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Oscillation criteria for second order nonlinear delay inequalities

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Oscillation criteria are obtained for the nonlinear delay differential inequality $u\big(u''+f(t,\ u(t),\ u(g(t)))\big) \le 0$. The main theorems give sufficient conditions (and in some cases sufficient and necessary conditions) for all solutions $u(t)$ to have arbitrary large zeros. Generalizations to more general cases are discussed.

1. Introduction

Oscillation criteria for the nonlinear delay differential equation

$$Lu = u'' + f\big(t,\ u(t),\ u(g(t))\big) = 0 \tag{1}$$

and more generally for the inequality $uLu \le 0$, will be derived. Suitable assumptions on $f(t,\ u,\ v)$ will be listed in Section 2.

Hereafter, "solution" means "solution on a half-axis". A solution of $uLu \le 0$ is called oscillatory if it has no largest zero. For a general discussion of existence and uniqueness properties of equations with delays, the reader is referred to El'sgol'ts [3].

Oscillation theory for equation (1) has been developed by many authors. We mention in particular the papers by Erbe [4], Gollwitzer [5], Ladas [6], Lillo [7], Norkin [8], Staïkos [9], Waltman [10], Wong [11], and the references therein.

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Our main results in §2 will extend known oscillation criteria to differential inequalities and sharpen the conclusion in certain cases. Our basic method will depend on the fact that, under appropriate conditions on f, nonoscillatory solutions of the inequality $uLu \le 0$ are positive solutions of a related differential inequality. We then use suitable Ricatti's transformations to derive sufficient conditions for the related inequality to have no solution which eventually becomes positive at ∞. As is customary in this area, we give some strictly nonlinear results (Theorems 2 and 6) and some results which can be specialized to the linear case (Theorems 4 and 7). It will be clear that this technique extends to more general inequalities in which the function f involves several retardations.

2.

In this section we obtain oscillation criteria for the delay differential inequality

$$u(t)Lu(t) \le 0 \ , \quad t \ge 0 \ , \tag{2}$$

with L as given in (1). The following assumptions on the function f and g will be retained in the sequel.

(3) ASSUMPTIONS. (a) $f(t, u, v) \in C\big([0, \infty) \times R \times R\big)$, u and $v > 0$ imply $f(t, u, v)$ is positive and nondecreasing in u and v for all $t \ge 0$.

(b) $f(t, u, v) \le -f(t, -u, -v)$, u and $v > 0$ and all $t \ge 0$;

(c) $g(t) \in C[0, \infty)$ with $0 < g(t) \le t$, $t > 0$, $\lim_{t\to\infty} g(t) = \infty$.

LEMMA 1. *Inequality* (2) *is oscillatory in* $[0, \infty)$ *if the delay differential inequality*

$$u''(t) + f\big(t, u(g(t)), u(g(t))\big) \le 0 \tag{4}$$

has no solution u *which is positive in* $[t_0, \infty)$ *for* $t_0 > 0$ *under the assumptions* (3).

Proof. Suppose to the contrary there exists a number $t_1 > 0$ such that $u(t)$ is a positive solution of (2) in $[t_1, \infty)$. Choosing t_1 sufficiently large if necessary, we can assume that $u(g(t)) > 0$ for $t \geq t_1$. Since $t \geq t_1$ implies that $u''(t) \leq 0$, a standard argument shows that $u'(t) > 0$ for sufficiently large t , say $t \geq t_0$. Using Assumption (3) (a) on f , it is easy to see that $u(t)$ is a positive solution of (4) in $[t_0, \infty)$ contradicting the hypothesis. Likewise, there cannot exist $t_0 > 0$ such that u is a negative solution of (2) in $[t_2, \infty)$ or else Assumptions (3) (a) and (b) would imply that $-u$ is a positive solution of (4).

Let $\phi(u)$ be a continuously differentiable function of u for $u \in [0, \infty)$ satisfying $\phi(u) > 0$ if $u > 0$, $\phi'(u) \geq 0$ for all $u \geq 0$, and

$$\int_1^\infty \frac{du}{\phi(u)} < +\infty .$$

We will say that $f(t, u, v)$ satisfies *condition* (A) provided there exists a $c > 0$ such that

$$\liminf_{u\to\infty} \frac{f(t,u,u)}{\phi(u)} \geq Kf(t, c, c) \tag{5}$$

for some positive constant K and all $t \geq T$.

THEOREM 2. *Assume that Assumptions* (3) *and condition* (A) *hold. Furthermore, assume* $g'(t) \geq 0$. *Then inequality* (2) *is oscillatory if*

$$\int_1^\infty g(t)f(t, \alpha, \alpha)dt = +\infty \tag{6}$$

for all $\alpha > 0$. *In addition, if*

$$\liminf_{t\to\infty} \frac{g(t)}{t} \geq a > 0 \quad \textit{for some} \quad a > 0 , \tag{7}$$

then (6) *is also necessary.*

Proof. That (6) is a sufficient condition will follow from Lemma 1 if we show that inequality (4) has no positive solution $u(t)$ in $[t_0, \infty)$ for any $t_0 \geq 0$. Suppose to the contrary that a solution $u(t) > 0$ of

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(4) exists and $t_0 > 0$ can be chosen such that $u(t)$ and $u(g(t)) > 0$ on $[t_0, \infty)$. Since $t \geq t_0$ implies that $u''(t) \leq 0$, a standard argument shows that $u'(t) > 0$ for sufficiently large t, say $t \geq t_1 \geq t_0$. We can assume that $\lim_{t\to\infty} u(t) = \infty$ since otherwise multiplication of (4) by t and integration by parts over $[t_1, t]$ would lead to

$$tu'(t) - t_1 u'(t_1) \leq -\int_{t_1}^{t} sf(s, u(g(s)), u(g(s)))ds + u(t) - u(t_1)$$

for all $t \geq t_1$, and consequently $u'(t) < 0$ for sufficiently large t on account of hypothesis (6).

Define

$$V(t) = \frac{g(t)u'(t)}{\phi(u(g(t)))}, \quad t \geq t_1 .$$

Then

$$(8) \quad V'(t) \leq -\frac{g(t)f(t,u(g(t)),u(g(t)))}{\phi(u(g(t)))} + \frac{g'(t)u'(t)}{\phi(u(g(t)))} - \frac{g(t)u'(t)u'(g(t))g'(t)\phi'(u(g(t)))}{(\phi(u(g(t))))^2} .$$

Condition (A), Assumptions (3), and inequality (8) imply that there exists a $T > 0$ and a $c > 0$ such that

$$V'(t) \leq -Kg(t)f(t, c, c) + \frac{g'(t)u'(g(t))}{\phi(u(g(t)))}$$

for $t \geq T$. Integration over $[T, t]$ yields

$$(9) \qquad V(t) - V(T) \leq -K\int_{T}^{t} g(s)f(s, c, c)ds + \int_{u(g(T))}^{u(g(t))} \frac{du}{(u)} .$$

It follows from (9) that $V(t) < 0$ for sufficiently large t on account of (6), and consequently $u'(t)$ is eventually negative. This contradiction proves the sufficiency part of Theorem 2.

Conversely, if (7) holds and (6) does not hold for some $\alpha > 0$, then the equation

$$(10) \qquad u''(t) + f(t, u(t), u(g(t))) = 0$$

has a bounded nonoscillatory solution by Theorem 3.2 of [4]. But a nonoscillatory solution of (10) is obviously a solution of the inequality (2).

REMARK. The requirement that $g'(t) \geq 0$ in Theorem 2 can be replaced by the less restrictive requirement that there exists a function $h(t) \in C'[0, \infty)$, $0 < h(t) \leq g(t)$, $h'(t) \geq 0$, and $\lim_{t\to\infty} h(t) = \infty$. In this case, Theorem 2 is valid with $h(t)$ replacing $g(t)$ in hypothesis (6).

As a simple corollary of Theorem 2, we obtain oscillation criteria for the inequality $uL_1u \leq 0$, where L_1 is defined by:

$$L_1u = u''(t) + p(t)\big(u(g(t))\big)^{\gamma}, \tag{11}$$

where $p(t) \geq 0$, $g'(t) \geq 0$ on $[T, \infty)$, and $\gamma > 1$ is the quotient of odd integers.

COROLLARY 3. *All solutions of* $uL_1u \leq 0$ *are oscillatory if*

$$\int_1^{\infty} g(t)p(t)dt = \infty . \tag{12}$$

The converse is true if condition (7) *holds.*

REMARK. Corollary 3 improves and extends previous results by Gollwitzer [5] and Erbe [4] for the equation $L_1u = 0$ in case $g'(t) \geq 0$, as the following example shows.

EXAMPLE. Let $p(t) = t^{-3/2}$ and $g(t) = t^{1/2}$. Condition (12) holds and the inequality $uL_1u \leq 0$ is oscillatory by Corollary 3. For this example condition (3.13) of [4] does not hold since

$$\int^{\infty} t^{1-\gamma}p(t)\big(g(t)\big)^{\gamma}dt = \int^{\infty} t^{-((\gamma+1)/2)}dt < \infty \quad \text{for all} \quad \gamma > 1 .$$

The next theorem is an analogue of Theorem 2 when condition (A) is not satisfied. Condition (A) is replaced by the following condition:

$f(t, u, v)$ is said to satisfy *condition* (B) if there exists a $c > 0$ and a number $\gamma \geq 1$ such that

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$$\liminf_{u\to\infty} \frac{f(t,u,u)}{u^\gamma} \geq Kf(t, c, c) \tag{13}$$

for some constant $K > 0$ and for all $t \geq T$.

This holds, for example, if

$$f(t, u, u) = u^\gamma(\log(|u|+1))^\beta, \quad \gamma \geq 1, \quad \beta \geq 0,$$

as well as in the linear $(\gamma = 1, \beta = 0)$ and superlinear $(\gamma \geq 1, \beta = 0)$ cases.

THEOREM 4. *Assume that Assumptions* (3) *and condition* (B) *hold. Furthermore, assume* $g'(t) \geq 0$. *Then inequality* (2) *is oscillatory in* $[0, \infty)$ *if*

$$\int_1^\infty (g(t))^\lambda f(t, \alpha, \alpha)dt = \infty \tag{14}$$

for all $\alpha > 0$ *and for some* $0 \leq \lambda < 1$.

Proof. Proceeding exactly as in the proof of Theorem 2 we find that $u'(t) > 0$, $u''(t) \leq 0$ for $t \geq t_1$, and

$$\frac{u(t)}{u'(t)} - \frac{u(t_1)}{u'(t_1)} = \int_{t_1}^t \frac{(u'(t))^2-u(t)u''(t)}{(u'(t))^2}\, dt \geq t - t_1 .$$

Hence there exists a number $t_2 \geq t_1$ such that

$$\frac{u'(g(t))}{u(g(t))} \leq \frac{2}{g(t)} \quad \text{for all } t \geq t_2 . \tag{15}$$

Choose $\gamma \geq 1$ and $c > 0$ such that (13) is satisfied for $t \geq T \geq t_2$, and let $0 \leq \lambda < 1$. Define

$$V(t) = \frac{(g(t))^\lambda u'(t)}{u^\gamma(g(t))}, \quad t \geq T .$$

Then

$$V'(t) \leq -\frac{(g(t))^\lambda f(t,u(g(t)),u(g(t)))}{u^\gamma(g(t))} + \frac{\lambda(g(t))^{\lambda-1}g'(t)u'(g(t))}{u^\gamma(g(t))}, \quad t \geq T . \tag{16}$$

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However, according to (13) and (15) we have

$$V'(t) \le -K\big(g(t)\big)^{\lambda} f(t,\, c,\, c) + \frac{2\lambda\big(g(t)\big)^{\lambda-2} g'(t)}{u^{\gamma-1}\big(g(t)\big)}$$

for some constant $K > 0$ and all $t \ge T$.

Using (16) and the positivity of u' we then obtain

$$V'(t) \le -K\big(g(t)\big)^{\lambda} f(t,\, c,\, c) + K_1 g'(t)\big(g(t)\big)^{\lambda-2} ,$$

where $K_1 = 2\lambda u\big(g(T)\big)^{1-\gamma}$. Integrating over $(T,\, t)$ we obtain

$$V(t) - V(T) \le -K \int_T^t \big(g(s)\big)^{\lambda} f(s,\, c,\, c)ds + K_1 \int_{g(T)}^{g(t)} s^{\lambda-2} ds .$$

As in the proof of Theorem 2, we arrive at the contradiction $u'(t) < 0$ for sufficiently large t.

The above result generalizes a result by Wong [11] where the special case $u'' + a(t)u\big(g(t)\big) = 0$, $a(t) \ge 0$, and $ct \le g(t) \le t$ for some constant $c > 0$ was considered.

COROLLARY 5. *The differential inequality*

$$u\big[u''+p(t)u^{\gamma}\big(g(t)\big)\big] \le 0 ,$$

where $p(t) \in C[0,\, \infty)$ *is nonnegative,* $\gamma \ge 0$ *is the quotient of odd integers, and* $g'(t) \ge 0$ *, is oscillatory in* $[0,\, \infty)$ *if*

$$\int_1^{\infty} \big(g(t)\big)^{\lambda} p(t)dt = \infty$$

for some $0 \le \lambda < 1$ *. Furthermore, if* $\gamma > 1$ *,* λ *can be taken to be* 1 *.*

The above corollary improves previous results by Erbe [4] in case $g'(t) \ge 0$.

We now give analogues of Theorems 2 and 4 when $g(t)$ is not necessarily differentiable. Conditions (A) and (B) are replaced by the following conditions.

Condition A_1. There exists $\phi(u) \in C^1[0,\, \infty)$ satisfying:

(a) $\phi(u) > 0$ if $u > 0$, $\phi'(u) \ge 0$ for all $u \ge 0$, and

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$$\int_1^\infty \frac{du}{\phi(u)} < \infty \ ;$$

(b) there exists a $c > 0$ and $0 < \alpha < 1$ such that

$$\liminf_{|u|\to\infty} \frac{f\left(t, |u|, \alpha \frac{g(t)}{t} |u|\right)}{\phi(|u|)} \geq Kf\left(t, c, \alpha \frac{g(t)}{t} c\right)$$

for some positive constant K and all $t \geq T$.

Condition B_1. There exists a $c > 0$, $0 < \alpha < 1$, and a number $\gamma \geq 1$ such that

$$\liminf_{|u|\to\infty} \frac{f\left(t, |u|, \alpha \frac{g(t)}{t} |u|\right)}{|u|^\gamma} \geq Kf\left(t, c, \alpha \frac{g(t)}{t} c\right)$$

for some constant $K > 0$ and for all $t \geq T$.

THEOREM 6. *Assume that Assumptions* (3) *and condition* A_1 *hold. Then inequality* (2) *is oscillatory in* $[0, \infty)$ *if*

$$\int_1^\infty tf\left(t, \alpha, \alpha \frac{g(t)}{t}\right) dt = \infty \tag{17}$$

for all $\alpha > 0$. *In addition, if* (7) *holds then* (17) *is also necessary.*

Proof. To prove the if part, assume to the contrary that $u(t)$ is a nonoscillatory solution of (2). Using the same argument as in Lemma 1, we can assume that $u(t)$ is a positive solution of the inequality

$$u'' + f\left(t, u(t), u\left(g(t)\right)\right) \leq 0 \tag{18}$$

in $[t_0, \infty)$ for some $t_0 \geq 0$. Obviously we can assume that $u\left(g(t)\right) > 0$ for $t \geq t_0$. Since $t \geq t_0$ implies $u''(t) \leq 0$, a standard argument shows that $u'(t) > 0$ for sufficiently large t, say $t \geq t_1 \geq t_0$. Since $u(t) > 0$, $u'(t) > 0$, $u''(t) \leq 0$ on $[t_1, \infty)$, Lemma 2.1 of [4] implies that for each $0 < k < 1$ there is a $T_k \geq t_1$ such that

$$u\left(g(t)\right) \geq ku(t) \frac{g(t)}{t}, \quad t \geq T_k . \tag{19}$$

From (18) and the assumptions (3) we then have that $u(t)$ satisfies

(20) $$u'' + f\left[t,\ u(t),\ k\,\frac{g(t)}{t}\,u(t)\right] \le 0\ ,\quad t \ge T_k\ .$$

We can assume that $\lim_{t\to\infty} u(t) = \infty$ since otherwise multiplication of (20) by t and integration by parts over $[T_k,\ t]$ would lead to

$$tu'(t) - T_k u'(T_k) \le -\int_{T_k}^{t} sf\left[s,\ u(s),\ k\,\frac{g(s)}{s}\,u(s)\right]ds + u(t) - u(T_k)$$

$$\le -\int_{T_k}^{t} sf\left[s,\ u(T_k),\ k\,\frac{g(s)}{s}\,u(T_k)\right]ds + u(t) - u(T_k)$$

for all $t \ge T_k$, and consequently $u'(t) < 0$ for sufficiently large t on account of hypothesis (17).

Choose $0 < \alpha < 1$ and $c > 0$ such that condition A_1 holds. Choose T_α sufficiently large such that (19) holds and

(21) $$f\left[t,\ u(t),\ \alpha\,\frac{g(t)}{t}\,u(t)\right] \ge K\phi\big(u(t)\big)f\left[t,\ c,\ \alpha\,\frac{g(t)}{t}\,c\right]$$

for some $K > 0$ and for all $t \ge T_\alpha \ge t_1$. Define

$$V(t) = \frac{tu'(t)}{\phi\big(u(t)\big)}\ ,\quad t \ge T_\alpha\ .$$

Using (19) and (21) we then obtain

(22) $$V'(t) \le -tKf\left[t,\ c,\ \alpha\,\frac{g(t)}{t}\,c\right] + \frac{u'(t)}{\phi\big(u(t)\big)} - \frac{\big(u'(t)\big)^2\phi'(u)}{\big(\phi(u)\big)^2}\ .$$

Integration over $[T_\alpha,\ t]$ yields

$$V(t) - V(T_\alpha) \le -K\int_{T_\alpha}^{t} sf\left[s,\ c,\ \alpha\,\frac{g(s)}{s}\,c\right]ds + \int_{u(T_\alpha)}^{u(t)} \frac{du}{\phi(u)}\ .$$

It follows from condition A_1 and the hypothesis (17) that $V(t) < 0$ for sufficiently large t , and consequently $u'(t)$ is eventually negative. This contradiction proves the sufficiency part of Theorem 6.

Conversely, if (7) holds and (17) does not hold for some $\alpha > 0$, then by Assumption (3) (a) the condition

$$\int_1^\infty tf(t, \alpha, \alpha)dt < \infty$$

must hold for some $\alpha > 0$. Then equation (10) has a bounded nonoscillatory solution by Theorem 3.2 of [4] which is obviously a solution of (3).

Theorem 6 extends results by Gollwitzer [5], Erbe [4], and Wong [11] to differential inequalities.

The proof of the following analogue of Theorem 4 is similar to the above proof and will be omitted.

THEOREM 7. *Assume that assumptions* (3) *and condition* B_1 *hold. Then inequality* (2) *is oscillatory in* $[0, \infty)$ *if*

$$\int_1^\infty t^\lambda f\left(t, \alpha, \alpha \frac{g(t)}{t}\right)dt = \infty \tag{23}$$

for all $\alpha > 0$ *and for some* $0 \le \lambda < 1$.

REMARKS. It is a very simple matter to write analogues of Theorems 2-6 when the operator L is replaced by the more general operator

$$Lu = u'' + \sum_{i=1}^{n} f_i\big(t, u(t), u(g_i(t))\big)$$

with f_i and g_i, $i = 1, 2, \ldots, n$, satisfying assumptions (3).

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