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Orbital Instability and Forming of Outer Part of Asteroidal Belt

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ABSTRACT

The orbital instability and forming of outer part of Asteroidal belt has been studied earlier in the Circular case. Here the same problem is studied in elliptic case.

FORMULATION OF THE PROBLEM

Here the case of planer restricted three-body problem will be considered. The mass of central body P_0 supposed as the unit, and the mass of perturbing body P' as $\mu << 1$. The semi-major axis of P' will be considered as a unit of length, but a unit of time will be selected so that the gravitational constant would become unity. For the Kepler elements of passive gravitating point we shall be using the well known terms but all elements of P' will be different.

Let us consider the first type of resonances. Thus initially we have the condition:

$$|\ell.n - (\ell+n)n'| < 0\ (\sqrt{\mu}), \tag{1}$$

where ℓ is a simple integer. Then we can introduce the Delamay anomaly as

$$S = \ell.M - (\ell+1)(M' - \psi) \tag{2}$$

where $\psi = \omega - \omega'$.

Then we carry out the expansion of perturbed function R over the fast variables M and M' and neglecting higher order terms in e and e'. We shall obtain

$$R = B_o + eB_1 \cos S + e' B_2 \cos(S+\psi) \qquad (3)$$

where $B_o = \mu/2\ L_1^{(0)}(a)$

$$B_1 = \mu/2\ [2(\ell+1)L_1^{(\ell+1)}(a) + a \frac{d}{da} L_1^{(\ell+1)}(a)],$$

$$B_2 = \mu/2[(2\ell+1)L_1^{(\ell+1)}(a) + a \frac{d}{da} L_1^{(\ell+1)}(a)$$

$$+ 2a\ \delta_\ell e'],\quad \delta_1 = 1,\quad \delta_\ell = 0 \text{ with } \ell \neq 0,$$

$L_1^{(\ell+1)}$ are the Laplace coefficients.

If we change our problem to circular problem (e' = 0), we have the following integral

$$\gamma = a(\ell+1 - \ell\sqrt{1-e^2})^2 \qquad (4)$$

At present the solution of this problem obtained with the Weierstrass-functions [2]. Some trajectories are shown in Figure 1.

On the lower part of range takes place the separatrix and trajectories types, but on the upper some results of numerical integration.

Stationary solutions of the problem are marked as e_1, e_2, e_3. It's behaviour depending on the values of γ have been shown in Figure 2.

From the conditions of first type trajectories there follows the impossibility of an approaching of asteroid and Jupiter in aphelion although that trajectories are in the instability range according to Hill [3]. Other trajectories in this case would be unstable.

Now we consider the following problem: What are the trajectories near the separatrix which change with variation to elliptic problem?

At first let us prove the existance of ergodic layer in this case in which the turbing of orbit type are happening. The method given in [6] is being used here.

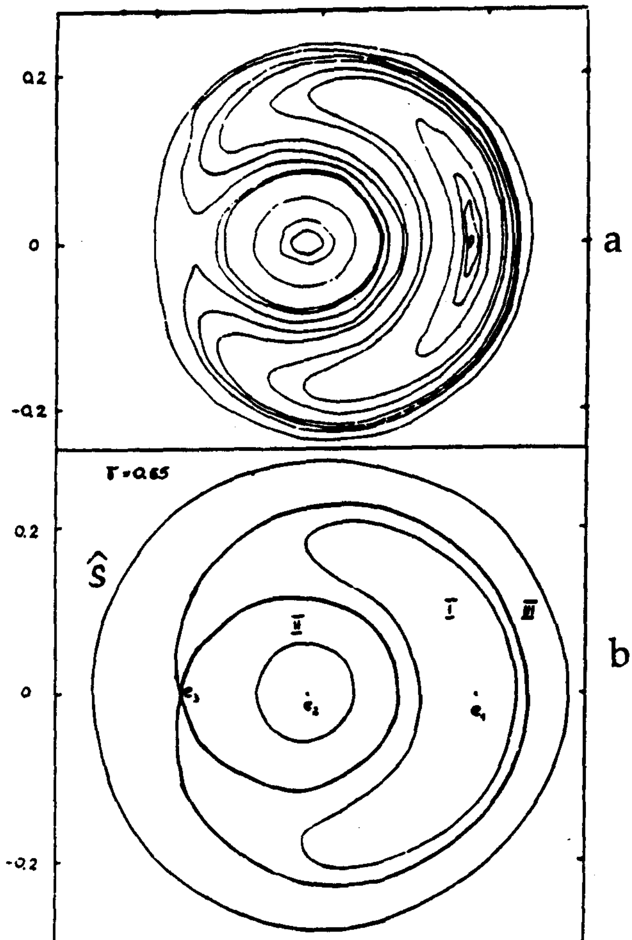

Figure 1: (a) The results of numerical integration.
(b) Separatrix $\hat{S}$ and some types of trajectories.

THE PROOF OF ERGODIC LAYAR EXISTANCE

For first type layar trajectories, if we mark the mean motion n_1 of e_1, solution for

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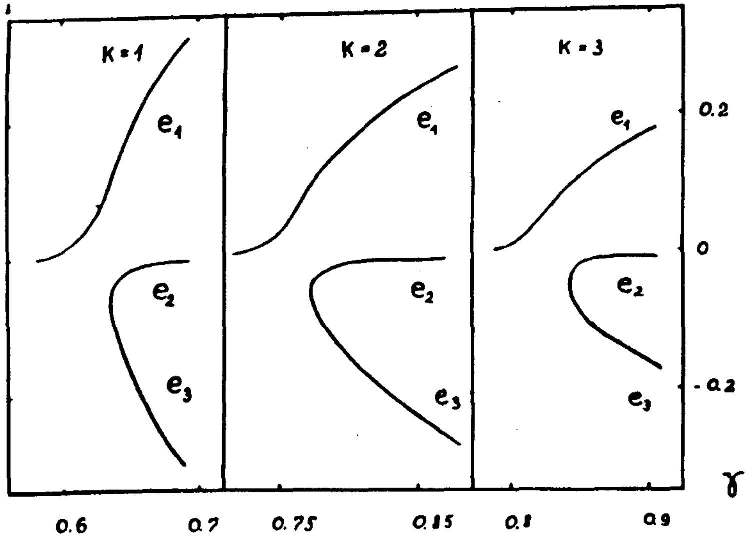

Figure 2: Behaviours of stationary solutions e_1, e_2, e_3 depended on the value of γ

$\Delta n = n - n_1$ is

$$(\Delta n)^2 = c + 2\varepsilon_1 \cos S, \tag{5}$$

where c is a constant, $\varepsilon_1 = \frac{3}{2}\ \gamma^{-2} B_1 e_1$.

Then, since $\dot{S} = \ell \Delta n$, for the second derivative we have

$$\ddot{S} = -\varepsilon_1 \sin S, \tag{6}$$

or, with $e' \neq 0$,

$$\ddot{S} = -\varepsilon_1 \sin S - \varepsilon_2 \sin(S+\psi) \tag{7}$$

Substitute the new variable $V = \tau\dot{S}$, where $\tau = \varepsilon_1^{-1}$, and obtain the equation (7) in the following form:

$$\tau\dot{V} = -\sin S - \varepsilon \sin(S + \psi). \tag{8}$$

Here it is supposed $\varepsilon = \varepsilon_2/\varepsilon_1$.

We make use of Hamiltonian of unperturbed motion:

$$H = \frac{1}{2} V^2 - \cos S, \tag{9}$$

and bring in new variables I-action and ψ-fase:

$$I = \frac{1}{2\pi} \oint V\, dS = \frac{8}{\pi}[E - (1-k^2)K]$$
$$\psi = \frac{\partial S}{\partial I}, \tag{10}$$

where $S = \frac{1}{2\pi} \oint dS\, V$.

Here K, E-are first and second type full integrals with module $k = \sqrt{\frac{1+H}{2}}$.

If perturbations are absent ($\mathcal{E} = 0$), then equations of motion would become

$$\dot{I} = 0, \quad \psi = \Omega(I), \tag{11}$$

where Ω is a frequency and

$$\Omega = \frac{1}{\tau}\frac{dH}{dI} = \frac{I}{2k}\ \frac{1}{\tau} \tag{12}$$

But, if perturbations are present, then

$$\dot{I} = \frac{dI}{dH}\left(\frac{\partial H}{\partial V}\dot{V} + \frac{\partial H}{\partial S}\dot{S}\right) = -\frac{\varepsilon}{\tau^2\Omega} V \sin(S + \psi) \tag{13}$$

Let us define the spectrum of speed V in the case when the perturbations are absent ($\varepsilon = 0$). So far as $V = \tau \ell (n-n_1)$, then with $k > 1$ we have to write [4]:

$$V + 2k\, cn(u,k), \tag{14}$$

where $u = \ell\tau^{-1}t$ is the new variable. Or, if we substitute the new variable $q = \exp(-\pi k'/k)$, where $k' = k(\sqrt{1-k^2})$, and expand it in Fourier-series, then we have the following form:

$$V = 8\Omega\tau \sum_{n=1}^{\infty} \frac{q^{n-\frac{1}{2}}}{1+q^{2n-1}} \cos[(2n-1)\Omega t] \tag{15}$$

We shall consider the motion near the separatrix. In this case we have

$$k \to 1,\ q \to 1,\ K \approx \frac{1}{2} \ln \frac{32}{1-H},\quad K' \approx \frac{\pi}{2},$$

$$\Omega_\tau \approx \frac{\pi}{\ln \frac{32}{1-H}},\quad q \approx \exp[-\pi-\Omega\tau] \tag{16}$$

Then, in the right hand side of equation (13) pick out some resonant term. Taking into account $\psi \approx \beta t$ we have the following resonant condition:

$$\tau\Omega(I_N) = \frac{\beta\tau}{N}, \tag{17}$$

where N is an odd number and I_N is an action by that resonance.

In the case of $\beta\tau \sim 1$ we have $\tau_\Omega(I_N) \sim 1/N$, but near the separatrix $\tau\Omega << 1$, therefore (17) would be carried out just for $N >> 1$. Let the distance between resonances N and (N+2) be

$$R_N = \Omega(I_N) - \Omega(I_{N+2}) \leq \frac{2}{\tau N^2} \sim \tau\Omega^2(I_N) \tag{18}$$

Then, the stochastic condition will have the following form:

$$\left| \frac{d\Omega(I_N)}{dI_N} \right| \delta I_N >> R_N \simeq \tau\Omega^2(I_N), \tag{19}$$

where δI_N is the maximum variation of particle's action in the N-resonance.

Assuming the variation of frequency have sufficiently small range, we have

$$\left| \frac{d\Omega}{dI}(I_N) \right| \delta I_N << \Omega(I_N). \tag{20}$$

After integration of (13) over time-interval, such that $\gtrsim \tau$, in the range of N-resonance we shall have

$$\delta I_N \sim \frac{q^{n-1/2}}{1+q^{\ell n+1}} \quad \frac{4\varepsilon}{\tau N \left| \frac{d\Omega}{dI} \right| \delta I_N} \tag{21}$$

but $q \sim 1$, therefore

$$\delta I_N \sim \left| \frac{2\varepsilon\Omega(I_N)}{\frac{d\Omega(I_N)}{dI}} \right|^{1/2} \tag{22}$$

Thus, the stochastic condition will be obtained in the form:

$$L = \frac{1}{R} \left| \frac{d\Omega}{dI} \right| \delta I = \sqrt{\frac{2\varepsilon}{\tau^2 \Omega^3} \left| \frac{d\Omega}{dI} \right|} >> 1 \tag{23}$$

But near the separatrix we have

$$\left| \frac{d\Omega}{dI} \right| = \frac{\tau^2 \Omega^3}{32\pi} \exp\left(\frac{\pi}{\tau\Omega} \right)$$

and

$$L^2 = \frac{\varepsilon}{16\pi} \exp\left(\frac{\pi}{\tau\Omega} \right) >> 1 \tag{24}$$

The boundary of stochastic case is approximately defined from such condition: $L \gtrsim 1$, that is within the frequency range $0 \leq \Omega \lesssim \bar{\Omega}$,

$$\bar{\Omega} = \frac{\pi}{\tau \ln \frac{\ell 6\pi}{\varepsilon}} \tag{25}$$

From (16) we find the stochastic boundary of energy such as

$$1 - \bar{H} \gtrsim \frac{2\varepsilon}{\pi} \sim \varepsilon \tag{26}$$

DEFINITION OF THE WIDTH OF SEPARATRIX SPLITTING

Let us define the range of separatrix splitting. In the case of elliptic problem, we have

$$\dot{\gamma} = 2\ell\sqrt{a}\ e'\ E_2\ \sin(S+\psi) \tag{27}$$

Therefore on the separatrix $\dot{S} << \dot{\psi}$, then

$$\gamma = \gamma_o + \frac{B}{\beta}\cos(\psi).$$

Using the analytic form for β, [4], we have to count the value $\Delta\gamma$

$$\gamma = \gamma_o \pm \Delta\gamma$$

Take that for $\ell = 1$, $e' = 0.048$, $e = 0{,}14$, $\mu = 1/1047$ and for the other conditions corresponding to [1].

We observe from Figure 3 that as γ changes it's value from 0.64 to 0.652, that to 0.18, the right boundary of projecting will be changed from 0,2 to 0.26.

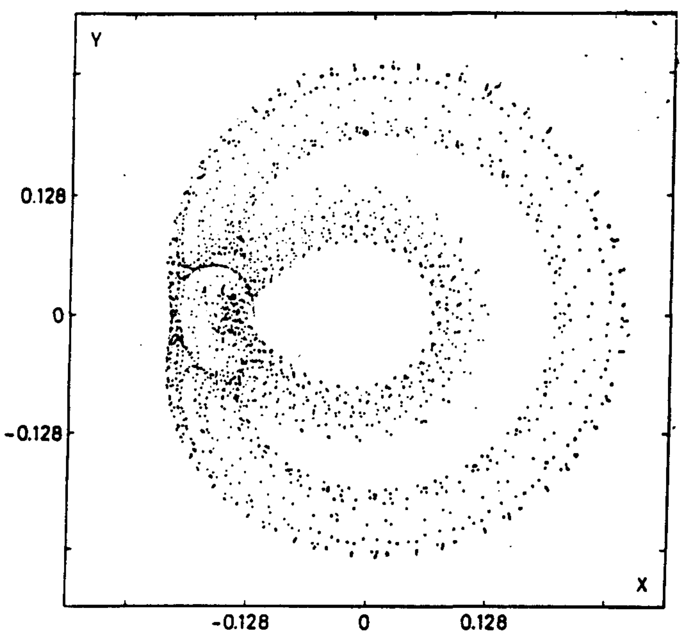

Figure 3: Variation of e_3-solution depending on the variation of $0.64 \leq \gamma \leq 0.652$. $X = e_3 \cos S$, $Y = e_3 \sin S$.

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Analogous change of $K = \sqrt{\gamma}$ are shown in Figure 4. Our future plan is to apply this method for asteroids of Gilda's group and in particular, for asteroid no.334, whose motion have been investigated by Shubart [5].

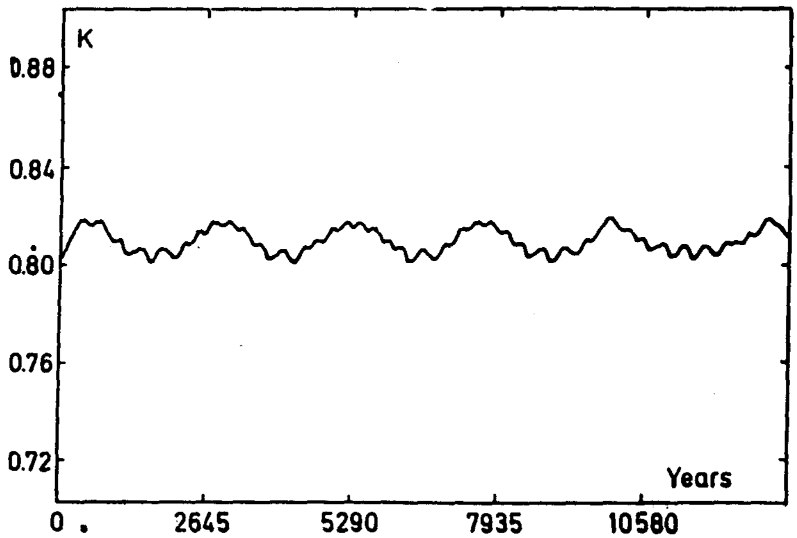

Figure 4: Variation of $K = \sqrt{\gamma}$ according to $0.64 \leq \gamma \leq 0.652$.

REFERENCES

[1] C.Froeschle, H.School, Astronomy and Astrophysics,V 51, 1977.

[2] I.A. Gerasimov, Astronomichesky Zhurnal, V.63,1986.

[3] I.A. Gekasimov, N.A. Solovaya, Meetino of Uppsala University, June 12-16,1989.

[4] I.A. Gekasimov, N.V.Kirushenkova, Kosmicheskie issledovanja, V.27, No.5, 1989.

[5] J.Schubart, Celestial Mechanics, V.43, 1988.

[6] N.A. Zaslavskyi, Stochastic instability in the nonlinear processes, Moscow,1970.

https://doi.org/10.1017/S0252921100066318 Published online by Cambridge University Press