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The Geometry of $d^2y^1/dt^2 = f(y, \dot{y}, t)$ and $d^2y^2/dt^2 = g(y, \dot{y}, t)$, and Euclidean Spaces

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Abstract. This paper investigates the relationship between a system of differential equations and the underlying geometry associated with it. The geometry of a surface determines shortest paths, or geodesics connecting nearby points, which are defined as the solutions to a pair of second-order differential equations: the Euler–Lagrange equations of the metric. We ask when the converse holds, that is, when solutions to a system of differential equations reveals an underlying geometry. Specifically, when may the solutions to a given pair of second order ordinary differential equations $d^2y^1/dt^2 = f(y, \dot{y}, t)$ and $d^2y^2/dt^2 = g(y, \dot{y}, t)$ be reparameterized by $t \to T(y, t)$ so as to give locally the geodesics of a Euclidean space? Our approach is based upon Cartan's method of equivalence. In the second part of the paper, the equivalence problem is solved for a generic pair of second order ordinary differential equations of the above form revealing the existence of 24 invariant functions.

1 Introduction

The geometry of a manifold is given by a metric, which defines a notion of distance between points. Paths of shortest length connecting points are obtained as the critical curves of the functional variation of the integral defining arclength. Functional variation of this integral yields Euler–Lagrange equations which are a system of ordinary differential equations of second order, whose solutions are the geodesics. Thus associated with geometry is a system of ODEs. This paper seeks to answer the inverse problem: when does a system of ODEs represent the paths of shortest length of a metric? That is, we wish to establish when ordinary differential equations exhibit an underlying geometry. We shall not be so ambitious as to attempt a solution on manifolds of arbitrary dimension and endowed with a general metric but shall restrict ourselves to the case of a pair of second order ODEs on a (two-dimensional) surface and ask when the underlying geometry is flat, that is a Euclidean space. We are concerned only with the solutions of the ODEs up to reparameterization since they serve merely to describe paths of shortest length on the surface. Geodesics however, are not invariant with respect to general changes of parameter, so it shall be necessary to incorporate reparmeterization in the precise definition of the problem. The formulation of the equivalence problem is the contents of sections 1 and 2. In section 3 Cartan's method of equivalence is employed up to the level of the first normalization for generic ODEs. In section 4 we obtain the solution in the Euclidean case. The equivalence method is carried through in section 5, for generic pairs of second order ODEs with the result that the symmetry of the system produces 24 invariant functions.

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Before proceeding, it is instructive to consider the simpler problem without considerations of parameterization: when are the solutions to a pair of second order ordinary differential equations,

$$d^2y^1/dt^2 = f(y, \dot{y}, t) \quad \text{and} \quad d^2y^2/dt^2 = g(y, \dot{y}, t), \tag{1}$$

locally the geodesics of some Euclidean metric on the plane? The geodesics of a Euclidean metric are the straight lines with respect to some coordinate system. Thus the problem may be formulated as follows: when does there exist a coordinate system $Y = Y(y)$ such that the solutions $y = y(t)$ of (1) correspond to straight lines $Y = Y(y(t)) = at + b$, $a, b \in R$? We therefore seek to determine the existence of a transformation $\Psi\colon R^2 \times R \to R^2 \times R$ of the form

$$\Psi(y, t) = (Y(y), t)$$

such that Ψ transforms the equations

$$d^2Y^1/dt^2 = 0 \quad \text{and} \quad d^2Y^2/dt^2 = 0 \tag{2}$$

into the equations (1). Any transformation of the form Ψ above transforms (2) into equations of the form

$$d^2y^i/dt^2 + \Gamma^i_{jk}(y)\dot{y}^j\dot{y}^k = 0.$$

Consequently, (1) must necessarily be of this form. The terms $\Gamma^i_{jk}(y)$ define a connection ∇ on the surface, hence the solutions to (1) are locally the geodesics of a Euclidean space if and only if ∇ is flat.

The problem above requires that the solutions to (1) already be parameterized in such a fashion that only a change in the coordinates of the surface is sufficient to straighten them out into lines. This paper is interested in whether the solutions to (1) may be reparameterized so as to be straight lines in some coordinate system. Specifically, do there exist coordinates $Y = Y(y)$ and a reparameterization of time $T = T(y, t)$ such that the solutions $y = y(t)$ of (1) correspond to straight lines $Y = Y(T) = aT + b$, $a, b \in R$? We therefore seek to determine the existence of a transformation $\Phi\colon R^2 \times R \to R^2 \times R$ of the form

$$\Phi(y, t) = (Y(y), T(y, t))$$

such that Φ transforms the equations

$$d^2Y^1/dT^2 = 0 \quad \text{and} \quad d^2Y^2/dT^2 = 0$$

into the equations (1).

Conceivably, other reparameterization criteria could be considered as well. For instance, one might investigate the more restricted transformations $T = T(t)$ where time is reparameterized in a manner independent of the point on the surface or the more general $T = T(y, \dot{y}, t)$. Here, we shall content ourselves with spacetime reparameterizations only and defer the other cases to another time and place.

It is convenient to place the problem in a more general setting: we consider the equivalence of two pairs of ordinary differential equations

$$\frac{d^2y^1}{dt^2} = f(y, \dot{y}, t) \quad \frac{d^2y^2}{dt^2} = g(y, \dot{y}, t) \tag{3}$$

and

$$\frac{d^2Y^1}{dT^2} = F(Y, \dot{Y}, T) \quad \frac{d^2Y^2}{dT^2} = G(Y, \dot{Y}, T). \tag{4}$$

under transformations of the form

$$\Phi(y, t) = (Y(y), T(y, t)).$$

The case $F = G = 0$ is solved in section 4. We obtain an e-structure on a 12-dimensional space with constant torsion. The solutions to a pair of ODE's belonging to this equivalence class have symmetries given by the group of fractal-linear transformations on the plane. In section 5 we make no restrictions on F, G, f and g and carry the equivalence through for the generic case.

A similar problem was studied by S. S. Chern [2]. He has considered the geometry of a system of second order ODEs

$$\frac{d^2y^i}{dt^2} = f^i(y, \dot{y}, t) \quad i = 1, \ldots, n,$$

under transformations of the form

$$\begin{cases} Y = Y(y, t), \\ T = t. \end{cases}$$

Prior to Chern, the local behaviour of systems of second order ODEs has been studied by M. D. D. Kosambi [4] and by E. Cartan [1].

2 The Equivalence Problem Formulated

The equations (3) may be represented by the Pfaffian system

$$I = \begin{cases} dy^1 - p^1 dt = 0, \\ dy^2 - p^2 dt = 0, \\ dp^1 - f dt = 0, \\ dp^2 - g dt = 0. \end{cases}$$

on $U \subseteq R^5$. Similarly the equations (4) may be represented by the Pfaffian system

$$J = \begin{cases} dY^1 - P^1 dT = 0, \\ dY^2 - P^2 dT = 0, \\ dP^1 - F dT = 0, \\ dP^2 - G dT = 0. \end{cases}$$

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on $V \subseteq R^5$.

Form the coframes

$$\omega = \begin{cases} \omega^1 = dy^1, \\ \omega^2 = dy^1 + p\,dy^2, \\ \omega^3 = dy^2 + q\,dt, \\ \omega^4 = dp + h\,dt, \\ \omega^5 = dq + g\,dt, \end{cases} \qquad \text{and} \quad \Omega = \begin{cases} \Omega^1 = dY^1, \\ \Omega^2 = dY^1 + P\,dY^2, \\ \Omega^3 = dY^2 + Q\,dT, \\ \Omega^4 = dP + H\,dT, \\ \Omega^5 = dQ + G\,dT, \end{cases}$$

where

$$\begin{cases} p = -p^1(p^2)^{-1}, \\ h = (p^2)^{-2}(p^2 f - p^1 g), \\ q = -p^2, \end{cases} \qquad \text{and} \quad \begin{cases} P = -P^1(P^2)^{-1}, \\ H = (P^2)^{-2}(P^2 F - P^1 G), \\ Q = -P^2. \end{cases}$$

Observe that I (resp., J) is spanned by $\omega^2, \dots, \omega^5$ (resp., $\Omega^2, \dots, \Omega^5$).

Consider those transformations

$$\Phi \colon U \to V$$

such that

$$(*) \qquad \begin{cases} (1) \quad \Phi^*(J) = I, \text{and} \\ (2) \quad \Phi(y,p,t) = (Y(y), P(y,p,t), T(y,t)). \end{cases}$$

This is an *overdetermined* equivalence problem (*cf.* [3]). Nevertheless the approach given in the above reference shall not be followed in that all the information contained in $(*)$ may be encoded by an appropriately chosen coframe and group, as follows.

Let G be the subgroup of $GL(5,R)$ whose elements are represented by

$$\left(\begin{array}{c|cc|c} a & b & 0 & 0 \\ \hline 0 & c & 0 & \multirow{2}{*}{0} \\ 0 & e & f & \\ \hline 0 & \multicolumn{2}{c|}{M} & N \end{array}\right).$$

It is easily shown that a diffeomorphism $\Phi \colon U \to V$ satisfies $(*)$ if and only if

$$\Phi^*(\Omega) = \gamma\omega$$

for some $\gamma \colon U \to G$. In order to avoid the awkward presence of the map γ form the lifted coframe η (resp., H) on the the G-bundle $G \times U$ (resp., $G \times V$):

$$\eta := S\omega, \quad (resp.,\ H := S\Omega),$$

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where $S\colon G \to GL(5,R)$ is the natural injection. It can be shown that

$$\Phi^*(\Omega) = \gamma\omega$$

if and only if

$$\bar{\Phi}^*(H) = \eta,$$

for some $\bar{\Phi}(g,u)\colon G \times U \to G \times V$. $\bar{\Phi}$ is related to Φ and γ by $\bar{\Phi}(g,u) := (g\gamma(u)^{-1}, \Phi(u))$, for all $(g,u) \in G \times U$. We arrive at the following formulation of the equivalence problem:

The two systems of ordinary differential equations (3) *and* (4) *are equivalent with respect to a diffeomorphism* $\Phi\colon U \to V$ *of the form* $\Phi(y,t) = (Y(y), T(y,t))$ *if and only if there exists a diffeomorphism* $\bar{\Phi}(g,u)\colon G \times U \to G \times V$ *satisfying*

$$\bar{\Phi}^*(H) = \eta.$$

With this characterization of equivalence we now proceed to reduce the group G.

3 The First Normalization

The structure equations, after making the obvious absorptions are

$$d\eta = \left(\begin{array}{c|cc|c} \alpha & \beta & 0 & 0 \\ \hline 0 & \gamma & 0 & \multirow{2}{*}{0} \\ 0 & \epsilon & \phi & \\ \hline 0 & \multicolumn{2}{c|}{\mu} & \nu \end{array}\right)\eta + \left(\begin{array}{c} 0 \\ \hline J\eta^1\eta^3 + A_1\eta^1\eta^4 + A_2\eta^1\eta^5 \\ B_1\eta^1\eta^4 + B_2\eta^1\eta^5 \\ \hline 0 \end{array}\right).$$

The equations

$$0 = d^2\eta^2\eta^2 \tag{5}$$
$$0 = d^2\eta^3\eta^2\eta^3$$

give the following infinitesimal group action on the torsion tensor:

$$\left.\begin{aligned} dJ + (\alpha - \gamma + \phi)J + A\mu_2 &\equiv 0 \\ dA + (\alpha - \gamma)A + A\nu &\equiv 0 \\ dB + (\alpha - \phi)B + B\nu - A\epsilon &\equiv 0 \end{aligned}\right\} \quad \text{mod base.}$$

Therefore the group action on the torsion tensor is:

$$\begin{cases} J = (J_0 - A_0 N^{-1} M_2) a^{-1} c f^{-1}, \\ A = a^{-1} c A_0 N^{-1}, \\ B = a^{-1}(f B_0 + e A_0) N^{-1}, \end{cases}$$

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where J_0, A_0, and B_0 denote the tensors J, A and B, respectively, at the group identity. A parametric calculation shows that

$$J_0 = -(pq)^{-1}h, \quad A_0 = (p^{-1}, 0), \quad B_0 = (0, -(pq)^{-1}).$$

We may therefore normalize

$$J = 0, \quad A = (0,1), \quad \text{and} \quad B = (1,0).$$

Then

$$\left.\begin{aligned} \mu_2^2 &\equiv 0 \\ \nu_1^2 &\equiv 0 \\ \epsilon - \nu_2^1 &\equiv 0 \\ \alpha - \gamma + \nu_2^2 &\equiv 0 \\ \phi - \gamma - \nu_1^1 + \nu_2^2 &\equiv 0 \end{aligned}\right\} \quad \text{mod base.}$$

Write

$$\mu_2^2 = A_i\eta^i, \quad \nu_1^2 = B_i\eta^i, \quad \epsilon = \nu_2^1 + C_i\eta^i,$$
$$\alpha = \gamma - \nu_2^2 + D_i\eta^i, \quad \phi = \gamma + \nu_1^1 - \nu_2^2 + E_i\eta^i.$$

Substituting these values into (5) results in

$$B_3 = A_4, \quad D_3 = A_5, \quad D_4 = B_5, \quad E_5 = C_4 + D_5.$$

The structure equations after absorbing the torsion become

$$d\eta = \left(\begin{array}{c|cc|cc} \gamma - \nu_2^2 & \beta & 0 & 0 & 0 \\ \hline 0 & \gamma & 0 & 0 & 0 \\ 0 & \nu_2^1 & \gamma + \nu_1^1 - \nu_2^2 & 0 & 0 \\ \hline 0 & \mu_1^1 & \mu_2^1 & \nu_1^1 & \nu_2^1 \\ 0 & \mu_1^2 & 0 & 0 & \nu_2^2 \end{array}\right)\eta + \left(\begin{array}{c} 0 \\ \hline \eta^1\eta^5 \\ \eta^1\eta^4 + A\eta^1\eta^2 + B\eta^1\eta^3 \\ \hline 0 \\ C\eta^1\eta^3 + D\eta^1\eta^4 \end{array}\right).$$

The equations

$$\begin{aligned} 0 &= d^2\eta^5\eta^2\eta^5, \\ 0 &= d^2\eta^3\eta^3\eta^4\eta^5 + d^2\eta^4\eta^2\eta^3\eta^4, \\ 0 &= d^2\eta^2\eta^3\eta^4\eta^5 + d^2\eta^3\eta^2\eta^4\eta^5 + d^2\eta^4\eta^2\eta^3\eta^5 + d^2\eta^5\eta^2\eta^3\eta^4, \end{aligned}$$

give the following infinitesimal group action on the torsion tensor:

$$\left.\begin{aligned} dA + A(\gamma - \nu_1^1) + B\nu_2^1 + 2\mu_1^1 &\equiv 0 \\ dB + (\gamma - \nu_2^2)B + 2\nu_2^1 D + 2\mu_2^1 - 2\mu_1^2 &\equiv 0 \\ dC + (2\gamma + \nu_1^1 - 3\nu_2^2)C + \mu_2^1 D &\equiv 0 \\ dD + (\gamma + \nu_1^1 - 2\nu_2^2)D &\equiv 0 \end{aligned}\right\} \quad \text{mod base.}$$

4 Geodesics of Flat, Symmetric Connections

In this section the equivalence problem is carried through for the case $F = G = 0$. This will lead to an e-structure with constant torsion on a 12-dimensional space. The only invariants therefore are constant invariants and thus we obtain a complete solution to the problem of equivalence.

4.1 The Second Normalization

A parametric calculation will show that at the identity, $A_0 = B_0 = C_0 = D_0 = 0$, hence $D \equiv C \equiv 0$. This leaves the following two equations:

$$\left.\begin{aligned} dA + A(\gamma - \nu_1^1) + B\nu_2^1 + 2\mu_1^1 &\equiv 0 \\ dB + (\gamma - \nu_2^2)B + 2\mu_2^1 - 2\mu_1^2 &\equiv 0 \end{aligned}\right\} \quad \text{mod base.}$$

Normalize $A \equiv B \equiv 0$. We then have

$$\left.\begin{aligned} \mu_1^1 &\equiv 0 \\ \mu_2^1 &\equiv \mu_1^2 \end{aligned}\right\} \quad \text{mod base.}$$

This produces new torsion by

$$\mu_1^1 = A_i\eta^i$$

$$\mu_2^1 = \mu_1^2 + B_i\eta^i.$$

After absorbing torsion the structure equations are

$$d\eta = \left(\begin{array}{c|cc|cc} \gamma - \nu_2^2 & \beta & 0 & 0 & 0 \\ \hline 0 & \gamma & 0 & 0 & 0 \\ 0 & \nu_2^1 & \gamma + \nu_1^1 - \nu_2^2 & 0 & 0 \\ \hline 0 & 0 & \mu_1^2 & \nu_1^1 & \nu_2^1 \\ 0 & \mu_1^2 & 0 & 0 & \nu_2^2 \end{array}\right)\eta + \left(\begin{array}{c} 0 \\ \hline \eta^1\eta^5 \\ \eta^1\eta^4 \\ \hline A\eta^1\eta^2 + B\eta^1\eta^3 \\ 0 \end{array}\right).$$

The equations

$$0 = d^2\eta^4\eta^3\eta^4\eta^5 \qquad 0 = d^2\eta^4\eta^2\eta^4\eta^5 + d^2\eta^5\eta^3\eta^4\eta^5$$

give

$$\left.\begin{aligned} dA + (2\gamma - \nu_1^1 - \nu_2^2)A + B\nu_2^1 &\equiv 0 \\ dB + (2\gamma - 2\nu_2^2)B &\equiv 0 \end{aligned}\right\} \quad \text{mod base.}$$

Now at the identity $A_0 = B_0 = 0$. Thus $A = B = 0$, and hence

$$d\eta = \left(\begin{array}{c|cc|cc} \gamma - \nu_2^2 & \beta & 0 & 0 & 0 \\ \hline 0 & \gamma & 0 & 0 & 0 \\ 0 & \nu_2^1 & \gamma + \nu_1^1 - \nu_2^2 & 0 & 0 \\ \hline 0 & 0 & \mu_1^2 & \nu_1^1 & \nu_2^1 \\ 0 & \mu_1^2 & 0 & 0 & \nu_2^2 \end{array}\right)\eta + \left(\begin{array}{c} 0 \\ \hline \eta^1\eta^5 \\ \eta^1\eta^4 \\ \hline 0 \\ 0 \end{array}\right).$$

Only constant torsion remains so the system must be prolonged.

4.2 Prolongation

Now, $\dim \mathcal{G}^{(1)} = 2$. The first prolongation corresponds to the following arbitrariness in the tableau:

$$
\begin{aligned}
\theta^1 &:= v^1_1 \\
\theta^2 &:= v^1_2 + a\eta^3 \\
\theta^3 &:= v^2_2 + a\eta^2 \\
\theta^4 &:= \gamma + 2a\eta^2 \\
\theta^5 &:= \beta + a\eta^1 + b\eta^2 \\
\theta^6 &:= \mu^2_1 + a\eta^5
\end{aligned}
$$

where $a, b \in R$ are arbitrary. The structure equations for η may be written,

$$
d\eta = \left(\begin{array}{c|cc|cc}
\theta^4 - \theta^3 & \theta^5 & 0 & 0 & 0 \\
\hline
0 & \theta^4 & 0 & 0 & 0 \\
0 & \theta^2 & \theta^4 + \theta^1 - \theta^3 & 0 & 0 \\
\hline
0 & 0 & \theta^6 & \theta^1 & \theta^1 \\
0 & \theta^6 & 0 & 0 & \theta^3
\end{array}\right) \eta + \left(\begin{array}{c}
0 \\
\hline
\eta^1\eta^5 \\
\eta^1\eta^4 \\
\hline
0 \\
0
\end{array}\right).
$$

Taking d^2 of the above structure equations we obtain the following

$$
\begin{aligned}
0 &= -d\theta^3\eta^1 + d\theta^4\eta^1 + d\theta^5\eta^2 + \theta^3\theta^5\eta^2 - \theta^5\eta^1\eta^5, \\
0 &= d\theta^4\eta^2 + \theta^5\eta^2\eta^5 + \theta^6\eta^1\eta^2, \\
0 &= d\theta^1\eta^3 + d\theta^2\eta^2 - d\theta^3\eta^3 + d\theta^4\eta^3 - \theta^1\theta^2\eta^2 + \theta^3\theta^2\eta^2 + \theta^5\eta^2\eta^4 + \theta^6\eta^1\eta^3, \\
0 &= d\theta^1\eta^4 + d\theta^2\eta^5 + d\theta^6\eta^3 - \theta^2\theta^3\eta^5 + \theta^6\theta^3\eta^3 + \theta^4\theta^6\eta^3 - \theta^6\eta^1\eta^4 - \theta^1\theta^2\eta^5, \\
0 &= d\theta^3\eta^5 + d\theta^6\eta^2 - \theta^3\theta^6\eta^2 + \theta^4\theta^6\eta^2 - \theta^6\eta^1\eta^5.
\end{aligned}
$$

It follows from a somewhat lengthy but straightforward calculation that the structure equations for θ are given by

$$
\begin{aligned}
d\theta^1 &= A_1\eta^5\eta^2 + A_2\eta^3\eta^2 + A_3\eta^4\eta^2 + A_4\eta^4\eta^3 + A_5\eta^5\eta^3 + \theta^5\eta^5 + \theta^6\eta^1, \\
d\theta^2 &= \Phi^1\eta^3 + A_1\eta^4\eta^2 + B\eta^5\eta^2 + A_5\eta^4\eta^3 + \theta^1\theta^2 - \theta^3\theta^2 + \theta^5\eta^4, \\
d\theta^3 &= \Phi^1\eta^2 + C\eta^5\eta^2 + 2\theta^5\eta^5 + \theta^6\eta^1, \\
d\theta^4 &= 2\Phi^1\eta^2 + D_1\eta^3\eta^2 + D_2\eta^4\eta^2 + (C - A_1)\eta^5\eta^2 + \theta^5\eta^5 - \theta^6\eta^1, \\
d\theta^5 &= \Phi^1\eta^1 + \Phi^2\eta^2 + D_1\eta^3\eta^1 + D_2\eta^4\eta^1 - A_1\eta^5\eta^1 - \theta^3\theta^5, \\
d\theta^6 &= \Phi^1\eta^5 + F\eta^3\eta^2 + A_2\eta^4\eta^2 + \theta^3\theta^6 - \theta^4\theta^6,
\end{aligned}
$$

where Φ^1 and Φ^2 are 1-forms in the new tableau. The equations

$$
\begin{aligned}
0 &= d^2\theta^1,\\
0 &= d^2\theta^2\eta^3,\\
0 &= d^2\theta^2\eta^2 + d^2\theta^3\eta^3,\\
0 &= 2d^2\theta^3 - d^2\theta^4,\\
0 &= d^2\theta^6\eta^5,
\end{aligned}
$$

give the following infinitesimal action on the torsion tensor

$$
\left.
\begin{aligned}
dA_1 - \Phi^2 &\equiv 0\\
dA_2 &\equiv 0\\
dA_3 &\equiv 0\\
dA_4 &\equiv 0\\
dA_5 &\equiv 0\\
dB &\equiv 0\\
dC - 2\Phi^2 &\equiv 0\\
dD_1 &\equiv 0\\
dD_2 &\equiv 0\\
dF &\equiv 0
\end{aligned}
\right\}\quad \text{mod base.}
$$

At the group identity,

$$A_2 = A_3 = A_4 = A_5 = B = D_1 = D_2 = F = 0,$$

therefore these terms are identically zero. Thus,

$$(dA_1 + A_1(\theta^3 + \theta^4) - \Phi^2)\eta^4\eta^2\eta^3 + (2A_1 - C)\theta^2\eta^5\eta^2\eta^3 = 0.$$

Consequently,

$$C = 2A_1.$$

We also have

$$dA_1 + A_1(\theta^3 + \theta^4) - \Phi^2 \equiv 0 \quad \text{mod } \eta^2, \eta^5.$$

Therefore,

$$dA_1 + A_1(\theta^3 + \theta^4) - \Phi^2 + l\eta^2 = 0$$

for some function l. We obtain the following structure equations

$$
\begin{aligned}
d\theta^1 &= A\eta^5\eta^2 + \theta^5\eta^5 + \theta^6\eta^1,\\
d\theta^2 &= \Phi^1\eta^3 + A\eta^4\eta^2 + \theta^1\theta^2 - \theta^3\theta^2 + \theta^5\eta^4,\\
d\theta^3 &= \Phi^1\eta^2 + 2A\eta^5\eta^2 + 2\theta^5\eta^5 + \theta^6\eta^1,\\
d\theta^4 &= 2\Phi^1\eta^2 + A\eta^5\eta^2 + \theta^5\eta^5 - \theta^6\eta^1,\\
d\theta^5 &= \Phi^1\eta^1 + \Phi^2\eta^2 - A\eta^5\eta^1 - \theta^3\theta^5,\\
d\theta^6 &= \Phi^1\eta^5 + \theta^3\theta^6 - \theta^4\theta^6,
\end{aligned}
$$

where A has been written for A_1.

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4.3 The Third Normalization

The infinitesimal group action on the torsion is given by

$$dA_1 + A_1(\theta^3 + \theta^4) - \Phi^2 \equiv 0 \quad \text{mod base.}$$

Normalize $A = 0$. Then $\Phi^2 = l\eta^2$ and hence

$$\begin{aligned}
d\theta^1 &= \theta^5\eta^5 + \theta^6\eta^1,\\
d\theta^2 &= \Phi^1\eta^3 + \theta^1\theta^2 - \theta^3\theta^2 + \theta^5\eta^4,\\
d\theta^3 &= \Phi^1\eta^2 + 2\theta^5\eta^5 + \theta^6\eta^1,\\
d\theta^4 &= 2\Phi^1\eta^2 + \theta^5\eta^5 - \theta^6\eta^1,\\
d\theta^5 &= \Phi^1\eta^1 - \theta^3\theta^5,\\
d\theta^6 &= \Phi^1\eta^5 + \theta^3\theta^6 - \theta^4\theta^6.
\end{aligned}$$

We have constant torsion and an e-structure. Thus $(\eta^1, \dots, \eta^5, \theta^1, \dots, \theta^6, \Phi)$ is an invariant coframe (here Φ is written for Φ^1.) The equations

$$(d\Phi - \Phi\theta^4 - \theta^5\theta^6)\eta^i = 0$$

for $i = 1, 2, 3$ and 5 give

$$d\Phi = \Phi\theta^4 + \theta^5\eta^6.$$

Thus the structure equations are

$$\begin{aligned}
d\eta^1 &= (\theta^4 - \theta^3)\eta^1 + \theta^5\eta^2,\\
d\eta^2 &= \theta^4\eta^2 + \eta^1\eta^5,\\
d\eta^3 &= \theta^2\eta^2 + (\theta^4 + \theta^1 - \theta^3)\eta^3 + \eta^1\eta^4,\\
d\eta^4 &= \theta^6\eta^3 + \theta^1\eta^4 + \theta^2\eta^5,\\
d\eta^5 &= \theta^6\eta^2 + \theta^3\eta^5,\\
d\theta^1 &= \theta^5\eta^5 + \theta^6\eta^1,\\
d\theta^2 &= \Phi\eta^3 + (\theta^1 - \theta^3)\theta^2 + \theta^5\eta^4,\\
d\theta^3 &= \Phi\eta^2 + 2\theta^5\eta^5 + \theta^6\eta^1,\\
d\theta^4 &= 2\Phi\eta^2 + \theta^5\eta^5 - \theta^6\eta^1,\\
d\theta^5 &= \Phi\eta^1 - \theta^3\theta^5,\\
d\theta^6 &= \Phi\eta^5 + (\theta^3 - \theta^4)\theta^6,\\
d\Phi &= \Phi\theta^4 + \theta^5\theta^6.
\end{aligned} \tag{6}$$

We have shown the following:

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Theorem 1 *The solutions $y(t)$ of*

$$d^2y^1/dt^2 = f(y, \dot{y}, t) \quad \textit{and} \quad d^2y^2/dt^2 = g(y, \dot{y}, t) \tag{7}$$

are the geodesics of a Euclidean space with respect to a transformation of the form $(Y, T) = (Y(y), T(y, t))$ *if and only if equations* (7) *yield the structure equations* (6).

The structure equations (6) are the Maurer–Cartan equations for a Lie-group: the group of fractal-linear transformations $\mathcal{F}$ on the plane. Recall that the fractal-linear transformations of the plane are those transformations A of the form

$$A = (\bar{y}, \bar{t}) = (\bar{y}(y), \bar{t}(y, t))$$

where

$$\bar{y}^1(y) = \frac{b_0^1 + b_1^1y^1 + b_2^1y^2}{a_0 + a_1y^1 + a_2y^2} \quad \bar{y}^2(y) = \frac{b_0^2 + b_1^2y^1 + b_2^2y^2}{a_0 + a_1y^1 + a_2y^2}$$
$$\bar{t}(y, t) = \frac{t + c_0 + c_1y^1 + c_2y^2}{a_0 + a_1y^1 + a_2y^2},$$

where $a_i, a_j^i, b_j^i \in R$ are constants. Let $\mathcal{G}$ denote the subgroup of $GL(4, R)$ consisting of those invertible matrices M whose first column is ${}^t(1, 0, 0, 0)$. We may identify the transformation A with the element $\phi(A)$ in $\mathcal{G}$ by

$$\phi(A) := \left(\begin{array}{c|c|cc} 1 & c_0 & c_1 & c_2 \\ \hline 0 & a_0 & a_1 & a_2 \\ \hline 0 & b_0^1 & b_1^1 & b_2^1 \\ 0 & b_0^2 & b_1^2 & b_2^2 \end{array}\right).$$

The map $\phi\colon \mathcal{F} \to \mathcal{G}$ is a group isomorphism and so the fractal-linear transformations may be identified with $\mathcal{G}$.

5 The Generic Case

In this section we continue the equivalence problem from the end of the first normalization (§3).

5.1 Thye Second Normalization

Recall that after the first normalization we have the structure equations

$$d\eta = \left(\begin{array}{c|cc|cc} \gamma - \nu_2^2 & \beta & 0 & 0 & 0 \\ \hline 0 & \gamma & 0 & 0 & 0 \\ 0 & \nu_2^1 & \gamma + \nu_1^1 - \nu_2^2 & 0 & 0 \\ \hline 0 & \mu_1^1 & \mu_2^1 & \nu_1^1 & \nu_2^1 \\ 0 & \mu_1^2 & 0 & 0 & \nu_2^2 \end{array}\right) \eta + \left(\begin{array}{c} 0 \\ \hline \eta^1\eta^5 \\ \eta^1\eta^4 + A\eta^1\eta^2 + B\eta^1\eta^3 \\ \hline 0 \\ C\eta^1\eta^3 + D\eta^1\eta^4 \end{array}\right)$$

with infinitesimal group action on the torsion tensor given by

$$\left.\begin{aligned} dA + A(\gamma - \nu_1^1) + B\nu_2^1 + 2\mu_1^1 &\equiv 0\\ dB + (\gamma - \nu_2^2)B + 2\nu_2^1 D + 2\mu_2^1 - 2\mu_1^2 &\equiv 0\\ dC + (2\gamma + \nu_1^1 - 3\nu_2^2)C + \mu_2^1 D &\equiv 0\\ dD + (\gamma + \nu_1^1 - 2\nu_2^2)D &\equiv 0 \end{aligned}\right\} \quad \text{mod base.}$$

In the generic case, $D \neq 0$. We thus normalize

$$A = 0, \quad B = 0, \quad C = 0, \quad D = 1.$$

This gives us

$$\left.\begin{aligned} \mu_1^1 &\equiv 0\\ \mu_2^1 &\equiv 0\\ \mu_1^2 &\equiv \nu_2^1\\ \gamma &\equiv 2\nu_2^2 - \nu_1^1 \end{aligned}\right\} \quad \text{mod base.}$$

Let

$$\mu_1^1 = A_i\eta^i, \quad \mu_2^1 = B_i\eta^i, \quad \mu_1^2 = \nu_2^1 + C_i\eta^i, \quad \gamma = 2\nu_2^2 - \nu_1^1 + D_i\eta^i.$$

It follows that

$$B_4 = D_3.$$

The new structure equations are:

$$d\eta = \left(\begin{array}{c|cc|cc} \nu_2^2 - \nu_1^1 & \beta & 0 & 0 & 0\\ \hline 0 & 2\nu_2^2 - \nu_1^1 & 0 & 0 & 0\\ 0 & \nu_2^1 & \nu_2^2 & 0 & 0\\ \hline 0 & 0 & 0 & \nu_1^1 & \nu_2^1\\ 0 & \nu_2^1 & 0 & 0 & \nu_2^2 \end{array}\right)\eta + \left(\begin{array}{c} 0\\ \hline \eta^1\eta^5\\ \eta^1\eta^4\\ \hline 0\\ \eta^1\eta^4 \end{array}\right) + T,$$

where

$$T = \left(\begin{array}{c} B_4\eta^3\eta^1 + D_4\eta^4\eta^1 + D_5\eta^5\eta^1 + D_2\eta^2\eta^1\\ \hline D_1\eta^1\eta^2 + B_4\eta^3\eta^2 + D_4\eta^4\eta^2 + D_5\eta^5\eta^2\\ D_1\eta^1\eta^3 + D_2\eta^2\eta^3 + D_4\eta^4\eta^3 + D_5\eta^5\eta^3\\ \hline A_1\eta^1\eta^2 + A_3\eta^3\eta^2 + A_4\eta^4\eta^2 + A_5\eta^5\eta^2 + B_1\eta^1\eta^3 + B_2\eta^2\eta^3\\ \qquad\qquad\qquad\qquad\qquad + B_4\eta^4\eta^3 + B_5\eta^5\eta^3\\ C_1\eta^1\eta^2 + C_3\eta^3\eta^2 + C_4\eta^4\eta^2 + C_5\eta^5\eta^2 \end{array}\right).$$

After absorbing the torsion the structure equations are

$$d\eta = \left(\begin{array}{c|cc|cc} \nu_2^2 - \nu_1^1 & \beta & 0 & 0 & 0 \\ \hline 0 & 2\nu_2^2 - \nu_1^1 & 0 & 0 & 0 \\ 0 & \nu_2^1 & \nu_2^2 & 0 & 0 \\ \hline 0 & 0 & 0 & \nu_1^1 & \nu_2^1 \\ 0 & \nu_2^1 & 0 & 0 & \nu_2^2 \end{array}\right)\eta + \left(\begin{array}{c} A\eta^5\eta^1 \\ \hline \eta^1\eta^5 + B\eta^1\eta^2 + A\eta^5\eta^2 \\ \eta^1\eta^4 + B\eta^1\eta^3 + C\eta^4\eta^3 + A\eta^5\eta^3 \\ \hline D\eta^1\eta^2 + E\eta^3\eta^2 + F\eta^1\eta^3 + G\eta^5\eta^3 \\ \eta^1\eta^4 + H\eta^1\eta^2 + I\eta^4\eta^2 + J\eta^5\eta^2 \end{array}\right).$$

5.2 The Third Normalization

The equations

$$\begin{aligned} 0 &= d^2\eta^1\eta^2\eta^3 + d^2\eta^2\eta^1\eta^3 + d^2\eta^3\eta^1\eta^2, \\ 0 &= d^2\eta^3\eta^2\eta^5 + d^2\eta^5\eta^2\eta^3, \\ 0 &= d^2\eta^4\eta^4\eta^5, \\ 0 &= d^2\eta^4\eta^2\eta^4 - d^2\eta^5\eta^4\eta^5, \\ 0 &= d^2\eta^3\eta^3\eta^5 - d^2\eta^5\eta^3\eta^5, \\ 0 &= -d^2\eta^3\eta^4\eta^5 - d^2\eta^3\eta^3\eta^4 + d^2\eta^4\eta^2\eta^4 + d^2\eta^5\eta^3\eta^4, \end{aligned} \tag{8}$$

give us the following infinitesimal group action on the torsion tensor:

$$\left.\begin{aligned} dA + A\nu_2^2 + C\nu_2^1 + 2\beta &\equiv 0 \\ dB + B(\nu_2^2 - \nu_1^1) &\equiv 0 \\ dC + C\nu_1^1 &\equiv 0 \\ dD + 3D(\nu_2^2 - \nu_1^1) + F\nu_2^1 - H\nu_2^1 &\equiv 0 \\ dE + E(3\nu_2^2 - 2\nu_1^1) - F\beta - G\nu_2^1 &\equiv 0 \\ dF + 2F(\nu_2^2 - \nu_1^1) &\equiv 0 \\ dG + G(2\nu_2^2 - \nu_1^1) &\equiv 0 \\ dH + 2H(\nu_2^2 - \nu_1^1) - B\nu_2^1 &\equiv 0 \\ dI + I\nu_2^2 - C\nu_2^1 &\equiv 0 \\ dJ + J(2\nu_2^2 - \nu_1^1) + B\beta + I\nu_2^1 &\equiv 0 \end{aligned}\right\} \mod \text{ base}.$$

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At this point, the group is the subgroup of $GL(5, R)$ whose elenents are of the form

$$\left(\begin{array}{c|cc|cc} ca^{-1} & e & 0 & 0 & 0 \\ \hline 0 & c^2a^{-1} & 0 & 0 & 0 \\ 0 & a^{-1}bc & c & 0 & 0 \\ \hline 0 & 0 & 0 & a & b \\ 0 & a^{-1}bc & 0 & 0 & c \end{array}\right).$$

The group action on the torsion is then given by

$$\begin{aligned}
A &= A_0c^{-1} - C_0a^{-1}bc^{-1} - 2ac^{-2}e, \\
B &= B_0ac^{-1}, \\
C &= C_0a^{-1}, \\
D &= D_0a^3c^{-3} + (1/2)B_0ab^2c^{-3} + (H_0 - F_0)a^2bc^{-3}, \\
E &= E_0a^2c^{-3} + F_0a^3c^{-4}e + G_0abc^{-3}, \\
F &= F_0a^2c^{-2}, \\
G &= G_0ac^{-2}, \\
H &= H_0a^2c^{-2} + B_0abc^{-2}, \\
I &= I_0c^{-1} + C_0a^{-1}bc^{-1}, \\
J &= J_0ac^{-2} - B_0a^2c^{-3}e - (1/2)C_0a^{-1}b^2c^{-2} - I_0bc^{-2}.
\end{aligned}$$

We may then normalize

$$A = 0, \quad B = 1, \quad C = 1, \quad I = 0.$$

It follows that

$$\nu_1^1 \equiv \nu_2^2 \equiv \nu_2^1 \equiv \beta \equiv 0 \mod \text{base}.$$

We write

$$\nu_1^1 = A_i\eta^i, \quad \nu_2^2 = B_i\eta^i, \quad \nu_2^1 = C_i\eta^i, \quad \beta = D_i\eta^i.$$

Substituting these values back into the equations (8) we obtain

$$A_5 = 2D_4 + C_4 \quad \text{and} \quad B_4 = A_1 + A_4.$$

The group has been reduced to the identity and so we have an e-structure. The structure equations are

$$
\begin{aligned}
d\eta^1 &= (D_1 + A_2 - B_2)\eta^1\eta^2 + (A_3 - B_3)\eta^1\eta^3 - A_1\eta^1\eta^4 + (A_5 - B_5)\eta^1\eta^5 - D_3\eta^2\eta^3 \\
&\quad + (1/2)(C_4 - A_5)\eta^2\eta^4 - D_5\eta^2\eta^5, \\
d\eta^2 &= (1 + 2B_1 - A_1)\eta^1\eta^2 + \eta^1\eta^5 + (A_3 - 2B_3)\eta^2\eta^3 - (A_1 + B_4)\eta^2\eta^4 \\
&\quad + (A_5 - 2B_5)\eta^2\eta^5, \\
d\eta^3 &= C_1\eta^1\eta^2 + (1 + B_1)\eta^1\eta^3 + \eta^1\eta^4 + (B_2 - C_3)\eta^2\eta^3 - C_4\eta^2\eta^4 - C_5\eta^2\eta^5 \\
&\quad - (1 + B_4)\eta^3\eta^4 - B_5\eta^3\eta^5, \\
d\eta^4 &= I_1\eta^1\eta^2 + I_3\eta^1\eta^3 + A_1\eta^1\eta^4 + C_1\eta^1\eta^5 - I_2\eta^2\eta^3 + A_2\eta^2\eta^4 + C_2\eta^2\eta^5 + A_3\eta^3\eta^4 \\
&\quad + (C_3 - I_4)\eta^3\eta^5 + (C_4 - A_5)\eta^4\eta^5, \\
d\eta^5 &= (C_1 + I_5)\eta^1\eta^2 + \eta^1\eta^4 + B_1\eta^1\eta^5 - C_4\eta^2\eta^4 + (B_2 - I_6 - C_5)\eta^2\eta^5 + B_3\eta^3\eta^5 \\
&\quad + B_4\eta^4\eta^5 - C_3\eta^2\eta^3.
\end{aligned}
$$

We obtain 24 local invariants.

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