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Boundaries of K-types, restriction of cohomology, and the multiplicity free case

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1. Introduction

Explicit formulas for K-types of a $(\mathfrak{g}, K)$ module, such as Blattner's formula, are well known. However, the formulas are often too complex to make the K-type structure transparent. In this note we make use of some ideas of Vogan linking the 'edges' of the set of K-types to a certain restriction map from $\mathfrak{u}$ to $\mathfrak{u}\cap\mathfrak{k}$ cohomology. It is hoped that such ideas will lead to tighter control of the set of K-types.

To get at the notion of an 'edge', a general definition of a $\mathfrak{q}$-edge of the set of K-types is given in Section 2 for $(\mathfrak{g}, K)$ modules. Section 3 proves a theorem that links the geometry of a $\mathfrak{q}$-edge to the algebra of the image of a restriction map from $\mathfrak{u}$ to $\mathfrak{u}\cap\mathfrak{k}$ cohomology. Section 4 applies these notions to the special case of finite dimesional representations in which the reduction is multiplicity free. This shows that the general definitions mesh well with the geometric motivating ideas and pin down the structure of the K-types in these cases.

This problem was suggested by D. Vogan and his conversations have been very helpful.

2. Extremal K-types

Let G be a real reductive Lie group with K a maximal compact subgroup. Write $\mathfrak{g}_0 = \mathrm{Lie}(G)$ and $\mathfrak{k}_0 = \mathrm{Lie}(K)$ for their respective Lie algebras, $\mathfrak{g} = (\mathfrak{g}_0)_{\mathbb{C}}$ and $\mathfrak{k} = (\mathfrak{k}_0)_{\mathbb{C}}$ for their respective complexifications, and $\mathfrak{g} = \mathfrak{k}+\mathfrak{p}$ for the corresponding Cartan decomposition.

Fix a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{k}$. Write W_K for its Weyl group and $l(w)$ for the length of $w \in W_K$. Denote the centralizer of $\mathfrak{t}$ in $\mathfrak{g}$ by $\mathfrak{h}$, a Cartan subalgebra of $\mathfrak{g}$. Write W_G for its Weyl group. Fix a positive root system $\Delta^+(\mathfrak{k}, \mathfrak{t})$ and write ρ_K for the half sum of positive roots in $\Delta^+(\mathfrak{k}, \mathfrak{t})$. Choose a positive root system $\Delta^+(\mathfrak{g}, \mathfrak{h})$ making ρ_K dominant and write ρ_G for the half sum of positive roots in $\Delta^+(\mathfrak{g}, \mathfrak{h})$.

Let X be a $(\mathfrak{g}, K)$ module. For μ a dominant weight in $\mathfrak{t}$ of $\mathfrak{k}$, write V_μ for the irreducible representation of K with highest weight μ. The multiplicity of V_μ in

$X|_K$ will be denoted $m(\mu)$. It is often useful (for instance in computing certain Euler characteristics) to extend the domain of this multiplicity function to include arbitrary weights in $\mathfrak{t}$ of $\mathfrak{k}$. The extention will be called $m_e(\mu)$. To do this, fix μ an arbitrary weight in $\mathfrak{t}$ of $\mathfrak{k}$. If $\mu + \rho_K$ is singular, set $m_e(\mu) = 0$. Otherwise, there is a unique element $w \in W_K$ so that $w(\mu + \rho_K) - \rho_K$ is dominant. In this case, define

$$m_e(\mu) = (-1)^{l(w)} m(w(\mu + \rho_K) - \rho_K).$$

For the final piece of notation, let $\mathfrak{q} \subseteq \mathfrak{g}$ be a parabolic subalgebra containing $\mathfrak{h}$. Its Levi decomposition will be denoted

$$\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$$

with $\mathfrak{u}$ the nilradical and $\mathfrak{l}$ reductive. Write L for the normalizer of $\mathfrak{q}$ in G.

Going back to Vogan's ideas in [7] and [6] about pencils of K-types and strongly $\mathfrak{u}$-minimal K-types, we want to be able to speak about 'edges' of the set of K-types of X. To describe an edge, two things ought to be given: a set of edge directions and a set of outward directions. One way to combine both pieces of data is to specify a parabolic subalgebra $\mathfrak{q}$ where $\mathfrak{l}$ represents the edge directions and $\mathfrak{u}$ represents the outward directions. So the rough idea for saying that a K-type μ of X lies on a '$\mathfrak{q}$-edge' of the set of K-types is that $\mu + \delta$ is not a K-type for any nonzero δ in the real positive span of $\Delta(\mathfrak{u}, \mathfrak{t})$. This idea needs to be refined by making use of the extended multiplicity function m_e instead of m and deciding how much of the outward direction needs to be free of further K-types. The technical additions below are chosen to make $\mathfrak{q}$-edges appear in the image of a certain restriction map of cohomology.

DEFINITION 2.1. Fix X a $(\mathfrak{g}, K)$ module and $\mathfrak{q}$ a θ-stable parabolic. Write

$$R = \dim(\mathfrak{u} \cap \mathfrak{p}).$$

Let $w \in W_K$ so that $w\Delta(\mathfrak{u} \cap \mathfrak{k}, \mathfrak{t}) \subseteq \Delta^+(\mathfrak{k}, \mathfrak{t})$. Write

$$m = \min\{R - 1, l(w)\}.$$

We say that $\mu \in \mathfrak{t}^*$ is on a $\mathfrak{q}$-**edge** of the set of K-types of X if

(i) μ appears as a K-type of X.
(ii) $m_e(\mu + \Sigma_{j=1}^{m+1} \delta_j) = 0$ for any $\delta_j \in \Delta(\mathfrak{u} \cap \mathfrak{p}, \mathfrak{t}) \cup \{0\}$, not all δ_j zero.

The second condition says that there are no K-types (with respect to m_e) beyond μ in the given direction for a certain distance. In the case that $\dim(\mathfrak{u} \cap \mathfrak{p}) = 1$, the condition is particularly easy to check.

We will eventually show that such $\mathfrak{q}$-edges are related to the process of restricting certain $w\mathfrak{u}$ cohomology to $w\mathfrak{u} \cap \mathfrak{k}$ cohomology. To show that the theory is non-trivial, the following definitions and lemma are necessary. Given $w \in W_K$ and $\mathfrak{u}_K$ a parabolic in $\mathfrak{k}$ containing $\mathfrak{t}$ with $\Delta(\mathfrak{u}_K, \mathfrak{t}) \subseteq \Delta^+(\mathfrak{k}, \mathfrak{t})$, recall from [3] the definitions

$$\Delta_w^+ = \{\alpha \in \Delta^+(\mathfrak{k}, \mathfrak{t}) | w^{-1}\alpha \in -\Delta^+(\mathfrak{k}, \mathfrak{t})\},$$

$$W_K^1(\mathfrak{u}_K) = \{w \in W_K | \Delta_w^+ \subseteq \Delta(\mathfrak{u}_K, \mathfrak{t})\}.$$

LEMMA 2.2. *In Definition* 2.1, w *may also be chosen so that* $w \in W_K^1(w\mathfrak{u} \cap \mathfrak{k})$.

Proof. Write $\mathcal{P}$ for the finite non-empty set of all $w \in W_K$ satisfying $w(\Delta(\mathfrak{u} \cap \mathfrak{k}, \mathfrak{t})) \subseteq \Delta^+(\mathfrak{k}, \mathfrak{t})$. We claim that any $w \in \mathcal{P}$ with $l(w)$ minimal satisfies the condition of the lemma. Let w be such an element. Suppose $w \notin W_K^1(w\mathfrak{u}\cap\mathfrak{k})$. Then there is an $\alpha \in \Delta^+(\mathfrak{k}, \mathfrak{t})$ so that $w^{-1}\alpha \in -\Delta^+(\mathfrak{k}, \mathfrak{t})$ with $\alpha \notin \Delta(w\mathfrak{u} \cap \mathfrak{k}, \mathfrak{t})$. Write $r_\alpha \in W_K$ for the reflection through α. Then $r_\alpha w$ is still in $\mathcal{P}$, but $l(r_\alpha w) = l(w^{-1} r_\alpha) = l(w^{-1}) - 1 = l(w) - 1$ which contradicts the choice of w. $\square$

3. Cohomology of $\mathfrak{q}$-edges

We recall some notation from the Hochschild–Serre spectral sequence that will allow us to relate $\mathfrak{q}$-edges to a statement about restriction in cohomology. Let $\mathfrak{q}$ be a θ-stable parabolic as above and carry over the notation from the last section. There is a spectral sequence

$$E_t^{a,b} \Rightarrow H^{a+b}(\mathfrak{u}, X)$$

with

$$E_1^{a,b} = H^{a+b-r(a)}(\mathfrak{u} \cap \mathfrak{k}, X) \otimes_{\mathbb{C}} V_a^*.$$

Here V_a is a certain summand of $\bigwedge(\mathfrak{u}\cap\mathfrak{p})$ decomposed under the $L \cap K$ action, V_a^* is the dual of V_a, and $r(a)$ is a certain integer between 0 and $R = \dim(\mathfrak{u} \cap \mathfrak{p})$ with $V_a \subseteq \bigwedge^{r(a)}(\mathfrak{u} \cap \mathfrak{p})$ and $r(0) = 0$. See [2], [8], or [6] for details. The differential, d_t, has bidegree $(t, 1-t)$ and is an $L \cap K$ module map.

Being a first quadrant spectral sequence, there are inclusions $E_{t+1}^{0,b} \hookrightarrow E_t^{0,b}$. A finite number of these yield the inclusion

$$E_\infty^{0,b} \hookrightarrow E_1^{0,b} = H^b(\mathfrak{u} \cap \mathfrak{k}, X).$$

Since $H^b(\mathfrak{u}, X) \cong \bigoplus_n E_\infty^{n,b-n}$, projecting onto the $E_\infty^{0,b}$ component followed by the above inclusion produces a map from $H^b(\mathfrak{u}, X)$ to $H^b(\mathfrak{u} \cap \mathfrak{k}, X)$.

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DEFINITION 3.1. Let

$$\tau\colon H^b(\mathfrak{u}, X) \to H^b(\mathfrak{u}\cap\mathfrak{k}, X)$$

be the map defined above. τ is an $L\cap K$ map and will be called the **restriction map**.

The reason for calling τ the restriction map is because it is alternately realized as the map from $H^b(\mathfrak{u}, X)$ to $H^b(\mathfrak{u}\cap\mathfrak{k}, X)$ induced by restricting $\mathrm{Hom}(\bigwedge^b \mathfrak{u}, X)$ to $\mathrm{Hom}(\bigwedge^b(\mathfrak{u}\cap\mathfrak{k}), X)$ (see [6] and [7]).

The next two propositions are needed to make the connection between $\mathfrak{q}$-edges and the restriction map. They are both old ideas from Vogan.

PROPOSITION 3.2. *If* $\tau\colon H^b(\mathfrak{u}, X) \to H^b(\mathfrak{u}\cap\mathfrak{k}, X)$ *is not surjective, there is an integer* m *with* $0 \leqslant m \leqslant \min\{b, R-1\}$ *so that*

$$H^{b-m}(\mathfrak{u}\cap\mathfrak{k}, X) \otimes \bigwedge^{m+1}(\mathfrak{u}\cap\mathfrak{p})^* \neq 0.$$

Proof. If τ is not surjective, then for some t, the inclusion $E_{t+1}^{0,b} \hookrightarrow E_t^{0,b}$ is not surjective. This happens only if d_t is nonzero on $E_t^{0,b}$. So at the very least, the range of d_t, $E_t^{t,b+1-t}$, must be nonzero. Again, at the very least, this requires that $E_1^{t,b+1-t}$ be nonzero. By definition, this implies that $H^{b+1-r(t)}(\mathfrak{u}\cap\mathfrak{k}, X) \otimes V_t^*$ is nonzero. Setting $m = r(t)-1$ and noting that $V_t \subseteq \bigwedge^{r(t)}(\mathfrak{u}\cap\mathfrak{p})$, the proof is complete. □

PROPOSITION 3.3. *If* $\tau\colon H^b(\mathfrak{u}, X) \to H^b(\mathfrak{u}\cap\mathfrak{k}, X)$ *is not injective, there is an integer* m *with* $0 < m \leqslant \min\{b, R\}$ *so that*

$$H^{b-m}(\mathfrak{u}\cap\mathfrak{k}, X) \otimes \bigwedge^{m}(\mathfrak{u}\cap\mathfrak{p})^* \neq 0.$$

Proof. If τ is not injective, then $E_\infty^{n,b-n}$ is nonzero for some $n > 0$. At the very least, this means that $E_1^{n,b-n}$ is nonzero. Hence $H^{b-r(n)}(\mathfrak{u}\cap\mathfrak{k}, X)\otimes V_n^*$ is nonzero. By setting $m = r(n)$, the proof is complete. □

These two lemmas lead to the main theorem of this section.

THEOREM 3.4. *Recalling the notation of Definition* 2.1*, let* μ *be on a* $\mathfrak{q}$*-edge. Then the restriction map*

$$\tau\colon H^{l(w)}(w\mathfrak{u}, X) \to H^{l(w)}(w\mathfrak{u}\cap\mathfrak{k}, X)$$

is bijective on the $w(\mu+\rho_K) - \rho_K$ $L\cap K$*-types.*

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Proof. We check that τ is surjective first. If not, then Proposition 3.2 says that the $L \cap K$-type

$$[H^{l(w)-m}(w\mathfrak{u} \cap \mathfrak{k}, X) \otimes \bigwedge^{m+1}(\mathfrak{u} \cap \mathfrak{p})^*]^{w(\mu+\rho_K)-\rho_K} \neq 0$$

for some $0 \leqslant m \leqslant \min\{l(w), R-1\}$. Thus there are $\delta'_j \in \Delta(w\mathfrak{u} \cap \mathfrak{p}, \mathfrak{t}) \cup \{0\}$ so that the $L \cap K$-type

$$[H^{l(w)-m}(w\mathfrak{u} \cap \mathfrak{k}, X)]^{w(\mu+\rho_K)-\rho_K+\Sigma_{j=1}^{m+1}\delta'_j} \neq 0.$$

But then Kostant's Borel–Weil theorem says that there is a certain $\sigma \in W_K^1(w\mathfrak{u} \cap \mathfrak{k})$ and a K-type μ_1 so that

$$w(\mu+\rho_K) - \rho_K + \sum_{j=1}^{m+1} \delta'_j = \sigma(\mu_1 + \rho_K) - \rho_K.$$

Setting $\delta_j = w^{-1}\delta'_j \in \Delta(\mathfrak{u} \cap \mathfrak{p}, \mathfrak{t}) \cup \{0\}$ and rearranging terms yields

$$\mu + \sum_{j=1}^{m+1} \delta_j = w^{-1}\sigma(\mu_1 + \rho_K) - \rho_K.$$

Hence, $m_e(\mu + \Sigma_{j=1}^{m+1}\delta_j) \neq 0$ which contradicts the fact that μ lies on a $\mathfrak{q}$-edge. Hence τ is surjective.

The argument that checks that τ is injective is exactly the same, except one uses Proposition 3.3, replaces $\Sigma_{j=1}^{m+1}$ by $\Sigma_{j=1}^{m}$, and has $1 \leqslant m \leqslant \min\{l(w), R\}$. □

As a final note, the results of these last two sections may be generalized to the case where K is any compact subgroup as long as a θ-stable parabolic is interpreted in the obvious way. Namely, the parabolic is constructed by choosing an element $x \in i(\mathfrak{t}_0)^*$ and following the same details as in [8] for the corresponding Hochschild–Serre spectral sequence.

4. Application to multiplicity free reduction

Let λ be a dominant weight in $\mathfrak{h}$ of $\mathfrak{g}$ and let F_λ be the corresponding finite dimensional representation of G of highest weight λ. We will apply the previous sections to the case $X = F_\lambda$.

To motivate the special cases we will examine, recall that there are essentially only three cases in which $m(\mu)$, for any λ and μ, is always either 0 or 1 – in other words, the reduction from G to K is multiplicity free. The three cases are $(\mathfrak{su}(l+1), \mathfrak{u}(l))$, $(\mathfrak{so}(2l+1), \mathfrak{so}(2l))$, and $(\mathfrak{so}(2l+2), \mathfrak{so}(2l+1))$. See [1] for details.

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Moreover, the multiplicity function $m(\mu)$ is well known in these three cases: $m(\mu)$ is 1 or 0 depending on whether μ lies in a certain precisely defined parallelepiped. As such, it makes good geometric sense in these cases to say that a K-type lies on the *geometric edge* of the set of K-types if it lies on one of the edges of the parallelepiped. The main theorem of this section will show that these geometrical edges of K-types are parameterized by the image of the restriction map τ when λ is regular and the geometric edge does not lie completely inside a Weyl chamber wall of K.

To be more precise, suppose that ν is an $L \cap K$-type appearing in the image of the restriction map $\tau\colon H^b(\mathfrak{u}, F_\lambda) \to H^b(\mathfrak{u} \cap \mathfrak{k}, F_\lambda)$. By Kostant's Borel–Weil theorem, this tells us that a certain K-type lies in $F_\lambda|_K$. Namely, let $w \in W_K$ so that $w(\nu + \rho_K)$ is in the positive Weyl chamber of K. Then the K-type

$$w(\nu + \rho_K) - \rho_K$$

appears in F_λ. The next theorem tells us that this procedure parameterizes the edges of the set of K-types.

THEOREM 4.1. *Let $(\mathfrak{g}, K)$ be one of the above multiplicity free reduction cases, let λ be dominant regular, and let E be a geometric edge of the set of K-types of F_λ not lying completely inside a Weyl chamber wall of K.*

If μ is a K-type lying in E, then μ may be written in the form $w(\nu+\rho_K)-\rho_K$ for some $L \cap K$-type ν appearing in the image of the restriction map $\tau\colon H^b(\mathfrak{u}, F_\lambda) \to H^b(\mathfrak{u} \cap \mathfrak{k}, F_\lambda)$ for some θ-stable parabolic $\mathfrak{q}$ and some $w \in W_K$.

Moreover, only K-types on the geometrical edge are realized by this procedure of associating K-types to the image of the restriction map.

Note that the restriction on the position of E is nontrivial only in the case of (D_{l+1}, B_l) since in the other two cases a nontrivial geometrical edge is never completely contained in a Weyl chamber wall.

The proof of the above theorem consists of a case by case check. Since the details are very similar (and easier) for each of the other two cases, we will only write explicitly the case (D_{l+1}, B_l). The details are contained in Propositions 5.2 and 5.3 of the next section.

5. (D_{l+1}, B_l)

Here we consider the case $\mathfrak{g} = \mathfrak{so}(2l + 2, \mathbb{C})$ and $\mathfrak{k} = \mathfrak{so}(2l + 1, \mathbb{C})$. The natural projections of $\mathfrak{g}^*$ onto $\mathfrak{k}^*$ and $\mathfrak{h}^*$ onto $\mathfrak{t}^*$ will be denoted by the map p. Let $\alpha_1, \ldots, \alpha_{l-1}, \alpha_-, \alpha_+$ be the standard basis of roots in $\mathfrak{h}^*$ for the root system of D_{l+1} given by $\alpha_j = \varepsilon_j - \varepsilon_{j+1}$ for $1 \leqslant j \leqslant (l-1)$ and $\alpha_\pm = \varepsilon_l \pm \varepsilon_{l+1}$. Likewise, let $\alpha_1, \ldots, \alpha_l$ be the standard basis of roots in $\mathfrak{t}^*$ for the root system of B_l given by $\alpha_j = \varepsilon_j - \varepsilon_{j+1}$ for $1 \leqslant j \leqslant (l-1)$ and $\alpha_l = p\alpha_- = p\alpha_+ = \varepsilon_l$.

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Write $(\cdot,\cdot)$ for an invariant symmetric non-degenerate two form on $\mathfrak{h}^*$ and $\mathfrak{t}^*$ and set

$$\langle x, y\rangle = \frac{2(x,y)}{(y,y)}.$$

If $\lambda \in \mathfrak{h}^*$, define

$$n_j = \langle \lambda, a_j\rangle \quad \text{for } 1 \leqslant j \leqslant l-1,$$

$$n_\pm = \langle \lambda, \alpha_\pm, \rangle, \quad n_l = \min\{n_\pm\}.$$

The dependence of these numbers on λ will be suppressed. If λ is a positive weight in $\mathfrak{h}^*$ and μ is a positive weight in $\mathfrak{t}^*$, it is well known (e.g. [1]) that $m(\mu)$ is 1 if μ is conjugate to $p\lambda$ by the projection of the weight lattice of $\mathfrak{h}^*$ and if μ lies in the parallelepiped

$$P = \left\{ x \in \mathfrak{t}^* | x = p\lambda - \sum_{j=1}^{l} t_j\varepsilon_j, \quad 0 \leqslant t_j \leqslant n_j \right\}.$$

$m(\mu)$ is 0 otherwise. From this, we say that the K-type μ is on a *geometrical edge* if $t_j = 0$ or $t_j = n_j$ in the above equation for some $j \in \{1, \ldots, l\}$.

The Weyl groups of D_{l+1} and B_l act by permutations and certain sign changes. For σ in one of the Weyl groups, we write its action as $\sigma\varepsilon_j = \eta_{\sigma j}\varepsilon_{\sigma j}$ where $\eta_{\sigma j} \in \{\pm 1\}$. The following calculation is straightforward.

LEMMA 5.1. *Continue the notation for σ and η above. For $1 \leqslant j \leqslant l$, write $k = \sigma^{-1}j$. If $\eta_j = +1$, the coefficient of ε_j in the expression $\lambda - \sigma\lambda$ is*

(i) $n_j + \cdots + n_{k-1}, \quad 1 \leqslant j < k \leqslant l,$
(ii) $-n_k - \cdots - n_{j-1}, \quad 1 \leqslant k < j \leqslant l,$
(iii) $n_j + \cdots + n_{l-1} + n_-, \quad 1 \leqslant j < k = (l+1),$
(iv) $0, \quad j = k.$

If $\eta_j = -1$, the coefficient of ε_j in the expression $\lambda - \sigma\lambda$ is

(v) $n_j + \cdots + n_{k-1} + 2n_k + \cdots + 2n_{l-1} + n_- + n_+, \quad 1 \leqslant j < k \leqslant (l-1),$
(vi) $-n_k - \cdots - n_{j-1} - 2n_j - \cdots - 2n_{l-1} - n_- - n_+, \quad 1 \leqslant k < j \leqslant (l-1),$
(vii) $n_j + \cdots + n_{l-1} + n_- + n_+, \quad 1 \leqslant j < k = l,$
(viii) $n_j + \cdots + n_{l-1} + n_+, \quad 1 \leqslant j < k = (l+1),$
(ix) $-n_k - \cdots - n_{l-1} - n_- - n_+, \quad 1 \leqslant k < j = l,$
(x) $2n_j + \cdots + 2n_{l-1} + n_- + n_+, \quad j = k.$

PROPOSITION. 5.2. *Let λ be dominant regular in $\mathfrak{h}^*$ and $\mathfrak{q}$ a θ-stable parabolic with $\Delta(\mathfrak{u} \cap \mathfrak{k}, \mathfrak{t}) \subseteq \Delta^+(\mathfrak{k}, \mathfrak{t})$. If ν is an $L \cap K$-type appearing in the image of the*

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restriction map $\tau\colon H^b(\mathfrak{u}, F_\lambda) \to H^b(\mathfrak{u}\cap\mathfrak{k}, F_\lambda)$ *and* $\sigma(\nu+\rho_K)$ *is dominant for* $\sigma \in W_K$*, then* $\mu = \sigma(\nu+\rho_K)-\rho_K$ *is a* K*-type of* F_λ *appearing on a geometrical edge of the set of* K*-types.*

Proof. Note the assumption $\Delta(\mathfrak{u}\cap\mathfrak{k},\mathfrak{t}) \subseteq \Delta^+(\mathfrak{k},\mathfrak{t})$ is always possible by using the W_K action. (In the case of (A_l, A_{l-1}) and (B_l, D_l), more notation is involved since the Weyl chambers of K contain a number of Weyl chambers for G, but the calculations are similar.)

To begin, observe that μ is a K-type by Kostant's theorem. Namely, if ν is an $L\cap K$-type in $H^b(\mathfrak{u}\cap\mathfrak{k}, F_\lambda)$, then there is a K-type μ_1 of F_λ and $w_1 \in W_K^1(\mathfrak{u}\cap\mathfrak{k})$ so that $\nu = w_1(\mu_1+\rho_K)-\rho_K$. Hence $\sigma(\nu+\rho_K) = \sigma w_1(\mu_1+\rho_K)$. But since $\sigma(\nu+\rho_K)$ is dominant and $\mu_1+\rho_K$ is dominant regular, $\sigma = w_1^{-1}$ so $\mu_1 = \sigma(\nu+\rho_K)-\rho_K$. In particular, $\mu = \mu_1$ is a K-type.

It remains to see that μ lies on a geometrical edge. For this, we may assume that $\mathfrak{u}\cap\mathfrak{k}$ is not empty or else the proposition is trivial. Since $\mathfrak{q}$ is θ-stable, we may thus fix $\alpha_{j_0} \notin \Delta(\mathfrak{l},\mathfrak{h})$ for some $1 \leqslant j_0 \leqslant l$.

Suppose that ν lies in the image of the restriction map τ. Recalling that τ is an $L\cap K$ map, this means that ν appears in $H^b(\mathfrak{u}, F_\lambda)$ restricted to $L\cap K$. However, since F_λ is finite dimensional, Kostant's theorem says that the highest weights of the L-types appearing in $H^b(\mathfrak{u}, F_\lambda)$ are of the form $w_G(\lambda+\rho_G)-\rho_G$ for certain $w_G \in W_G$. Examining Kostant's multiplicity formula in [5] or in [1] where the desired case is explicitly worked out in Section 4, we see that if ν is an $L\cap K$ type appearing in the L-representation of highest weight $w_G(\lambda-\rho_G)-\rho_G$, then ν must be of the form

$$p(w_G(\lambda+\rho_G)-\rho_G)-\xi,$$

where $\xi = \Sigma_{i=j_0+1}^{l} c_i\varepsilon_i$ with $c_i \geqslant 0$. Though ξ may be described even more precisely, all we will need is that the the ε_{j_0} term does not appear in ξ.

Since we are really interested in the K-type associated to ν, we must examine when

$$w_K[p(w_G(\lambda+\rho_G)-\rho_G)-\xi+\rho_K]-\rho_K$$

lies in the parallelepiped P for some $w_K \in W_K$. To this end, look at the coefficient of ε_j, $1 \leqslant j \leqslant l$, in the expression

$$p(\lambda)-[w_K[p(w_G(\lambda+\rho_G)-\rho_G)-\xi+\rho_K]-\rho_K]. \tag{1}$$

In order for the associated K-type to lie in P, each coefficient of ε_j in Equation 1 must be in the closed interval $[0, n_j]$. If the K-type is in P, it lies on the edge if and only if at least one of the coefficients of ε_j is either 0 or n_j. Below, we demonstrate that if such an associated K-type lies in P, it must lie on the edge. This will finish the proposition.

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Equation 1 may be rewritten as p applied to the sum of the following four equations:

$$\lambda - w_K w_G \lambda,$$

$$\rho_G - w_K w_G \rho_G,$$

$$(\rho_K - \rho_G) - w_K(\rho_K - \rho_G),$$

$$-w_K \xi.$$

Using Lemma 5.1, it will be easy to show that *every* coefficient of ε_j from the the sum of the first three equations lies outside the open interval $(0, n_j)$. Since the fourth equation leaves at least one coefficient unchanged, for instance $w_K j_0$, at least one coefficient lies outside $(0, n_j)$. This implies that if the K-type lies in P, it must lie on the boundary as desired.

Since the above statement follows trivially from Lemma 5.1, we give a sample calculation and omit the rest. Recalling that λ is dominant regular, each n_i is a positive integer in Lemma 5.1 applied to the first equation. To apply the Lemma to the second equation, use $n_j = 1$ for $1 \leqslant j \leqslant l + 1$ and to apply it to the third equation, use $n_j = 0$ for $1 \leqslant j \leqslant l - 1$ and $n_j = -\frac{1}{2}$ for $j = l, l + 1$. Consider case (i) of the Lemma. Here the contribution to the coefficient from the first three equations is

$$(n_j + \cdots + n_{k-1}) + (k - j) + 0.$$

Since $k > j$ and $n_i \geqslant 1$, this number is clearly greater than n_j. In particular, it lies outside $(0, n_j)$. The other cases are similar and as easy. □

PROPOSITION 5.3. *Let λ be dominant regular and let E be a geometrical edge of the set of K-types not contained in a Weyl chamber wall. If μ is a K-type of F_λ in E, then there is θ-stable parabolic $\mathfrak{q}$ and $w \in W_K$ so that $\mu = \omega(\nu + \rho_K) - \rho_K$ where ν is a $L \cap K$ type appearing in the image of the restriction map $\tau\colon H^b(\mathfrak{u}, F_\lambda) \to H^b(\mathfrak{u} \cap \mathfrak{k}, F_\lambda)$.*

Proof. Let μ be on E. This means that $\mu = p\lambda - \Sigma_{j=1}^{l} t_j \varepsilon_j$ with $t_j \in [0, n_j]$ and for some j_0, $t_{j_0} \in \{0, n_{j_0}\}$. Pick $x \in \mathfrak{t} \subseteq \mathfrak{h}$ of the form $x = (0, \ldots, 0, \pm 1, 0, \ldots, 0)$ where the 1 is in the j_0 place with a $+$ if $t_{j_0} = 0$ and a $-$ if $t_{j_0} = n_{j_0}$. Let $\mathfrak{q}_1$ be the associated θ-stable parabolic in $>$. In particular, $\Delta(\mathfrak{l}_1, \mathfrak{h}) = \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}) \mid \alpha(x) = 0\}$ and $\Delta(\mathfrak{u}_1, \mathfrak{h}) = \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}) \mid \alpha(x) > 0\}$. It is easy to check in this case that $\Delta(\mathfrak{u} \cap \mathfrak{p}, \mathfrak{t}) = \{\varepsilon_{j_0}\}$ if $t_{j_0} = 0$ and $\Delta(\mathfrak{u} \cap \mathfrak{p}, \mathfrak{t}) = \{-\varepsilon_{j_0}\}$ if $t_{j_0} = n_{j_0}$. In either case, $R = \dim(\mathfrak{u} \cap \mathfrak{p}) = 1$.

Let $w_1 \in W_K$ so that $w_1(\Delta(\mathfrak{u}_1 \cap \mathfrak{k}, \mathfrak{t})) \subseteq \Delta^+(\mathfrak{k}, \mathfrak{t})$. Put $\mathfrak{q} = w_1 \mathfrak{q}_1$, $\mathfrak{u} = w_1 \mathfrak{u}_1$, $\mathfrak{l} = w_1 \mathfrak{l}_1$, $w = w_1^{-1}$, and $b = l(w_1)$. In the next paragraphs we will show that μ is on a $\mathfrak{q}_1$-edge according to Definition 2.1. Thus Theorem 3.4 tells us that

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$\tau\colon H^b(\mathfrak{u}, F_\lambda) \to H^b(\mathfrak{u}\cap\mathfrak{k}, F_\lambda)$ is bijective on the $\nu = w_1(\mu+\rho_K) - \rho_K$ $L\cap K$-types. By Lemma 2.2, we may also assume that $w_1 \in W_K^1(\mathfrak{u}\cap\mathfrak{k})$ so that by Kostant's theorem, ν actually appears in $H^b(\mathfrak{u}\cap\mathfrak{k}, F_\lambda)$ since μ is a K-type. Writing $\mu = w(\nu+\rho_K) - \rho_K$, we will be done.

It remains to show that μ is on a $\mathfrak{q}_1$-edge. Since $R = 1$, we need to show that $m_e(\mu \pm \varepsilon_{j_0}) = 0$ where the $\pm$ depends on x as above. (The case of (A_l, A_{l-1}) does not even need to use the restriction on R.) Write $\mu \pm \varepsilon_{j_0} = p\lambda - \Sigma_{j=1}^{l} c_j\varepsilon_j$. It is enough to show that

$$w\left(p\lambda - \sum_{j=1}^{l} c_j\varepsilon_j + \rho_K\right) - \rho_K \tag{2}$$

does not lie in P for any $w \in W_K$. Write this last expression as

$$p\lambda - \left[\lambda - w\left(p\lambda - \sum_{j=1}^{l} c_j\varepsilon_j + \rho_K\right) + \rho_K\right].$$

To be in P, all the coefficients of ε_j in

$$\lambda - w\left(p\lambda - \sum_{j=1}^{l} c_j\varepsilon_j + \rho_K\right) + \rho_K,$$

must lie $[0, n_j]$ for $1 \leqslant j \leqslant l$. This last equation can be written as the projection of the sum of the three equations

$$\lambda - w\lambda,$$

$$\rho_K - w\rho_K,$$

$$w\sum_{j=1}^{l} c_j\varepsilon_j.$$

By assumption, $c_j \in [0, n_j]$ for $j \neq j_0$ and c_{j_0} is in either $[-1, 0)$ or in $(n_{j_0}, n_{j_0}+1]$ depending on x as above. Recalling that λ is dominant regular and examining the cases of Lemma 5.1 where $j < k = w^{-1}j$ applied to the first two equations, it is easy to check that if (2) lies in P, then there is an i_0, $1 \leqslant i_0 \leqslant j_0$, so that w^{-1} is a cyclic permutation of the ε_j of the form $(j_0, j_0 - 1, \ldots, i_0 + 1, i_0)$ with certain sign changes. Examining the Lemma again in the cases where $\eta_k = -1$, it is easy to see that if (2) lies in P, then every $\eta_j = +1$ for $j \neq i_0$.

If $i_0 < j_0$, then part (ii) of the Lemma applied to the above three equations with $j = i_0 + 1$ yields $-n_{i_0} - 1 + s$ where $s \in [0, n_{i_0}]$ so that the number is negative. In particular, it is not in $[0, n_{i_0+1}]$ so that (2) is not in P.

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Hence, if (2) is in P then $i_0 = j_0$ so that w is the identity except perhaps for a sign change of ε_{j_0}. By definition, if $\eta_{j_0} = +1$, w is the identity and (2) cannot lie in P. If $\eta_{j_0} = -1$, then the Lemma applied to the three equations with $j = j_0$ yields

$$[2n_{j_0} + \cdots + 2n_{l-1} + n_- + n_+] + [2(l - j_0) + 1] - t, \tag{3}$$

where t is -1 or $n_{j_0} + 1$ depending on whether a $\pm$ appears in x. If $l > j_0$, then (3) lies outside of $[0, n_{j_0}]$ so that if (2) is in P, then $l = j_0$. In this case, (3) reduces to

$$n_- + n_+ + 1 - t.$$

Recalling that $n_l = \min\{n_-, n_+\}$, the above equation lies in $[0, n_l]$ if and only if $t = n_{j_0} + 1$ and $n_- = n_+$. But it is easy to see that $n_- = n_+$ if and only if E lies in the Weyl chamber wall perpendicular to ε_l. But since we have assumed that E is not in a chamber wall, we have achieved our goal of showing that Equation 2 does not lie in P. □

6. Concluding remarks

The examination of the finite dimensional multiplicity free case has shown that the idea of a geometric edge of the set of K-types fits very well with the idea of a $\mathfrak{q}$-edge which is in turn related to the restriction map on cohomology. The assumption of finite dimensionality can hopefully be removed. For instance, it seems likely that similar results are reasonable for the discrete series since one can replace Kostant's theorem by theorems of Schmid ([4]).

When the multiplicity free assumption is removed, the idea of a geometrical edge is no longer easy to describe. In fact, even in the special case we examined, new techniques are needed for geometrical edges that lie on a Weyl chamber wall. However, various examples still suggest that further study of $\mathfrak{q}$-edges and the restriction map will help to pin down the structure of the K-types.

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