

$$(5.5) \qquad w(\lambda_y{}^i, u_y{}^i) = 0 \leqq w(\lambda_y{}^i, u)$$

whence

$$\lambda_y{}^i = \lambda(u_y{}^i) \leqq \lambda(u)$$

for all $u \in H_y$. Thus $\lambda_y{}^i = \lambda^i(y)$.

We defer the combination of Theorem 5.2 with Assumption I_u until the next section, and continue with examples illustrating the roles played by some of our assumptions. Let $H = l_2$ with orthonormal basis $e_1, e_2, \ldots$.

Example 5.5. Let $V(\lambda)e_m = (\lambda - (\lambda)^3)e_m$, $Te_m = (m - 1)e_m$, $m = 1, 2, \ldots$. Then the conditions of Theorem 5.2 are satisfied, but those of Corollary 5.3 are not, since I fails. Eigenvalues λ^i may not be unique, e.g. $\lambda^0 = -1$, 0 or 1.

Example 5.6. Let $V(\lambda)e_m = e^{-\lambda}e_m$, $Te_m = me_m$, $m = 1, 2, \ldots$. Then all the conditions of Corollary 5.3 are satisfied except III_u. Nevertheless, (1.1) has no solutions. Indeed, $\rho^i(\lambda) = i + 1 + e^{-\lambda}$ follows easily from (2.1), so $\rho^i(\lambda) = 0$ has no roots.

Example 5.7. Let $V(\lambda)e_m = (-\lambda - 2 \operatorname{sgn} \lambda)e_m$, $Te_m = me_m$, $m = 1, 2, \ldots$. Then all the conditions of Corollary 5.3 are satisfied except IV_u. This time each ρ^i is discontinuous, and $\rho^0(\lambda) = 1 - \lambda - 2 \operatorname{sgn} \lambda$, so λ^0 does not exist.

Example 5.8. If $\lambda < 0$, let $V(\lambda)e_1 = -\lambda e_1$, $V(\lambda)e_m = -(\lambda)^2 e_m$, $m = 2, 3, \ldots$
If $\lambda \geqq 0$, let $V(\lambda)e_m = -\lambda e_m$, $Te_m = (m - 2)e_m$, $m = 1, 2, \ldots$

Thus, if $(u, e_1) = \eta$, then

$$\begin{aligned} \nu(\lambda, u) &= -\lambda \quad \text{if } \lambda \geqq 0 \\ &= -\lambda\eta^2 - (\lambda)^2(1 - \eta)^2 \quad \text{if } \lambda < 0. \end{aligned}$$

Therefore, II (with $u^* = e_1$) and IV_u are clearly satisfied.

Also, it is readily seen that $|v(\lambda, u)| \to \infty$ as $|\lambda| \to \infty$. The limit is not uniform in u, however, so we cannot apply Theorem 5.2. Indeed,

$$w(\lambda, e_2) = (\lambda)^2 < 0 \text{ if } \lambda < 0$$
$$w(\lambda, e_1) = -1 - \lambda < 0 \text{ if } \lambda \geqq 0.$$

Thus $\rho^0(\lambda) < 0$ for all λ, and so again λ^0 does not exist.

In connection with this example, we should perhaps point out that II guarantees $\rho^0(\lambda) \to -\infty$ as $\lambda \to +\infty$ (say). The need for uniformity is then only in $v(\lambda, u) \to +\infty$ as $\lambda \to -\infty$. Also, we are using infinite limits primarily for ease of generalisation to $k > 1$. It is enough, of course, to ensure in Theorem 5.2 that $\rho^i(\lambda)$ takes both signs. Relatively sharp hypotheses, involving T explicitly, are easily given.

6. Equivalent assumptions. If, in Corollary 5.3, we strengthen I to I_u $[\mathrm{I}_L]$ then we obtain continuous [Lipschitz] parametric dependence as well; see the end of Section 4. Since this is the version of the theory that we shall extend to $k > 1$, we now give an alternative set of assumptions possessing more obvious multiparameter analogues.

Assumption I_u'. Either *Assumption* I_u holds, or else $\alpha(\mu, \lambda) < 0$ whenever $\lambda > \mu$.

The second alternative in I_u' corresponds to $v(\lambda, u)$ increasing in λ, "uniformly" in u.

Assumption V_u. $v(\ \ , u)$ is a homeomorphism of $\mathbf{R}^1$, and the continuity of $v(\ \ , u)^{-1}$ is uniform in u.

Most of this section will be devoted to showing that III_u, IV_u and V_u are essentially equivalent to the assumptions used previously; we merely have to replace I_u, by I_u'.

THEOREM 6.1. III_u, IV_u *and* V_u *are equivalent* to I_u', III_u *and* IV_u.

Proof. Suppose that I_u, III_u and IV_u hold. Evidently $v(\ \ , u)$ is a homeomorphism for each u, so it remains to establish the uniformity condition. If this fails for continuity on the left then for some $\epsilon > 0$ and q there exist u_m and $q_m \uparrow q$ so that

$$(6.1) \qquad v(\ \ , u_m)^{-1}(q_m) > v(\ \ , u_m)^{-1}(q) + \epsilon.$$

Let $v(\lambda_m, u_m) = q$ and $v(\mu_m, u_m) = q_m$, so (6.1) can be rewritten

$$(6.2) \qquad \mu_m > \lambda_m + \epsilon.$$

Observe that the λ_m are bounded by virtue of III_u. Thus by passing to a subsequence, if necessary, we may assume that $\lambda_m \to \lambda$, say. Let $v(\lambda, u_m) = p_m$, so $p_m \to q$ by virtue of IV_u. Evidently

$$\liminf \mu_m \geqq \lambda + \epsilon$$

from (6.2), yet

$$\alpha(\mu_m, \lambda) \geqq v(\mu_m, u_m) - v(\lambda, u_m) = q_m - p_m \to 0$$

as $m \to \infty$.

This contradicts I_u, so the continuity of the $v(\ \ , u)^{-1}$ on the left is uniform in u. Continuity on the right is handled similarly, as is the second alternative in Assumption I_u'.

For the converse, we note that V_u implies II. Thus III_u and the argument of Lemma 5.1 show that each $v(\ \ , u)$ is strictly monotonic with the same sense, and it remains to prove that V_u implies the uniformity condition in I_u'. If this uniformity fails then, for some λ and μ with $\lambda \neq \mu$, there exist u_m so that, with $v(\lambda, u_m) = p_m$ and $v(\mu, u_m) = q_m$,

$$q_m - p_m \to 0 \tag{6.3}$$

as $m \to \infty$. Noting that the $|q_m|$ are bounded by $\|V(\mu)\|$, we may assume (by passing to a subsequence if necessary) that $q_m \to q$, say.

Now if we define λ_m by $v(\lambda_m, u_m) = q$, then V_u yields $\lambda_m \to \mu$. Thus

$$v(\ \ , u_m)^{-1}(q) - v(\ \ , u_m)^{-1}(p_m) = \lambda_m - \lambda$$

has limit $\mu - \lambda$, hence is bounded away from zero, yet $p_m \to q$ from (6.3). This contradicts V_u, so I_u is established.

COROLLARY 6.2. *Assuming* III_u, IV_u *and* V_u, (1.1) *has exactly one eigenvalue* λ^i *for each* i, *and parametric dependence of* W *is reflected in* λ^i *as at the end of Section* 4.

This is, of course, just a combination of Theorem 6.1 with Corollary 5.3 and the end of Section 4; see Corollary 4.5 for an explicit continuous dependence result.

For the purposes of the next section, we shall restate our assumptions for general k in terms of the following.

Definition 6.3. $v(\ \ , u)$ is a *homeomorphism of* $\mathbf{R}^k$ *uniformly in* u if (*i*) the continuity of $v(\ \ , u)$ (*ii*) the continuity of $v(\ \ , u)^{-1}$ and (*iii*) the limit of $\|v(\lambda, u)\|$ as $\|\lambda\| \to \infty$ are all uniform in u.

We point out that the three uniformity assumptions here are independent, as one may show by means of simple examples.

7. The general case. We are now in a position to tackle (1.1) for $k > 1$. Roughly, we shall solve $W_1(\lambda)x_1 = 0$ for λ_1 in terms of $\lambda_2, \ldots, \lambda_k$. Then we substitute for λ_1 into $W_2(\lambda)x_2 = 0$ and solve for λ_2 in terms of $\lambda_3, \ldots, \lambda_k$, and so on. It will suffice to carry out these first two steps in order to indicate what is involved. The formal inductive step (from λ_n to λ_{n+1}) would merely require more notation. We state the basic result in terms of Definition 6.3.

THEOREM 7.1. *If $v(\ \ , u)$ is a homeomorphism uniformly in u then* (1.1) *has exactly one eigenvalue λ^i for each i.*

Proof. Step 1. We may assume that, for some $u^* = (u_1^*, \ldots, u_k^*)$,

$$v(\lambda, u^*) = -\lambda. \tag{7.1}$$

In order to see this, we fix any u^*, and consider the preliminary transformation $\lambda \rightarrow \lambda^*$, where

$$\lambda^* = -v(\lambda, u^*).$$

We then define operators V_n^* by

$$V_n^*(\lambda^*) = V_n(\lambda) \quad 1 \leqq n \leqq k$$

and we obtain

$$v^*(\lambda^*, u^*) = -\lambda^*.$$

For convenience we shall suppress the asterisks (except on u^*).

Step 2. If $\hat{v}_1$ is defined by

$$\hat{v}_1(\lambda_1, u_1) = v_1(\lambda, u), \tag{7.2}$$

then $\hat{v}_1(\ \ , u_1)$ satisfies Definition 6.3 for $k = 1$, for each fixed $\lambda_2, \ldots, \lambda_k$. (Note that $v_1(\lambda, u)$ depends on u_1 but not on $u_2, \ldots, u_k$.)

Indeed, consider $y = (u_1, u_2^* \ldots, u_k^*)$. Evidently

$$\hat{v}_1(\lambda_1, u_1) = v_1(\lambda, y) \tag{7.3}$$

and

$$-\lambda_n = v_n(\lambda, y) \quad 2 \leqq n \leqq k. \tag{7.4}$$

Since $\lambda \leftrightarrow v(\lambda, y)$ is a homeomorphism uniformly in y, so is $\lambda \leftrightarrow (\hat{v}_1(\lambda_1, u_1), -\lambda_2, \ldots, -\lambda_k)$, uniformly in u_1. Thus so is $\lambda_1 \leftrightarrow \hat{v}_1(\lambda_1, u_1)$, uniformly in u_1.

Step 3. $\hat{v}_1$ satisfies I_u uniformly in $\lambda_2, \ldots, \lambda_k$; see Corollary 4.5, at least for convergent sequences $\lambda_{2m}, \ldots, \lambda_{km}$.

In order to see this, note that

$$\hat{v}_1(\lambda_1, u_1^*) = -\lambda_1$$

follows from (7.1) and (7.2). By virtue of Lemma 5.1, we may replace u_1^* by arbitrary u_1, so each $\hat{v}_1(\ \ , u_1)$ decreases. Thus Theorem 6.1 shows that I_u is satisfied. If $\hat{\alpha}_1$ (defined for $\hat{v}_1$ analogously to (3.1)) is not uniform as stated, then there exist convergent sequences λ_{nm} $(2 \leqq n \leqq k)$ so that for some $y_m = (u_{1m}, u_2^*, \ldots, u_k^*)$ and $\lambda, \mu \in \mathbf{R}$ with $\lambda > \mu$,

$$\lim_{m \to \infty} v_1(\lambda, \lambda_{2m}, \ldots, \lambda_{km}, y_m) - v_1(\mu, \lambda_{2m}, \ldots, \lambda_{km}, y_m) = 0.$$

But the $v(\quad, y_m)$ are homeomorphisms uniformly in m, so (7.3) and (7.4) yield

$$\lim_{m\to\infty}(\lambda, \lambda_{2m}, \ldots, \lambda_{km}) - (\mu, \lambda_{2m}, \ldots, \lambda_{km}) = 0,$$

which is absurd.

Step 4. For each $\lambda_2, \ldots, \lambda_k$ and i_1 there exists a unique $\lambda_1 = \lambda_1^{i_1}(\lambda_2, \ldots, \lambda_k)$, continuously dependent on $\lambda_2, \ldots, \lambda_k$ and satisfying $\rho_1^{i_1}(\lambda) = 0$.

Existence and uniqueness follow from Corollary 5.3, Theorem 6.1 and Step 2. Continuous dependence comes from Step 3 and Corollary 4.5.

Step 5. If $\lambda_3, \ldots, \lambda_k$ and i_1 are fixed, if λ_1 and λ_1' correspond via Step 4 to λ_2 and λ_2' and if we write $\lambda' = (\lambda_1', \lambda_2', \lambda_3, \ldots, \lambda_k)$, then there exists $u_1 = u_1(\lambda_2, \lambda_2')$ so that

$$(7.5) \qquad v_1(\lambda, u_1) = v_1(\lambda', u_1).$$

In order to prove this, we set $W(\lambda_1) = W_1(\lambda)$, $W'(\lambda_1') = W_1(\lambda')$, $\lambda = \lambda_2$ and $\mu = \lambda_2'$ in (3.3) which gives, for some $\hat{u}_1$,

$$0 \leqq v_1(\lambda, \hat{u}_1) - v_1(\lambda', \hat{u}_1).$$

In a similar way, with $\lambda = \lambda_2'$ and $\mu = \lambda_2$, we may find a u_1' so that

$$0 \leqq v_1(\lambda', u_1') - v_1(\lambda, u_1').$$

(7.5) now follows because

$$v_1(\lambda, u) - v_1(\lambda', u)$$

is continuous in u.

Step 6. If

$$(7.6) \qquad \hat{v}_2(\lambda_2, u_2) = v_2(\lambda_1^{i_1}(\lambda_2, \ldots, \lambda_k), \lambda_2, \ldots, \lambda_k, u)$$

then $\hat{v}_2$ satisfies Definition 6.3 for $k = 1$, for each fixed $\lambda_3, \ldots, \lambda_k$ and i_1.

Indeed, consider λ_2 and λ_2' as in Step 5, with corresponding λ, λ' and u_1. If $z = (u_1, u_2, u_3^* \ldots, u_k^*)$, where u_2 is arbitrary, then

$$(7.8) \qquad v_1(\lambda, z) = v_1(\lambda', z)$$

$$(7.8) \qquad v_2(\lambda, z) = \hat{v}_2(\lambda_2, u_2), v_2(\lambda', z) = \hat{v}_2(\lambda_2', u_2)$$

and

$$(7.9) \qquad v_n(\lambda, z) = -\lambda_n \quad 3 \leqq n \leqq k$$

using (7.5), (7.6) and (7.1) in turn.

We therefore obtain

$$|\hat{v}_2(\lambda_2, u_2) - \hat{v}_2(\lambda_2', u_2)| = \|v(\lambda, z) - v(\lambda', z)\|$$

and the desired conclusion follows because $v(\quad, z)$ is a homeomorphism uniformly in z.

Step 7. For each $\lambda_3, \ldots, \lambda_k$, i_1 and i_2 there exists a unique λ_2, continuously dependent on $\lambda_3, \ldots, \lambda_k$, and satisfying $\rho_2^{i_2}(\lambda) = 0$.

This is almost a repeat of Steps 3 and 4. Existence and uniqueness come from Corollary 5.3, Theorem 6.1 and Step 6. The analogue of Step 3, that $\hat{v}_2$ satisfies I_u uniformly for convergent sequences $\lambda_{3m}, \ldots, \lambda_{km}$, is proven via (7.7)–(7.9) instead of (7.3) and (7.4). Corollary 4.5 again completes the proof.

We remark that the analogue of Step 5, for proving existence of λ_n, involves $n - 1$ equations of the form (7.5).

Various parametric dependence results follow by combining Theorem 7.1 with the end of Section 4. For example, if the continuity of $v(\quad, u, \epsilon)^{-1}$ is uniform in u and ϵ, or if the continuity of $w(\lambda, u, \epsilon)$ in ϵ is uniform in λ, then λ^i is continuous in ϵ. Lipschitz dependence is similar. For monotonic dependence see Corollary 3.4, and note also that each $\hat{v}_n$ satisfies I_u. Thus the full apparatus of Section 4 is available for the dependence of λ_n on parameters affecting $\hat{v}_n$; e.g. λ_n^i increases with i_n.

Finally, we may generalise Corollary 5.4 to $k > 1$ as follows. Recall the definition (3.6) of the sets $C_\sigma(\lambda)$; we shall fix $\sigma = (1, 1, \ldots, 1)$ and suppress it.

COROLLARY 7.2. *Under the conditions of Theorem* 7.1, *for each* u *with* $u_n \in (T_n)$, *there exists* $\lambda(u)$ *such that* $w(\lambda(u), u) = 0$. *Further, for each set* y *of* $y_{nj} \in H_n$, $1 \leqq j \leqq i_n$, $1 \leqq n \leqq k$, *there exists* $\lambda^i(y)$ *with the following properties.*

(a) $\lambda(u) \in C(\lambda^i(y))$ *whenever* $u_n \in (T_n)$, $u_n \perp y_{nj}$, $1 \leqq j \leqq i_n$, $1 \leqq n \leqq k$.

(b) $\lambda(u) = \lambda^i(y)$ *for some* $u = u(y)$.

(c) $\lambda^i \in C(\lambda^i(y))$ *for all choices of* y.

(d) $\lambda^i = \lambda^i(y)$ *for some* y.

Proof. Existence of $\lambda(u)$ is trivial. Essentially we now apply Corollary 5.4 to each component W_n in turn. From (5.5) we obtain

$$w(\lambda^i, u_y^{\,i}) = 0 \leqq w(\lambda_y^{\,i}, u) \tag{7.10}$$

for all u as in (a), for some $\lambda_y^{\,i} \in \mathbf{R}^k$ and for some $u_y^{\,i}$ satisfying the conditions on u in (a). Now set $\lambda^i(y) = \lambda_y^{\,i}$ and $u(y) = u_y^{\,i}$ to give (b). (a) follows from (7.10) and $w(\lambda(u), u) = 0$.

Likewise, (5.1) yields

$$w(\lambda^i, u) \leqq 0$$

for all u as in (a), and this, with (7.10), gives (c). Finally (d) follows from setting y_{nj} as the jth eigenvector of $W_n(\lambda^i)$.

8. Notes and remarks. Historically, the method of reducing (1.1) for $k > 1$ to a sequence of one parameter problems was used first. Klein [**11**],

https://doi.org/10.4153/CJM-1981-018-4 Published online by Cambridge University Press

and more analytically Bôcher [5], solved Lamé's equation (for $k = 2$) this way. Bôcher [6] extended this analysis to general k, and, as Ince [10, § 10.9] remarks, it works for the system

$$y_n''(x_n) + q_n(x_n)y_n(x_n) + \sum_{m=1}^{k} y_n(x_n)a_{nm}(x_n)\lambda_m = 0, \quad a_n \leqq x_n \leqq b_n,$$

of Sturm-Liouville equations with continuous coefficients, provided

(8.1) $\det [a_{nm}(x_n)]_{m,n=1}^{k} > 0$

for all possible values of x_n.

One parameter problems, both linear and non-linear, have been treated by real variable methods many times, the emphasis being on existence and uniqueness rather than on parametric dependence. Prüfer's (polar co-ordinate) transformation was, for example, used by Tal [15], Estabrooks and Macki [9] and Sleeman [12] under assumptions related to ours. (In fact Sleeman discussed a Sturm-Liouville problem with three boundary conditions and two parameters. He extended this to two coupled equations in [13], under assumptions and methods broadly but not directly comparable with ours.)

Variational methods have been used for non-linear one parameter problems before: see [17] and the references there for a bibliography. The approach seems to have been via the λ function of Corollary 5.4 rather than the ρ^i functions used here. (Turner [17] in fact treats non-self-adjoint polynomial eigenvalue problems mostly by non-variational methods. His earlier work [16] on a quadratic eigenvalue problem from hydrodynamics is closer to our setting, and will be discussed further at the end.)

Variational methods for $k > 1$ have, with one exception that I know of, been used in conjunction with a problem reformulation which replaces $\bigoplus_{n=1}^{k} H_n$ by $\bigotimes_{n=1}^{k} H_n$ (in the Sturm-Liouville case, ordinary de are replaced by partial de). We shall show that our results here include the corresponding ones in [3], which is the exception noted above. The assumptions used here have more in common with [2] than [3], however, and since these two works are felt to be the closest to the one at hand, we shall employ extra assumptions to facilitate a more detailed comparison.

Assumption V$_l$.

$$V_n(\lambda) = \sum_{m=1}^{k} V_{nm}\lambda_m \quad \text{and}$$

$$\inf_u |\det [(u_n, V_{nm}u_m)]_{m,n=1}^{k}| > 0.$$

This, of course, generalises (8.1).

Assumption V$_d$. $v(\lambda, u)$ has a non-singular partial derivative

$\partial v(\lambda, u)/\partial\lambda$, jointly continuous in λ and u, and $\|[\partial v(\lambda, u)/\partial\lambda]^{-1}\| \leqq f(\|\lambda\|)$ for all λ and u, where f is nondecreasing and $\int^\infty dr/f(r)$ diverges.

We shall need the following relation beteeen V_d and the assumptions used here.

LEMMA 8.1. *V_d implies that $v(\ \ , u)$ is a homeomorphism uniformly in u.*

Proof. This is a slight extension of [**2**, Corollary 2] where it is shown that V_d implies that each $v(\ \ , u)$ is a homeomorphism satisfying the analogue of III_u for $k > 1$. (As the referee has pointed out, the fact that $v(\ \ , u)$ is a homeomorphism is proved in [**17**, p. 222]. This problem was first discussed in 1906 by Hadamard.) IV_u is obvious, so it remains to establish V_u.

For fixed u, b and c let

$$v(p(s), u) = (1 - s)b + sc, \quad 0 \leqq s \leqq 1.$$

As in [**2**, Corollary 2], the chain rule gives

$$p'(s) = [\partial v(p(s), u)/\partial\lambda]^{-1}(c - b)$$

whence

$$d\|p(s)\|/ds \leqq f(\|p(s)\|)\|c - b\|.$$

Setting $b = 0$ we obtain

$$\int_0^{\|p(1)\|} dr/f(r) \leqq \|c\|$$

so $\|v(\ \ , u)^{-1}(c)\|$ is bounded in terms of $\|c\|$, independently of u.

Now let $c_m \to c$ and take $b = c_m$. Then

$$\|p(1) - p(0)\| \leqq \int_0^1 f(\|p(s)\|)ds\|c_m - c\|.$$

By our earlier remarks, the integral is bounded independently of u and m. It follows therefore that $\|v(\ \ , u)^{-1}(c_m) - v(\ \ , u)^{-1}(c)\|/\|c_m - c\|$ is bounded independently of u.

Using the above, we shall consider [**3**] first, where V_l is assumed. It is shown in [**2**] that V_l implies V_d (actually, that the assumptions are equivalent in the linear case). Thus Lemma 8.1 shows that this study includes the corresponding results in [**3**]. Specifically, Theorem 7.1 and the remarks that follow include [**3**, Theorems 2, 3, 9, 11] and the part of [**3**, Theorem 10] dealing with eigenvalues. While our proof of Theorem 7.1 is modelled on that of [**3**, Theorem 2], we have extended the latter in various ways and also made some simplifications.

Next we consider [**2**] under V_d. In this case [**2**, Corollary 2] corresponds to the existence part of Theorem 7.1, so we extend [**2**] in this respect.

Indeed, a prime motivation for this study was a lack of uniqueness and dependence theory in [**2**]. Of course, both [**2**] and the analysis here give results under weaker assumptions than V_d. When $k = 1$, it turns out that the existence results are essentially equivalent, and moreover, we do not require degree theory here. When $k > 1$, however, [**2**, Theorem 1] proves existence under weaker conditions than those used here.

As mentioned in the introduction, our conditions on T_n and $V_n(\lambda)$ include Sturm-Liouville systems and difference equation analogues. They do not, however, include integral equations, nor cases where the $V_n(\lambda)$ are unbounded (e.g. when they contain differentiations or unbounded coefficients). We assume that each T_n is positive definite. (Note that under the conditions of Theorem 7.1 we can shift the λ origin by $v(\ \ , u^*)^{-1}(r, r, \ldots, r)$ for sufficiently large r, to ensure that the T_n are indeed positively bounded below.) It follows that (1.1) may be rewritten

$$(8.2) \qquad A_n(\lambda)x_n = x_n \neq 0, \quad A_n(\lambda) = -T_n^{-1/2}V_n(\lambda)T_n^{-1/2}$$

where $\mathscr{D}(A_n(\lambda)) \supseteq \mathscr{D}(V_n(\lambda))$. It is realistic to assume that the $A_n(\lambda)$ are compact. Then with the appropriate continuity conditions in λ, the problem may be solved in the form $\alpha_n^{i_n}(\lambda) = 1$, $1 \leqq n \leqq k$, where $\alpha_n^{i_n}(\lambda)$ is the i_n-th eigenvalue of $A_n(\lambda)$.

(Alternatively it may be possible to make definiteness assumptions on the $V_n(\lambda)$ and analyse

$$(8.3) \qquad B_n(\lambda)x_n = x_n \neq 0,$$

on the assumption that $B_n(\lambda) = -V_n(\lambda)^{-1/2}T_nV_n(\lambda)^{-1/2}$ has compact resolvent.)

When T_n^{-1} is compact and $V_n(\lambda)$ is bounded, it is easily shown that $A_n(\lambda)$ is compact. This situation has been analysed by Sleeman [**14**] in the form (8.2), via the $\bigotimes_{n=1}^k H_n$ reformulation mentioned earlier. Turner's quadratic eigenvalue problem for $k = 1$ [**16**] was analysed via (8.2); in this case $V(\lambda)$ was also definite so (8.3) could have been used. $A(\lambda)$ was compact because $V(\lambda)^{1/2}$ was compact relative to $T^{1/2}$.

References

1. M. Berger, *Nonlinearity and functional analysis* (Academic Press, 1977).

2. P. Binding, *On the use of degree theory for non-linear multiparameter eigenvalue problems*, J. Math. Anal. Appl. *73* (1980), 381–391.

3. P. Binding and P. J. Browne, *A variational approach to multiparameter eigenvalue problems in Hilbert space*, SIAM J. Math. Anal. *9* (1978), 1054–1067.

4. ——— *A variational approach to multiparameter eigenvalue problems for differential equations* (to appear).

5. M. Böcher, *The theorems of oscillation of Sturm and Klein 1*, Bull. Amer. Math. Soc. *4* (1898), 295–313.

6. ——— *The theorems of oscillation of Sturm and Klein 2*, Bull. Amer. Math. Soc. *4* (1898), 365–376.

7. P. Browne, *A completeness theorem for a non-linear multiparameter eigenvalue problem*, J. Differential Equations, *23* (1977), 285–292.
8. P. Browne and B. Sleeman, *Nonlinear multiparameter Sturm-Liouville problems* (to appear).
9. M. Estabrooks and J. Macki, *A nonlinear Sturm-Liouville problem*, J. Differential Equations *10* (1971), 181–187.
10. E. Ince, *Ordinary differential equations* (Dover, 1944).
11. F. Klein, *Über Lamésche Funktionen*, Math. Ann. *18* (1881), 237–246.
12. B. Sleeman, *The two parameter Sturm-Liouville problem for ordinary differential equations*, Proc. Roy. Soc. Edin. A*69* (1971), 139–148.
13. ——— *The two parameter Sturm-Liouville problem for ordinary differential equations II*, Proc. Amer. Math. Soc. *34* (1972), 165–170.
14. ——— *Multiparameter spectral theory in Hilbert space*, J. Math. Anal. Appl. *65* (1978), 511–530.
15. A. Tal, *Eigenfunctions for a class of non-linear differential equations.* J. Differential Equations *3* (1967), 112–134.
16. R. Turner, *Some variational principles for a non-linear eigenvalue problem*, J. Math. Anal. Appl. *17* (1967), 151–160.
17. ——— *A class of nonlinear eigenvalue problems*, J. Functional Anal. *2* (1968), 297–322.

University of Calgary,
Calgary, Alberta