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SUBLATTICES AND INITIAL SEGMENTS OF THE DEGREES OF UNSOLVABILITY

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In this paper we shall prove that every finite lattice is isomorphic to a sublattice of the degrees of unsolvability, and that every one of a certain class of finite lattices is isomorphic to an initial segment of degrees.

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1. Representation of lattices. The equivalence lattice $\mathscr{E}(S)$ of the set S consists of all equivalence relations on S, ordered by setting $\theta \leqq \theta'$ if for all a and b in S, $a\,\theta\,b \Rightarrow a\,\theta'\,b$. The least upper bound and greatest lower bound in $\mathscr{E}(S)$ are given by the $\bigcup$ and $\bigcap$ operations:

$$a(\theta \bigcap \theta')b \Leftrightarrow a\,\theta\,b \wedge a\,\theta'\,b,$$
$$a(\theta \bigcup \theta')b \Leftrightarrow a\,\theta\,a_1\,\theta'\,a_2\,\theta \ldots \theta\,a_n\,\theta'\,b \quad \text{for some } a_1, \ldots, a_n \in S.$$

By Whitman's theorem [**7**], every finite lattice is isomorphic to a sublattice of $\mathscr{E}(N_1)$, where N_1 is either N or $\{0, \ldots, m-1\}$ for some $m \in N$, depending on the lattice. For the purposes of this paper, a refinement of Whitman's theorem is more convenient.

The usual notation for sequence numbers will be used. In particular, if $\sigma = \langle a_0, \ldots, a_{m-1}\rangle$, then $lh(\sigma) = m$ is the length of σ, and $(\sigma)_x = a_x$ if $0 \leqq x < m$. If also $\tau = \langle b_0, \ldots, b_{k-1}\rangle$, then

$$\sigma * \tau = \langle a_0, \ldots, a_{m-1}, b_0, \ldots, b_{k-1}\rangle.$$

A finite lattice L will be said to be in normal form if there is a number m such that: the elements of L are subsets of $\{0, \ldots, m-1\}$; the zero of L is $\emptyset$; the unit of L is $\{0, \ldots, m-1\}$; the order relation of L is set inclusion; and the meet operation of L is set intersection. Then the meet and join of the elements α and β of L may be written $\alpha \cap \beta$ and $\alpha \vee \beta$, respectively. It is easily seen that every finite lattice is isomorphic to a lattice in normal form.

If S is a set of sequences all of length m and $\alpha \subseteq \{0, \ldots, m-1\}$, then $\equiv_\alpha$ is the equivalence relation defined on S by

$$\sigma \equiv_\alpha \tau \Leftrightarrow [(\sigma)_x = (\tau)_x \text{ for all } x \in \alpha].$$

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THEOREM 1. *If L is a finite lattice in normal form, then there is a recursive set S of sequences, all of length m (where $\{0, \ldots, m-1\}$ is the unit of L), such that the correspondence $\alpha \to \equiv_\alpha$ is a dual isomorphism of L onto a sublattice of $\mathscr{E}(S)$.*

Proof. (A simple direct proof is to appear in [**6**].) By Whitman's theorem, applied to the dual of L, L is dually isomorphic to a sublattice of $\mathscr{E}(N_1)$, where N_1 is either N or $\{0, \ldots, k-1\}$. Let $\alpha \to \theta_\alpha$ be the dual isomorphism. It may be assumed that $\theta_\emptyset$ is the unit in $\mathscr{E}(N_1)$, i.e. that $x\,\theta_\emptyset\,y$ for all x and y in N_1. For if this is not the case, let N_2 be the Cartesian product of the $\theta_\emptyset$-classes in N_1, choose $f_0 \in N_2$, and let $N_1^* = \{f \mid f \in N_2 \wedge f(C) = f_0(C)$ for all but finitely many $\theta_\emptyset$-classes $C\}$. Set $f\,\theta_\alpha{}^*g \Leftrightarrow f(C)\,\theta_\alpha\,g(C)$ for all C. Then it may be verified that $\alpha \to \theta_\alpha{}^*$ is a dual isomorphism of L onto a sublattice of $\mathscr{E}(N_1^*)$ and that $\theta_\emptyset{}^*$ is the unit in $\mathscr{E}(N_1^*)$. (This proof is due to McKenzie.)

Let there be fixed for each $\alpha \in L$ a (finite or infinite) enumeration $C_0{}^\alpha, C_1{}^\alpha, \ldots$ of the equivalence classes into which θ_α partitions N_1. For each $x \in N_1$ let $\sigma(x) = \langle x_0, \ldots, x_{m-1}\rangle$, where $x_i = j$ if $x \in C_j{}^\gamma$ and γ is the least element of L such that $i \in \gamma$. Let $S = \{\sigma(x) \mid x \in N_1\}$. Jónsson's proof [**1**] of Whitman's theorem is so effective that one may assume that S is recursive. To show that $\alpha \to \equiv_\alpha$ is a dual isomorphism of L onto a sublattice of $\mathscr{E}(S)$, it is sufficient to prove that

$$x\,\theta_\alpha\,y \Leftrightarrow \sigma(x) \equiv_\alpha \sigma(y). \tag{1.1}$$

Now

$$x\,\theta_\alpha\,y \Rightarrow (x\,\theta_\gamma\,y \text{ for all } \gamma \subseteq \alpha) \Rightarrow [(\sigma(x))_i = (\sigma(y))_i \text{ for all } i \in \alpha] \Rightarrow \sigma(x) \equiv_\alpha \sigma(y).$$

The reverse implication is proved by induction on the number of elements of α. If $\alpha = \emptyset$, then the proof is easy. If $\alpha \neq \emptyset$, then either α is the least element of L such that $i \in \alpha$, for some i, or else $\alpha = \beta \vee \gamma$ for some β and γ in L both strictly contained in α. In the first case,

$$[\sigma(x) \equiv_\alpha \sigma(y)] \Rightarrow [(\sigma(x))_i = (\sigma(y))_i] \Rightarrow x\,\theta_\alpha\,y,$$

by definition of σ. In the second case,

$$\begin{aligned}(\sigma(x) \equiv_\alpha \sigma(y)) &\Rightarrow (\sigma(x) \equiv_\beta \sigma(y) \wedge \sigma(x) \equiv_\gamma \sigma(y)) \\ &\Rightarrow (x\,\theta_\beta\,y \wedge x\,\theta_\gamma\,y) \Rightarrow x\,\theta_\alpha\,y.\end{aligned}$$

For the initial segment problem it is necessary to consider even more special representations.

Definition 1. A finite lattice L is well-representable if there is a finite set $T = \{t_0, \ldots, t_{k-1}\}$ and dual isomorphism $\alpha \to \theta_\alpha$ of L onto a sublattice of $\mathscr{E}(T)$ such that for every $u, v \in T$ and $i_1, i_2 = 0, \ldots, k-1$, if

$$(\alpha)_{\alpha \in L}(t_{i_1}\,\theta_\alpha\,t_{i_2} \Rightarrow u\,\theta_\alpha\,v),$$

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then there are $t_0', \ldots, t_{k-1}' \in T$ such that $t_{i_1}' = u$ and $t_{i_2}' = v$ and for every $i, j = 0, \ldots, k-1$ and $\alpha \in L$,

$$t_i \,\theta_\alpha\, t_j \Rightarrow t_i' \,\theta_\alpha\, t_j'.$$

Given a well-representation T of a lattice in normal form, McKenzie's construction (proof of Theorem 1) produces a well-representation T^* such that $\theta_\emptyset^*$ is the unit in $\mathscr{E}(T^*)$; then the final step of the proof yields a well-representation by a set of sequences in the manner of Theorem 1.

THEOREM 2. *Either of the following conditions is sufficient for the finite lattice L to be well-representable:*
(a) *L is distributive,*
(b) *$L = \mathrm{PG}(F, m-1) =$ (the subspace lattice of an m-dimensional vector space over the finite field F).*

Proof. (a) L may be assumed to be a sublattice of $\mathscr{P}(\{0, \ldots, m-1\})$. Let S be the set of all sequences of 0s and 1s of length m and let θ_α be $\equiv_\alpha$. The computations which verify well-representation are straightforward. (Indeed this representation satisfies the condition: if $t_1, \ldots, t_p \in S$ and $t_i \equiv_{\alpha_i \cap \alpha_j} t_j$ for all i, j, then there is a $t \in S$ such that $t \equiv_{\alpha_i} t_i$ for all i; this is even stronger than the well-representability condition. The stronger condition was used in [**5**] to prove that every finite distributive lattice is isomorphic to an initial segment of hyperdegrees; that proof could probably be modified for the weaker condition, to prove that every well-representable lattice is isomorphic to an initial segment of hyperdegrees.)

(b) Let S be the set of all linear transformations of V into itself, where V is an m-dimensional vector space over the finite field F. For each $\alpha \in \mathrm{PG}(F, m-1)$, set $\Phi \,\theta_\alpha\, \Psi \Leftrightarrow (x)\ (x \in \alpha \Rightarrow \Phi x = \Psi x)$. Direct computation shows that the correspondence $\alpha \rightarrow \theta_\alpha$ is a dual isomorphism. For example, if $\Phi \,\theta_{\alpha \cap \beta}\, \Psi$, there are sets B_0, B_1, B_2, B_3 which are bases of $\alpha \cap \beta$, α, β, and V, respectively, and such that $B_0 \subseteq B_i \subseteq B_3$ $(i = 1, 2)$. Then

$$\Phi' b = \begin{cases} \Phi b & \text{if } b \in B_1, \\ \Psi b & \text{if } b \in B_2 - B_1, \\ 0 & \text{if } b \in B_3 - (B_1 \cup B_2), \end{cases}$$

defines a member Φ' of S such that

$$\Phi \,\theta_\alpha\, \Phi' \,\theta_\beta\, \Psi.$$

If $S = \{\Phi_0, \ldots, \Phi_{k-1}\}$ and if Ψ and X are members of S such that $\Psi x = \mathrm{X} x$ whenever $\Phi_{i_1} x = \Phi_{i_2} x$, then the null space of $\Psi - \mathrm{X}$ contains that of $\Phi_{i_1} - \Phi_{i_2}$ so that there is an $\Omega \in S$ such that $\Omega(\Phi_{i_1} - \Phi_{i_2}) = \Psi - \mathrm{X}$. Let $\Phi_i' = \Omega(\Phi_i - \Phi_{i_2}) + \mathrm{X}$. Then a simple series of computations shows that $\Phi_{i_1}' = \Psi$, $\Phi_{i_2}' = \mathrm{X}$, and $\Phi_i \,\theta_\alpha\, \Phi_j \Rightarrow \Phi_i' \,\theta_\alpha\, \Phi_j'$. Thus the well-representability condition is satisfied.

Lachlan [**2**] has shown that every countable distributive lattice is isomorphic to an initial segment of degrees. In [**3**] Lerman proved that if $n - 1$ is a prime power, then the lattice L_n having n incomparable elements together with least and greatest elements is isomorphic to an initial segment of degrees. The key Lemma 2.3 in Lerman's proof expresses what is here called the well-representability of L_n. These lattices are exactly the lattices $\mathrm{PG}(F, 1)$. McKenzie has observed that every well-representable lattice is modular, but not conversely. Lerman also proved that the five-element non-modular lattice is isomorphic to an initial segment of degrees, using a weaker form of well-representability. We first became aware of Lerman's methods during the final stages of preparing this paper; we have not chosen to make the revisions necessary to include the weaker form of well-representability, largely because we cannot prove any very general theorems about it.

2. Factorization of partial recursive functionals. If $\Sigma_0, \Sigma_1, \ldots$ are sets of sequences, then $\Sigma_0 * \ldots * \Sigma_n$ is the set of all $\sigma_0 * \ldots * \sigma_n$ such that $\sigma_i \in \Sigma_i$ for $i = 0, \ldots, n$, and $\Sigma_0 * \Sigma_1 * \ldots$ is the union of the $\Sigma_0 * \ldots * \Sigma_n$ over all $n \in N$, together with the null sequence. If $\Sigma = \Sigma_0 * \Sigma_1 * \ldots$, then Σ will be called regular if $\{\langle \sigma, n\rangle | \sigma \in \Sigma_n\}$ is recursive and

$$(n)(\sigma)(\tau)(\sigma, \tau \in \Sigma_n \rightarrow lh(\sigma) = lh(\tau) > 0).$$

If in addition each Σ_n is finite, and the cardinality of Σ_n is a recursive function of n, then Σ will be called compact.

Let $\bar{f}(x) = \langle f(0), \ldots, f(x-1)\rangle$. Associated with a regular set Σ is the class $F(\Sigma) = \{f \mid \bar{f}(x) \in \Sigma$ for infinitely many $x\}$. If Σ is compact, then $F(\Sigma)$ is compact as a subspace of N^N (topologized as a product of discrete spaces). In any case the classes $[\sigma] = \{f \mid f \in F(\Sigma) \wedge \bar{f}(lh(\sigma)) = \sigma\}$ for $\sigma \in \Sigma$ form a basis for $F(\Sigma)$. A class $F \subseteq N^N$ will be called regular if $F = F(\Sigma)$ for some regular Σ. Several useful properties of regular Σ are evident. If σ, τ, and $\sigma * \rho$ are all members of Σ, then so is $\tau * \rho$. If $\sigma \in \Sigma$, define

$$\Sigma \cap \sigma = \{\tau | \tau \in \Sigma \wedge \tau \text{ extends } \sigma\};$$

then $\Sigma \cap \sigma$ is again regular. Finally, if $f \in F(\Sigma)$, $\sigma \in \Sigma$, and

$$g(x) = \begin{cases} (\sigma)_x & \text{if } x < lh(\sigma), \\ f(x) & \text{if } x \geqq lh(\sigma), \end{cases}$$

then $g \in F(\Sigma)$.

Let e^f be the partial function which results when the partial recursive (p.r.) functional with index e is applied to the function f, and let $e^{\bar{f}(x)}$ be that part of e^f which is computed using a knowledge of $f(z)$ for $z < x$ only; formally, $e^{\bar{f}(x)}(y)$ is defined exactly when $(Ew)_{w \leqq x} T_1{}^1(\bar{f}(w), e, y)$, in which case $e^{\bar{f}(x)}(y)$ is $U(w_0)$, where w_0 is the least such w.

The theory of factorization of p.r. functionals is naturally complicated by the partial nature of the functionals, so that many theorems require suitable

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hypotheses restricting this partialness. One such restriction will be accomplished by a notational convention. The symbol π (often with a subscript) will be used only to denote a p.r. functional having either the form $\pi f(x) = 0$ or the form $\pi f(x) = f(h(x))$, where h is recursive, $h^{-1}(y)$ is finite for each y, and $k(y) = \max\{x \mid h(x) = y\}$ is recursive. In the former case, $\pi\sigma$ is the partial function

$$\pi\sigma(x) = \begin{cases} 0 & \text{if } x < lh(\sigma), \\ \text{undefined otherwise,} \end{cases}$$

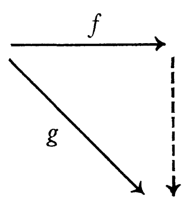
and in the latter case,

$$\pi\sigma(x) = \begin{cases} (\sigma)_{h(x)} & \text{if } h(x) < lh(\sigma), \\ \text{undefined otherwise.} \end{cases}$$

If θ and θ' are partial functions, then θ *extends* θ' if $\theta(x)$ is defined and equal to $\theta'(x)$ whenever the latter is defined, and $\theta = \theta'$ if each of θ, θ' extends the other. Also, $\theta(x) = \theta'(x)$ if both $\theta(x)$ and $\theta'(x)$ are defined and the values are equal, $\theta(x) \blacktriangle \theta'(x)$ if $\theta(x)$ and $\theta'(x)$ are both defined but $\theta(x) \neq \theta'(x)$, and $\theta \blacktriangle \theta'$ if $(Ex)(\theta(x) \blacktriangle \theta'(x))$.

Definition 2. The p.r. functional with index e factors through the functional π on the regular class F if there is a d such that $d^{\pi f}$ extends e^f for all $f \in F$.

Theorem 3 is a partial analogue for p.r. functionals of the well-known theorems which state that the commutative diagram



can be completed provided f does not identify more things than g does.

THEOREM 3. *If Σ is regular, then e factors through π on $F(\Sigma)$ if and only if whenever f and g are in $F(\Sigma)$ and $e^f \blacktriangle e^g$, then $\pi f \neq \pi g$.*

Proof. The "only if" part is trivial, and has nothing to do with recursiveness. On the other hand, if the condition is satisfied, then $d^{\pi f}$ extends e^f for all $f \in F(\Sigma)$, where d is an index of the p.r. functional defined by the following instructions.

Given πf (for an unknown $f \in F(\Sigma)$) and given x, search for a $\sigma \in \Sigma$ such that πf extends $\pi\sigma$ and such that $e^\sigma(x)$ is defined. (The special form of π makes $\pi\sigma$ a finite partial function such that one may determine, effectively relative to πf, whether πf extends $\pi\sigma$.) If and when such a σ is found, give $e^\sigma(x)$ as output.

If $f \in F(\Sigma)$ is such that $e^f(x)$ is defined, then such σ exist (e.g. $\bar{f}(y)$ for sufficiently large y) and one eventually will be found. If $e^f(x) \blacktriangle e^\sigma(x)$, then let

$$h(y) = \begin{cases} (\sigma)_y & \text{if } y < lh(\sigma), \\ f(y) & \text{if } y \geqq lh(\sigma). \end{cases}$$

Then $h \in F(\Sigma)$ by regularity of Σ, and $\pi h = \pi f$ because of the special form of π and the choice of σ so that πf extends $\pi\sigma$; but $e^f \blacktriangle e^h$, contrary to hypothesis.

Write $\sigma \equiv_\pi \tau$ if $lh(\sigma) = lh(\tau)$ and $\pi\sigma = \pi\tau$, and $f \equiv_\pi g$ if $\pi f = \pi g$. The equivalence relation $\equiv_\pi$ is more or less the kernel of π and will turn out to have more or less the properties expected of kernels. We shall write ker π instead of $\equiv_\pi$ sometimes, especially when we have in mind the properties of the relation as a member of $\mathscr{E}(\Sigma)$ or $\mathscr{E}(F(\Sigma))$. The lattice properties of the kernels of p.r. functionals are of considerable importance, as witnessed by the two corollaries of Theorem 3 which follow. Let $\mathbf{deg}(g)$ be the degree of g.

Corollary 1. *If* Σ *is regular and* $\ker \pi = (\ker \pi_1) \cap (\ker \pi_2)$ *in* $\mathscr{E}(\Sigma)$, *then* $\mathbf{deg}(\pi f) = \mathbf{deg}(\pi_1 f) \cup \mathbf{deg}(\pi_2 f)$ *for all* $f \in F(\Sigma)$.

Proof. If either of π_1, π_2 is the zero functional, then the result is easy; thus assume that $\pi_i f(x) = f(h_i(x))$ for $i = 1, 2$. Setting

$$h_3(x) = \begin{cases} h_1(x/2) & \text{if } x \text{ is even}, \\ h_2((x-1)/2) & \text{if } x \text{ is odd}, \end{cases}$$

and $\pi_3 f(x) = f(h_3(x))$ yields $\mathbf{deg}(\pi f) = \mathbf{deg}(\pi_3 f)$ by two applications of Theorem 3. But plainly $\mathbf{deg}(\pi_3 f) = \mathbf{deg}(\pi_1 f) \cup \mathbf{deg}(\pi_2 f)$.

The following corollary has much the effect of a dual of Corollary 1. Corollaries 1 and 2, since they make $\cap$ in $\mathscr{E}(\Sigma)$ correspond to $\cup$ in the degrees and dually, explain why dual representations were discussed in § 1 instead of representations.

Let us say that e is *densely defined* on Σ if for each $\sigma \in \Sigma$ and $z \in N$ there is a $\tau \in \Sigma$ extending σ such that $e^\tau(z)$ is defined. It is easily seen that e is densely defined on Σ if and only if for each $\sigma \in \Sigma$ there is an $f \in [\sigma]$ such that e^f is total.

Corollary 2. *If* Σ *is regular*, $\ker \pi = (\ker \pi_1) \cup (\ker \pi_2)$ *in* $\mathscr{E}(\Sigma)$, *and* e *is densely defined on* Σ, *then* e *factors through* π *on* $F(\Sigma)$ *if and only if* e *factors through both* π_1 *and* π_2 *on* $F(\Sigma)$.

Proof. If e factors through π, then

$$\begin{aligned} \pi_1 f = \pi_1 g &\Rightarrow \pi_1 \bar{f}((y)) = \pi_1(\bar{g}(y)) \quad \text{for all } y \\ &\Rightarrow \pi(\bar{f}(y)) = \pi(\bar{g}(y)) \quad \text{for all } y \\ &\Rightarrow \pi f = \pi g \\ &\Rightarrow \sim(e^f \blacktriangle e^g), \end{aligned}$$

and similarly for π_2.

Conversely, if e factors through both π_1 and π_2, suppose (for *reductio ad absurdum* via Theorem 3) that $f, g \in F(\Sigma)$, $\pi f = \pi g$, and $e^f \blacktriangle e^g$. Let z, $\sigma = \bar{f}(y) \in \Sigma$ and $\tau = \bar{g}(y) \in \Sigma$ be such that $e^\sigma(z) \blacktriangle e^\tau(z)$. Then $\sigma \equiv_\pi \tau$, and hence by definition of $\bigcup$ in $\mathscr{E}(\Sigma)$ there are $\sigma_1, \ldots, \sigma_n \in \Sigma$ such that

$$\sigma = \sigma_1 \equiv_{\pi_1} \sigma_2 \equiv_{\pi_2} \ldots \equiv_{\pi_1} \sigma_{n-1} \equiv_{\pi_2} \sigma_n = \tau.$$

If $e^{\sigma_i}(z)$ were defined for each i, then the way would be clear; instead, one accomplishes the same effect as follows. Set $\sigma_i{}^0 = \sigma_i$. Given $\sigma_i{}^j$ for some $j < n$ and all i, choose σ_{j+1}^{j+1} extending σ_{j+1}^j so that $e^\omega(z)$ is defined ($\omega = \sigma_{j+1}^{j+1}$) and set $\sigma_i{}^{j+1} = \sigma_i{}^j * \rho$ (where $\sigma_{j+1}^{j+1} = \sigma_{j+1}^j * \rho$) for all $i \neq j+1$. If j is even, then $e^\omega(z) = e^{\omega'}(z)$ ($\omega = \sigma_j{}^{j+1}$, $\omega' = \sigma_{j+1}^{j+1}$) for the alternative would violate the factorability of e through π_2, and similarly if j is odd, using π_1 in place of π_2. Then after a while one has $e^\sigma(z) = \sigma^{\sigma_1'}(z) = \ldots = e^{\sigma_n'}(z) = e^\tau(z)$ (where $\sigma_i' = \sigma_i{}^{n+1}$) contrary to the choice of σ, τ, z.

It follows immediately from Theorem 3 that

$$\ker \pi_1 = \ker \pi_2 \Rightarrow \mathbf{deg}(\pi_1 f) = \mathbf{deg}(\pi_2 f) \text{ for all } f \in F(\Sigma).$$

For the initial segment problem one needs to approximate this result with an arbitrary p.r. functional in place of one of the given functionals.

Definition 3. (The p.r. functional with index) e factors exactly through π on F if

$$(f)(g)[f, g \in F \Rightarrow (\pi f \neq \pi g \Leftrightarrow e^f \blacktriangle e^g)].$$

By Theorem 3, if e factors exactly through π on F, then e factors through π on F.

THEOREM 4. *If Σ is compact and e factors exactly through π on $F(\Sigma)$, then $\pi f \equiv_T e^f$ for all $f \in F(\Sigma)$ such that e^f is total.*

Proof. It is clear that $e^f \leqq_T \pi f$ for all the f in question, and if π is the zero functional, then $\pi f \leqq_T e^f$ trivially. Suppose then that $\pi f(x) = f(h(x))$ for all f and x. Then given the total function e^f for some (unknown) $f \in F(\Sigma)$ one can effectively calculate $(\overline{\pi f})(x)$ for an arbitrary number x as follows. Let n be the least number such that whenever $\sigma \in \Sigma_0 * \ldots * \Sigma_n$, then $lh(\sigma) \geqq \max\{h(y) | y < x\}$. Make a list of all $\sigma \in \Sigma_0 * \ldots * \Sigma_n$. For each σ in the list search for $\{\rho_1, \ldots, \rho_m\} \subseteq \Sigma$ such that $e^{\rho_i} \blacktriangle e^f$ for $i = 1, \ldots, m$ and such that $[\sigma] = [\rho_1] \cup \ldots \cup [\rho_m]$ (working on all the σ in the list simultaneously). If and when such $\rho_1, \ldots, \rho_m$ is found, erase σ from the list. Continue until $(\overline{\pi\sigma})(x) = (\overline{\pi\tau})(x)$ for all σ, τ remaining in the list.

(*Note.* $(\overline{\pi\sigma})(x) = \langle \pi\sigma(0), \ldots, \pi\sigma(x-1)\rangle$ is defined for all σ in the list because of the choice of $lh(\sigma)$.) Then (we claim) $(\overline{\pi f})(x)$ is the common value of $(\overline{\pi\sigma})(x)$ for σ still remaining in the list.

There can be no doubt that the procedure just outlined is effective (relative to the given total function e^f), but it must be shown that the procedure terminates and that the last remaining $(\overline{\pi\sigma})(x)$ is in fact $(\overline{\pi f})(x)$. I.e., it must be shown that if σ is in the original list and $(\overline{\pi\sigma})(x) \neq (\overline{\pi f})(x)$, then σ is eventually erased.

If $(\overline{\pi\sigma})(x) \neq (\overline{\pi f})(x)$, then $\pi g \neq \pi f$, and hence by exactness $e^g \blacktriangle e^f$, for every $g \in [\sigma]$. This means that the set of $[\rho]$ such that $\rho \in \Sigma$, ρ extends σ, and $e^\rho \blacktriangle e^f$ constitutes an open covering of the compact set $[\sigma]$. A finite subcovering yields the desired $\rho_1, \ldots, \rho_m$.

LEMMA 5. *If Σ is regular, e does not factor through π on $[\sigma]$ for any $\sigma \in \Sigma$, and e is densely defined on $F(\Sigma)$, then for every $\sigma_1 \equiv_\pi \sigma_2$ in Σ there are $\tau_1 \equiv_\pi \tau_2$ in Σ extending σ_1 and σ_2, respectively, such that $e^{\tau_1} \blacktriangle e^{\tau_2}$.*

Proof. Choose $z \in N$ and $\sigma_1{}^1 \equiv_\pi \sigma_1{}^2$ in Σ extending σ_1 such that $e^{\sigma_1{}^1}(z) \blacktriangle e^{\sigma_1{}^2}(z)$, by Theorem 3. Let $\sigma_2{}^1 = \sigma_2 * \rho$, where $\sigma_1{}^1 = \sigma_1 * \rho$. Let $\tau_2 \in \Sigma$ extending $\sigma_2{}^1$ be such that $e^{\tau_2}(z)$ is defined. Then for $i = 1$ or $i = 2$, $e^{\tau_2}(z) \blacktriangle e^{\sigma_1{}^i}(z)$. For such i, let $\tau_1 = \sigma_1{}^i * \rho$, where $\tau_2 = \sigma_2{}^1 * \rho$. Then $\tau_1 \equiv_\pi \tau_2$ and $e^{\tau_1} \blacktriangle e^{\tau_2}$.

3. The sublattice theorem. By a sublattice of the degrees is meant a set D of degrees such that the greatest lower bound of any two degrees in D exists, and both it and the least upper bound are again in D.

THEOREM 6. *If L is any finite lattice, then there is a sublattice of degrees isomorphic to L.*

It may be assumed that L is in normal form, with unit $\{0, \ldots, m-1\}$. Apply Theorem 1 to obtain a set S of sequences, all of length m, such that $\alpha \to \equiv_\alpha$ is a dual isomorphism of L onto a sublattice of $\mathscr{E}(S)$. Consider the regular set $\Sigma = S * S * \ldots$.

Extend the relations $\equiv_\alpha$ to all of Σ by setting $\sigma \equiv_\alpha \tau$ if and only if

$$lh(\sigma) = lh(\tau) \wedge (k)(i)((km + i < lh(\sigma) \wedge i \in \alpha) \Rightarrow (\sigma)_{km+i} = (\tau)_{km+i}).$$

Thus if $\sigma_0, \ldots, \sigma_{k_0}, \tau_0, \ldots, \tau_{k_0} \in \Sigma$, then $\sigma_0 * \ldots * \sigma_{k_0} \equiv_\alpha \tau_0 * \ldots * \tau_{k_0}$ if and only if $\sigma_k \equiv_\alpha \tau_k$ for $0 \leqq k \leqq k_0$. Also for $\alpha \neq \emptyset$ define $(\pi_\alpha f)(x) = f(h(x))$, where $h(x) =$ the xth member of $\{km + i \mid k \in N, i \in \alpha\}$, so that $\pi_\alpha\sigma = \pi_\alpha\tau \Leftrightarrow \sigma \equiv_\alpha \tau$. For $\alpha = \emptyset$ let $(\pi_\alpha f)(x) = 0$, so that the last equivalence will still hold.

LEMMA 7. *The map $\alpha \to \equiv_\alpha$ is a dual isomorphism of L onto a sublattice of $\mathscr{E}(\Sigma)$.*

Proof. The only possible problem is to show that $\equiv_{\alpha\cap\beta} = (\equiv_\alpha) \cup (\equiv_\beta)$ in $\mathscr{E}(\Sigma)$. If $\sigma, \tau \in \Sigma$ and $\sigma \equiv_{\alpha\cap\beta} \tau$, then $lh(\sigma) = lh(\tau) = k_0 m$ say (even if $\alpha \cap \beta = \emptyset$). Say also that $\sigma = \sigma_0 * \ldots * \sigma_{k_0-1}$ and $\tau = \tau_0 * \ldots * \tau_{k_0-1}$, where each σ_k and τ_k is a member of S. Then for each k there is a chain

$$\sigma_k = \sigma_k{}^0 \equiv_\alpha \sigma_k{}^1 \equiv_\beta \ldots \equiv_\beta \sigma_k{}^{n(k)} = \tau_k \tag{3.1}$$

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in S. Set $\rho_k = \sigma_0 * \ldots * \sigma_{k_0-(k+1)} * \tau_{k_0-k} * \ldots * \tau_{k_0-1}$. Then $\rho_0 = \sigma$ and $\rho_{k_0-1} = \tau$ and (3.1) states how to build a chain

$$\rho_k = \rho_k^{\,0} \equiv_\alpha \rho_k^{\,1} \equiv_\beta \ldots \equiv_\beta \rho_k^{\,n(k)} = \rho_{k+1}$$

in Σ, for $k = 0, \ldots, k_0 - 2$. Taken together, these latter chains constitute a chain

$$\sigma \equiv_\alpha \sigma^1 \equiv_\beta \sigma^2 \equiv_\alpha \ldots \equiv_\alpha \sigma^n \equiv_\beta \tau,$$

which is exactly what is required to show that $\equiv_{\alpha\cap\beta} = (\equiv_\alpha) \bigcup (\equiv_\beta)$ in $\mathscr{E}(\Sigma)$.

If Σ is regarded as a relational system with relations $\equiv_\alpha$ for $\alpha \in L$, then there is an obvious isomorphism of Σ onto $\Sigma \cap \rho$ for any $\rho \in \Sigma$. Hence Lemma 7 applies to $\Sigma \cap \rho$ as well.

Now in order that $\{\mathbf{d}_\alpha | \alpha \in L\}$ be a sublattice of the degrees isomorphic to L via the correspondence $\alpha \rightarrow \mathbf{d}_\alpha$, it is necessary and sufficient that:

$$\alpha \leqq \beta \Rightarrow \mathbf{d}_\alpha \leqq \mathbf{d}_\beta, \tag{3.2}$$

$$\alpha \nleqq \beta \Rightarrow \mathbf{d}_\alpha \nleqq \mathbf{d}_\beta, \tag{3.3}$$

$$\mathbf{d}_{\alpha\vee\beta} = \mathbf{d}_\alpha \bigcup \mathbf{d}_\beta, \tag{3.4}$$

$$\mathbf{d}_{\alpha\cap\beta} = \mathbf{d}_\alpha \bigcap \mathbf{d}_\beta. \tag{3.5}$$

Implicit in (3.5) is the assertion that $\mathbf{d}_\alpha \bigcap \mathbf{d}_\beta$ exists.

The strategy is to define $\sigma_0, \sigma_1, \ldots \in \Sigma$ such that σ_{n+1} extends σ_n for each n and such that $\lim_n lh(\sigma_n) = \infty$. Then $\bigcap\{[\sigma_n] | n \in N\}$ consists of a single point, say h, and $\mathbf{d}_\alpha$ is defined to be $\mathbf{deg}(\pi_\alpha h)$. Then (3.2) is automatic, and (3.4) follows from Lemma 7 and Corollary 1 of Theorem 3, since $h \in F(\Sigma)$. To satisfy (3.3) and (3.5), it is necessary to define the σ_n carefully. Consider the following requirements which might be met by $\sigma \in \Sigma$:

$R_{d,\alpha,\beta}$: if $\alpha \nleqq \beta$, then for every $f \in [\sigma]$, if $d^{\pi_\beta f}$ is total, then $\pi_\alpha f \neq d^{\pi_\beta f}$;
$R_{d,\alpha,e,\beta}$: for every $f \in [\sigma]$, if $d^{\pi_\alpha f}$ and $e^{\pi_\beta f}$ are total and equal, then $d^{\pi_\alpha f} \leqq_T \pi_{\alpha\cap\beta} f$.

Suppose that these requirements are enumerated $R_0, R_1, \ldots$ and that σ_n is always chosen so as to meet R_n. Then (3.3) will hold (since $R_{d,\alpha,\beta}$ will have been met for every d) and so will (3.5) (since $R_{d,\alpha,e,\beta}$ will have been met for every d and e). Thus the proof of Theorem 6 is reduced to showing that given any $\rho \in \Sigma$ and any one of the requirements $R_{d,\alpha,\beta}$ or $R_{d,e,\alpha,\beta}$ there is a $\sigma \in \Sigma$ extending ρ which meets the given requirement. (It is easy to ensure that $\lim_n lh(\sigma_n) = \infty$.)

$R_{d,\alpha,\beta}$; Case I: $\alpha \leqq \beta$. Trivial.

$R_{d,\alpha,\beta}$; Case II: e is not densely defined on $[\rho]$, where $e^f = d^{\pi_\beta f}$. Let $z \in N$ and $\sigma \in \Sigma$ extending ρ be such that for no $\tau \in \Sigma$ extending σ is $e^\tau(z)$ defined. Then $R_{d,\alpha,\beta}$ is trivially met by σ.

$R_{d,\alpha,\beta}$; Case III: otherwise.
Since $(\equiv_\beta) \nleq (\equiv_\alpha)$ in $\mathscr{E}(\Sigma)$, there are $\rho_1, \rho_2 \in \Sigma$ extending ρ such that $\rho_1 \equiv_\beta \rho_2$ but $\rho_1 \not\equiv_\alpha \rho_2$. Let z be such that $(\pi_\alpha\rho_1)(z) \blacktriangle (\pi_\alpha\rho_2)(z)$, $\tau_1 \in \Sigma$ extending ρ_1 such that $d^{\pi_\beta\tau_1}(z)$ is defined, $i = 1$ or 2 so that $d^{\pi_\beta\tau_1}(z) \blacktriangle (\pi_\alpha\rho_i)(z)$, and $\sigma = \rho_i * \rho'$ where $\tau_1 = \rho_1 * \rho'$. Then $\pi_\beta\sigma = \pi_\beta\tau_1$ and

$$(\pi_\alpha\sigma)(z) = (\pi_\alpha\rho_i)(z) \blacktriangle d^{\pi_\beta\sigma}(z),$$

so that $R_{d,\alpha,\beta}$ is met by σ.

$R_{d,\alpha,e,\beta}$; Case I: there is a $\sigma \in \Sigma$ extending ρ such that any of the following hold:

$$\text{for no } f \in [\sigma] \text{ is } d^{\pi_\alpha f} \text{ total,} \tag{3.6}$$

$$\text{for no } f \in [\sigma] \text{ is } e^{\pi_\beta f} \text{ total,} \tag{3.7}$$

$$d^{\pi_\alpha\sigma} \blacktriangle e^{\pi_\beta\sigma}. \tag{3.8}$$

Then clearly such a σ meets the requirement.

$R_{d,\alpha,e,\beta}$; Case II: otherwise.
Consider the p.r. functional with index c defined by:

$$c^\sigma(z) = \begin{cases} d^{\pi_\alpha\sigma}(z) & \text{if } \sigma \in \Sigma, \sigma \text{ extends } \rho, \text{ and } d^{\pi_\alpha\sigma}(z) \text{ is defined,} \\ e^{\pi_\beta\sigma}(z) & \text{if } \sigma \in \Sigma, \sigma \text{ extends } \rho, \text{ and } e^{\pi_\beta\sigma}(z) \text{ is defined,} \\ \text{undefined otherwise.} \end{cases}$$

The failure of (3.8) ensures the consistency of the definition of c, c is densely defined on $[\rho]$ because (3.6) and (3.7) fail, and finally c factors through both π_α and π_β on $[\rho]$ by Theorem 3. Since $(\ker \pi_\alpha) \cup (\ker \pi_\beta) = (\ker \pi_{\alpha\cap\beta})$ in $\mathscr{E}(\Sigma \cap \rho)$ by the remark after Lemma 7, Corollary 2 of Theorem 3 applies and c factors through $\pi_{\alpha\cap\beta}$ on $[\rho]$. But then $R_{d,\alpha,e,\beta}$ is met by ρ already. This completes the proof of Theorem 6.

4. The initial segment theorem.

An initial segment of degrees is a set D of degrees such that $c \leqq d \in D \Rightarrow c \in D$.

THEOREM 8. *If L is a well-representable lattice, then L is isomorphic to an initial segment of the degrees.*

By Theorem 3 it is safe to assume that L is in normal form with unit $\{0, \ldots, m-1\}$ and that $S = \{\tau_0, \ldots, \tau_{k-1}\}$ is a set of sequences of length m which, with the relations $\equiv_\alpha$, satisfies the well-representability condition (Definition 1). Extend the relations $\equiv_\alpha$ to all of Σ, and define p.r. functionals π_α, as in § 3.

A regular set $\Sigma = \Sigma_0 * \Sigma_1 * \ldots$ is said to be *admissible* if each Σ_n can be enumerated $\{\tau_0^n, \ldots, \tau_{k-1}^n\}$ so that for all $\alpha \in L$ and $i, j = 0, \ldots, k-1$, we have $\tau_i^n \equiv_\alpha \tau_j^n \Leftrightarrow \tau_i \equiv_\alpha \tau_j$. For example, $S * S * \ldots$ is admissible. If Σ is admissible and $\sigma \in \Sigma$, then $\Sigma \cap \sigma$ is admissible, since

$$\Sigma \cap \sigma = \Sigma_0' * \Sigma_{n+1} * \Sigma_{n+2} * \ldots$$

if $\sigma \in \Sigma_0 * \ldots * \Sigma_{n-1}$ and $\Sigma_0' = \{\sigma * \tau | \tau \in \Sigma_n\}$. If Σ is admissible, then the relations $\equiv_\alpha$ form a sublattice of $\mathscr{E}(\Sigma)$ dually isomorphic to L by a proof similar to that of Lemma 7. Indeed, Σ satisfies the condition for well-representability, in the following sense. If $\rho_0, \rho_1 \in \Sigma$, $i_0, i_1 \in \{0, \ldots, k-1\}$, and

$$(\alpha)_{\alpha \in L}(\tau_{i_0} \equiv_\alpha \tau_{i_1} \Rightarrow \rho_0 \equiv_\alpha \rho_1),$$

then there are $\tau_0', \ldots, \tau_{k-1}' \in \Sigma$ such that $\tau_{i_0}' = \rho_0$, $\tau_{i_1}' = \rho_1$, and for every $i, j = 0, \ldots, k-1$, we have

$$\tau_i \equiv_\alpha \tau_j \Rightarrow \tau_i' \equiv_\alpha \tau_j'.$$

For the hypothesis implies that $lh(\rho_0) = lh(\rho_1)$, since $\rho_0 \equiv_\emptyset \rho_1$; writing $\rho_i = \rho_i^0 * \ldots * \rho_i^n$ $(i = 0, 1)$, where $\rho_i^l \in \Sigma_l$, one obtains the required $\tau_0', \ldots, \tau_{k-1}'$ by applying the well-representability condition to the "factors" ρ_0^l and ρ_1^l for each l.

Actually, Σ has an even stronger well-representation property: if $\sigma_0, \ldots, \sigma_{k-1} \in \Sigma$ are such that for all $\alpha \in L$ and $i, j \in \{0, \ldots, k-1\}$, $\tau_i \equiv_\alpha \tau_j \Rightarrow \sigma_i \equiv_\alpha \sigma_j$, and if $\rho_0, \rho_1 \in \Sigma$ extend $\sigma_{i_0}, \sigma_{i_1}$, respectively, and satisfy

$$(\alpha)_{\alpha \in L}(\tau_{i_0} \equiv_\alpha \tau_{i_1} \Rightarrow \rho_0 \equiv_\alpha \rho_1),$$

then the τ_i' can be chosen so as to extend the respective σ_i. For i_0 and i_1 this is already the case; for all other i replace the first $lh(\sigma_i)$ terms of τ_i' by the terms of σ_i.

Let $R(\sigma, \tau)$ be a predicate satisfying

(4.1) $(\sigma)(\tau)\{lh(\sigma) = lh(\tau) \Rightarrow (E\sigma')(E\tau')[(\sigma', \tau'$ extend σ, τ, respectively$) \wedge R(\sigma', \tau') \wedge (\alpha)_{\alpha \in L}(\sigma \equiv_\alpha \tau \Rightarrow \sigma' \equiv_\alpha \tau')]\}$

(4.2) $(\sigma)(\tau)(\sigma')(\tau')\{[(\sigma', \tau'$ extend σ, τ, respectively$) \wedge R(\sigma, \tau)] \Rightarrow R(\sigma', \tau')\}$

(where the variables σ, τ, σ', τ' range over Σ). Also let $\sigma_0, \ldots, \sigma_{k-1} \in \Sigma$ satisfy $\sigma_i \equiv_\alpha \sigma_j$ whenever $\tau_i \equiv_\alpha \tau_j$. Then by k^2 applications of the argument of the preceding paragraph one obtains $\sigma_0', \ldots, \sigma_{k-1}' \in \Sigma$ extending $\sigma_0, \ldots, \sigma_{k-1}$, respectively, satisfying $R(\sigma_i', \sigma_j')$, and still satisfying $\sigma_i' \equiv_\alpha \sigma_j'$ whenever $\tau_i \equiv_\alpha \tau_j$.

Now suppose that one needs to define an admissible $\Sigma' \subseteq \Sigma$ such that $R(\sigma, \tau)$ for all $\sigma, \tau \in \Sigma_n'$, $n \in N$. It can be done by repeated applications of the above construction, provided R is recursively enumerable. Namely, let $\Sigma' = \Sigma_0' * \Sigma_1' * \ldots$ where the Σ_n' are defined by induction on n as follows. Let $\{\sigma_0, \ldots, \sigma_{k-1}\}$ be the enumeration of Σ_0 by which $\Sigma = \Sigma_0 * \Sigma_1 * \ldots$ is admissible and obtain $\Sigma_0' = \{\sigma_0', \ldots, \sigma_{k-1}'\}$ by the construction. To go from Σ_n' to Σ_{n+1}' is similar but requires k^{n+1} applications; say $\Sigma_n' = \{\rho_0, \ldots, \rho_{k^{n+1}-1}\}$, where each $\rho_i \in \Sigma_0 * \ldots * \Sigma_{n_1}$. The first application proceeds from $\rho_0 * \sigma_0, \ldots, \rho_0 * \sigma_{k-1}$ (where $\Sigma_{n_1+1} = \{\sigma_0, \ldots, \sigma_{k-1}\}$ in the right order) to $\rho_0 * \sigma_0^1, \ldots, \rho_0 * \sigma_{k-1}^1$. The second application proceeds

$$\text{from} \quad \rho_1 * \sigma_0^1, \ldots, \rho_1 * \sigma_{k-1}^1 \quad \text{to} \quad \rho_1 * \sigma_0^2, \ldots, \rho_1 * \sigma_{k-1}^2.$$

We proceed in the same way through all the ρ_i. If $q = n^{k+1} - 1$ and $\rho_q * \sigma_0{}^{q+1}, \ldots, \rho_q * \sigma_{k-1}^{q+1}$ is the result of the last application, then

$$\Sigma'_{n+1} = \{\sigma_0{}^{q+1}, \ldots, \sigma_{k-1}^{q+1}\}.$$

The above results can be summarized in the following lemma.

LEMMA 9. *If Σ is admissible and R is a recursively enumerable predicate satisfying* (4.1) *and* (4.2), *then there is an admissible $\Sigma' \subseteq \Sigma$ such that*

$$(f)(g)[f, g \in F(\Sigma) \Rightarrow (En)R(\bar{f}(n), \bar{g}(n))].$$

In order that $\{\mathbf{d}_\alpha | \alpha \in L\}$ be an initial segment of the degrees isomorphic to L via the obvious correspondence, it is necessary and sufficient that (3.2) and (3.3) hold and in addition

$$\text{(4.3)} \qquad \mathbf{d} \leqq \mathbf{d}_\beta \Rightarrow \mathbf{d} = \mathbf{d}_\beta \vee (E\alpha)(\alpha < \beta \wedge \mathbf{d} \leqq \mathbf{d}_\alpha).$$

(The adequacy of (4.3) is a consequence of the finiteness of L.)

The strategy this time is to define a sequence $\Sigma^0 \supseteq \Sigma^1 \supseteq \ldots$ of admissible sets so that $\bigcap\{F(\Sigma^n) | n \in N\}$ consists of a single point h, and to define $\mathbf{d}_\alpha = \mathbf{deg}(\pi_\alpha h)$. Again (3.2) is automatic. Consider the following requirements:

$R_{d,\alpha,\beta}$: if $\alpha \nleqq \beta$, then for every $f \in F(\Sigma)$, if $d^{\pi_\beta f}$ is total, then $\pi_\alpha f \neq d^{\pi_\beta f}$;

$R_{e,\beta}$: if e^f is total for some $f \in F(\Sigma)$, e factors through π_β on $F(\Sigma)$, and for no $\alpha < \beta$ does e factor through π_α on $F(\Sigma)$, then e factors exactly through π_β on $F(\Sigma)$.

If for each requirement there is an n such that Σ^n meets the requirement, then (3.3) will hold because of $R_{d,\alpha,\beta}$, just as in § 3. Because of $R_{e,\beta}$, (4.3) will hold: if g is a function such that $\mathbf{deg}(g) = \mathbf{d}$ and $g = d^{\pi_\beta h} = e^h$ (where e is chosen so that $e^f = d^{\pi_\beta f}$ uniformly) and Σ^n meets $R_{e,\beta}$, then either e factors through π_α on $F(\Sigma^n)$ for some $\alpha < \beta$ (whence $\mathbf{d} \leqq \mathbf{d}_\alpha$) or else e factors exactly through π_β on $F(\Sigma^n)$ (whence $\mathbf{d} = \mathbf{d}_\beta$ by Theorem 5).

Thus the proof of the theorem reduces to showing that given an admissible Σ and one of the requirements, there is an admissible $\Sigma' \subseteq \Sigma$ which meets the requirement. $R_{d,\alpha,\beta}$ can be handled just as in § 3; thus consider $R_{e,\beta}$.

It may be assumed that

$$\text{(4.4)} \qquad e \text{ is densely defined on } \Sigma,$$

$$\text{(4.5)} \qquad e \text{ factors through } \pi_\beta \text{ on } \Sigma,$$

$$\text{(4.6)} \qquad \text{for each } \alpha < \beta \text{ and } \sigma \in \Sigma, e \text{ does not factor through } \pi_\alpha \text{ on } \Sigma \cap \sigma,$$

for otherwise $R_{e,\beta}$ can easily be met by $\Sigma' = \Sigma$ or $\Sigma' = \Sigma \cap \sigma$ for appropriate $\sigma \in \Sigma$. As a consequence of (4.4) and (4.6), we have:

$$\text{(4.7)} \qquad \text{for each } \alpha \ngeqq \beta \text{ and } \sigma \in \Sigma, e \text{ does not factor through } \pi_\alpha \text{ on } \Sigma \cap \sigma.$$

For $\ker \pi_{\alpha\cap\beta} = (\ker \pi_\alpha) \bigcup (\ker \pi_\beta)$ in $\mathscr{E}(\Sigma \cap \sigma)$, and hence by Corollary 2 of Theorem 3, if e factored through π_α, then e would factor through $\pi_{\alpha\cap\beta}$; but $\alpha \cap \beta < \beta$.

Thus, by Lemma 5, given $\sigma, \tau \in \Sigma$, there are $\sigma', \tau' \in \Sigma$ extending σ, τ, respectively, such that

$$(\alpha)_{\alpha\in L}(\sigma \equiv_\alpha \tau \Rightarrow \sigma' \equiv_\alpha \tau') \quad \text{and} \quad \sigma' \not\equiv_\beta \tau' \Rightarrow e^{\sigma'} \blacktriangle e^{\tau'}.$$

If $R(\sigma, \tau)$ is $\sigma \not\equiv_\beta \tau \Rightarrow e^\sigma \blacktriangle e^\tau$, then R is a recursively enumerable predicate satisfying (4.1) and (4.2). Hence by Lemma 9 there is an admissible $\Sigma' \subseteq \Sigma$ such that for every $f, g \in F(\Sigma')$, $R(\bar{f}(n), \bar{g}(n))$ holds for infinitely many n. Then e factors exactly through π_β on Σ' and $R_{e,\beta}$ is met by Σ'. This completes the proof of Theorem 8.

5. Conclusions. We hold that the key to verifying Sacks' conjecture (that every finite lattice is isomorphic to an initial segment of degrees [**4**, p. 171]) is Whitman's theorem. We believe that the consideration of factorization of p.r. functionals reduces the notational complexity of the problem, and we suggest that such considerations might be appropriate elsewhere in recursion theory as well. Finally we recommend a new approach to Sacks' conjecture: instead of proving the best result for larger classes of lattices, prove better results for the class of all finite lattices. An initial segment which is a lattice is just a sublattice with some strong additional properties; perhaps our Theorem 6 can be improved to include some additional properties.

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