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Existence of Solutions to Poisson's Equation

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Abstract. Let Ω be a domain in $\mathbb{R}^n$ $(n \geq 2)$. We find a necessary and sufficient topological condition on Ω such that, for any measure μ on $\mathbb{R}^n$, there is a function u with specified boundary conditions that satisfies the Poisson equation $\Delta u = \mu$ on Ω in the sense of distributions.

1 Introduction

Let Ω be an unbounded domain in Euclidean space $\mathbb{R}^n$ $(n \geq 2)$, and let $f\colon \overline{\Omega} \to \mathbb{R}$ be a locally bounded measurable function. The purpose of this paper is to characterize the domains Ω for which there exists a solution to the Poisson equation $\Delta u = f$ (in the sense of distributions) with boundary values 0. (As usual, the boundary condition is relaxed at the polar set of irregular boundary points.) We will also address the corresponding problem where f is replaced by a measure. We impose the minor restriction that Ω be *Greenian*, that is, Ω possesses a Green function. Thus, when $n = 2$, the set Ω must have non-polar complement.

Theorem 1 *Let Ω be an unbounded, Greenian domain in $\mathbb{R}^n$. The following statements are equivalent:*

(i) *For each locally bounded measurable function $f\colon \overline{\Omega} \to \mathbb{R}$ there exists a continuous function $u\colon \Omega \to \mathbb{R}$ satisfying:*
  - (a) *$\Delta u = f$ (in the distributional sense) on Ω;*
  - (b) *u has limit 0 at regular points of $\partial\Omega$;*
  - (c) *u is bounded near each irregular point of $\partial\Omega$.*

(ii) *For each compact set K in $\mathbb{R}^n$ there is a compact set L which contains the bounded components of $\Omega \setminus K$ whose closure intersects K.*

We note that the topological condition (ii) has arisen previously in connection with the Dirichlet problem on unbounded domains [2]. It is also reminiscent of the "long islands" condition first introduced by Gauthier [3] in connection with Carleman approximation by holomorphic functions.

Before stating the next result, we make some definitions. We denote the open ball in $\mathbb{R}^n$ with centre x and radius r by $B(x, r)$. The *fine topology* is the coarsest topology on $\mathbb{R}^n$ that makes every superharmonic function on $\mathbb{R}^n$ continuous in the extended sense. A set $E \subset \mathbb{R}^n$ is said to be *thin* at a point y if there is a superharmonic function u on a neighbourhood of y such that $\liminf_{x\to y, x\in E} u(x) > u(y)$. We will say that a function $f\colon \Omega \to [-\infty, +\infty]$ has *fine limit* l at a point $y \in \partial\Omega$ if there is a set $E \subset \Omega$ that is thin at y and $f(x) \to l$ as $x \to y$ along $\Omega \setminus E$. Finally, if a proposition

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concerning a point y in a set A is true for all $y \in A$ apart from a polar set, then it is said to hold for *quasi-every* point $y \in A$.

Theorem 2 *Let Ω be an unbounded Greenian domain in $\mathbb{R}^n$. The following statements are equivalent:*

(i) *For each measure μ on $\overline{\Omega}$ there is a subharmonic function u on Ω satisfying:*

(a) $\Delta u = \mu$ *on Ω (in the distributional sense),*

(b) $\limsup_{x\to y} u(x) \leq \operatorname{fine\,lim}_{x\to y} u(x) = 0$ *at quasi-every point $y \in \partial\Omega$,*

(c) $\limsup_{x\to y} u(x) < +\infty$ *at each point $y \in \partial\Omega$,*

(d) $\liminf_{x\to y} u(x) > -\infty$ *whenever $y \in \partial\Omega$ and $\int_{B(y,\varepsilon)\cap\Omega} U(\cdot, z)\, d\mu(z)$ is bounded above for some $\varepsilon > 0$.*

(ii) *For each compact set K in $\mathbb{R}^n$ there is a compact set L which contains the bounded components of $\Omega \setminus K$ whose closure intersects K.*

In the proofs of these theorems we will make use of the function U on $\mathbb{R}^n \times \mathbb{R}^n$ defined by

$$U(x, y) = \begin{cases} -\log \|x - y\| & (x \neq y; n = 2), \\ \|x - y\|^{2-n} & (x \neq y; n \geq 3) \\ +\infty & (x = y). \end{cases}$$

2 Proof of Theorem 1

Suppose that Ω is an unbounded Greenian domain in $\mathbb{R}^n$ satisfying the topological condition (ii) and that $f\colon \overline{\Omega} \to \mathbb{R}$ is a locally bounded measurable function. Without loss of generality, we can assume that $f \geq 0$. Further, we can assume that f is defined on all of $\mathbb{R}^n$ and valued 0 outside Ω.

We will apply the method used to prove Theorem 4.3.10 of [1] to get a subharmonic function s on $\mathbb{R}^n$ which has Riesz measure $f\lambda_n$, where λ_n denotes Lebesgue measure on $\mathbb{R}^n$. We define a sequence of functions (f_k) by

$$f_1(x) = \begin{cases} f & \text{on } \overline{B(0, 2)}, \\ 0 & \text{elsewhere.} \end{cases}$$

and, for $k \geq 2$,

$$f_k(x) = \begin{cases} f & \text{on } \overline{B(0, k + 1)} \setminus \overline{B(0, k)}, \\ 0 & \text{elsewhere.} \end{cases}$$

The function $u_k(x) = -\int U(x, y) f_k(y)\, d\lambda_n(y)$ $(x, y \in \mathbb{R}^n)$ is subharmonic on $\mathbb{R}^n$ and harmonic on $B(0, k)$ when $k \geq 2$. The harmonic function u_k on $B(0, k)$ has an expansion in terms of homogeneous harmonic polynomials. By suitably truncating this expansion we see that there is a harmonic polynomial h_k on $\mathbb{R}^n$ such that $|u_k - h_k| < 2^{-k}$ on $\overline{B(0, k - 1)}$ when $k \geq 2$. We define

$$s = u_1 + \sum_{m=2}^{\infty} (u_m - h_m).$$

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Since, for $k \geq 2$, the series $\sum_{m=k}^{\infty}(u_m - h_m)$ converges uniformly on $B(0,k)$ to a harmonic function, the function s is subharmonic on $\mathbb{R}^n$.

Let ψ be a real-valued infinitely differentiable function on $\mathbb{R}^n$ with compact support in Ω. We can choose k such that the support of ψ is contained in $B(0,k)$. We write the function s in the form

$$s(x) = -\int_{\overline{B(0,k)}} U(x,y) f(y)\, d\lambda_n(y) - \sum_{m=2}^{k-1} h_m(x) + \sum_{k}^{\infty}(u_m - h_m)(x).$$

Now the function $\int_{\overline{B(0,k)}} U(\,\cdot\,,y) f(y)\, d\lambda_n(y)$ is a potential on $\mathbb{R}^n$ when $n \geq 3$, and a logarithmic potential on $\mathbb{R}^2$, with associated Riesz measure $f|_{\overline{B(0,k)}}\lambda_n$. Let $a_n = \max\{1, n-2\}\sigma(\partial B)$, where $\sigma(\partial B)$ denotes surface area measure of the unit sphere in $\mathbb{R}^n$. Since $\sum_{m=2}^{k-1} h_m$ and $\sum_{m=k}^{\infty}(u_m - h_m)$ are harmonic on $B(0,k)$, and the support of ψ is contained in $B(0,k)$, it follows that

$$\int_{\Omega} s\Delta\psi\, d\lambda_n = a_n \int_{\overline{B(0,k)}} \psi f\, d\lambda_n = a_n \int_{\Omega} \psi f\, d\lambda_n,$$

that is, $\Delta s = a_n f$ in the sense of distributions.

Since f_k is a bounded Lebesgue integrable function with compact support in $\mathbb{R}^n$, it follows from [1, Theorem 4.5.3] that u_k is a continuous function on $\mathbb{R}^n$. Hence $\sum_{k=2}^{\infty}(u_k - h_k)$, which converges locally uniformly on $\mathbb{R}^n$, is also continuous on $\mathbb{R}^n$, and so s is continuous on $\mathbb{R}^n \supset \partial\Omega$. Since the topological condition (ii) is precisely that required to solve the Dirichlet problem on unbounded domains for arbitrary continuous boundary data (see [2]), there is a harmonic function h_s on Ω such that $h_s(x) \to s(y)$ as $x \to y$ when y is a regular point of $\partial\Omega$ and $\limsup_{x\to y} |h_s(x)| < +\infty$ when y is an irregular point of $\partial\Omega$. If we define $u = 1/a_n(s - h_s)$, then u satisfies the conditions in (i).

Conversely, suppose that the topological condition (ii) fails. Then there is a compact set K in $\mathbb{R}^n$ and a sequence (Ω_k) of bounded components of $\Omega \setminus K$ such that $\overline{\Omega}_k \cap K \neq \varnothing$ and $\Omega_k \setminus B(0,k) \neq \varnothing$. We choose $x_k \in \Omega_k$, for each k, such that $\operatorname{dist}(x_k, K) < 1/k$. Then, by replacing (Ω_k) by a suitable subsequence if necessary, we can assume that (x_k) converges to some point x^* of $\partial\Omega \cap K$. Let B_k be an open ball such that $\overline{B}_k \subset \Omega_k \setminus B(0,k)$, and $x_k \notin \overline{B}_k$. For each k, there exists $\varepsilon_k > 0$ such that $G_{\Omega_k}(x_k, y) > \varepsilon_k$ when $y \in B_k$, where $G_{\Omega_k}(x,y)$ is the Green function for Ω_k. Let $c_k = \lambda_n(B_k)$ and define

$$f(y) = \begin{cases} \dfrac{k}{G_{\Omega_k}(x_k, y)c_k} & (y \in B_k;\ k \geq 1), \\ 0 & \text{elsewhere.} \end{cases} \tag{2.1}$$

For each k, we define

$$v_k(x) = \int_{\Omega_k} G_{\Omega_k}(x,y) f(y)\, d\lambda_n(y) \quad (x \in \Omega_k).$$

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Then v_k is a potential on Ω_k since f is bounded ($0 \leq f < k(c_k\varepsilon_k)^{-1}$) and has compact support in Ω_k for each k. Also, $\Delta v_k = -a_n f$ on Ω_k in the sense of distributions.

Now suppose, for the sake of contradiction, that u is a continuous function on Ω that satisfies the conditions in (i) for the function f. Since $\Delta u = f \geq 0$ on Ω in the sense of distributions, it follows that u is also subharmonic on Ω (see [4, Theorem 2.5.8]). Let

$$m = \sup_{K\cap\Omega} u^+. \tag{2.2}$$

Since u is bounded near each point of $\partial\Omega\cap K$ and is a continuous function on $K\cap\Omega$, we see that m is finite. Similarly, u is bounded above on Ω_k since $\partial\Omega_k \subset \partial\Omega\cup(K\cap\Omega)$. By our supposition $\lim_{x\to y} u(x) = 0$ for every point y in $\partial\Omega_k\cap\partial\Omega$ except the polar set of irregular points and, by (2.2), $\lim_{x\to y} u(x) \leq m$ for every y in $\partial\Omega_k \cap K$, so we can apply the maximum principle (the general form provided in [1, Theorem 5.2.6(i)]) to the subharmonic function $u - m$ to see that $u \leq m$ on Ω_k for each k. By the Riesz decomposition theorem, since v_k is the potential on Ω_k of the Riesz measure $f = \Delta u$, it follows that $u = -a_n^{-1}v_k + H_k$ on Ω_k for each k, where H_k is the least harmonic majorant of u on Ω_k. Clearly $H_k \leq m$ on Ω_k. Hence,

$$\begin{aligned} u(x_k) &= -\frac{1}{a_n}\int_{\Omega_k} G_{\Omega_k}(x_k, y) f(y)\, d\lambda_n(y) + H_k(x_k), \\ &= -\frac{k}{a_n c_k}\int_{B_k} d\lambda_n(y) + H_k(x_k) \\ &\leq -\frac{k}{a_n} + m \quad \to -\infty \text{ as } k \to \infty, \end{aligned} \tag{2.3}$$

contradicting the supposition that u satisfies condition (c) of (i).

3 Proof of Theorem 2

Suppose that Ω satisfies the topological condition in (ii). Let μ be a positive measure on $\mathbb{R}^n$. We can assume that $\mu(\mathbb{R}^n \setminus \Omega) = 0$ and, without loss of generality, that $\overline{B(0,2)} \subset \Omega$. For each j and k in $\mathbb{N}$, we define

$$\Omega_{j,k} = \{x \in \Omega \cap (\overline{B(0,k+1)} \setminus B(0,k)) : \operatorname{dist}(x, \mathbb{R}^n \setminus \Omega) \geq 1/j\}.$$

Then as $j \to \infty$, $\mu(\Omega_{j,k}) \uparrow \mu\big(\Omega \cap (\overline{B(0,k+1)} \setminus B(0,k))\big)$. For each k, we can choose j_k such that $\mu\big((\Omega \setminus \Omega_{j_k,k}) \cap (\overline{B(0,k+1)} \setminus B(0,k))\big) < 2^{-k}$. Let $\Omega_1 = \big(\bigcup_{k\geq 1} \Omega_{j_k,k}\big) \cup \overline{B(0,1)}$. Then Ω_1 is a relatively closed subset of Ω since $\overline{B(0,2)} \subset \Omega$ and $j_k \geq 1$. Further,

$$\int_{\Omega\setminus\Omega_1} d\mu(y) \leq \sum_{k=1}^{\infty} 2^{-k} = 1. \tag{3.1}$$

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Now we define $\mu_1 = \mu|_{\Omega_1 \cap \overline{B(0,2)}}$ and $\mu_k = \mu|_{\Omega_1 \cap (\overline{B(0,k+1)} \setminus \overline{B(0,k)})}$ when $k \geq 2$. Let

$$u_k(x) = -\int U(x, y)\, d\mu_k(y)$$

and let h_k ($k \geq 2$) be a harmonic function on $\mathbb{R}^n$ such that $|u_k - h_k| < 2^{-k}$ on $\overline{B(0, k-1)}$. Then, as in the proof of the previous theorem, the function

$$s = u_1 + \sum_{k=2}^{\infty} (u_k - h_k)$$

is subharmonic on $\mathbb{R}^n$, harmonic (and so, in particular, continuous) on $\mathbb{R}^n \setminus \overline{\Omega}_1 \supset \partial\Omega$, and has associated Riesz measure $\mu|_{\Omega_1}$. By hypothesis, Ω satisfies the topological condition required to solve the Dirichlet problem for continuous boundary data (see [2]), so there is a harmonic function h_s on Ω such that $h_s(x) \to s(y)$ as $x \to y$ for each regular point $y \in \partial\Omega$ and $\limsup_{x \to y} |h_s(x)| < +\infty$ for each point $y \in \partial\Omega$. Clearly $\Delta(s - h_s) = a_n \mu|_{\Omega_1}$,

$$\text{fine}\lim_{x \to y} (s - h_s)(x) = \lim_{x \to y} (s - h_s)(x) = 0 \tag{3.2}$$

at each regular point y of $\partial\Omega$, and $\limsup_{x \to y} |s - h_s|(x) < +\infty$ at each $y \in \partial\Omega$. Thus, since the set of irregular points of $\partial\Omega$ is polar, the conditions in (i) are fulfilled for the measure $\mu|_{\Omega_1}$ by the function $a_n^{-1}(s - h_s)$.

Let $B = B(0, 1)$. By (3.1), the function defined by

$$v(x) = \begin{cases} \int U(x, y)\, d\mu|_{\Omega \setminus \Omega_1}(y) & (x \in \mathbb{R}^n, n \geq 3), \\ \int G_{\mathbb{R}^2 \setminus \overline{B}}(x, y) d\mu|_{\Omega \setminus \Omega_1}(y) & (x \in \mathbb{R}^2 \setminus \overline{B}, n = 2), \\ 0 & (x \in \overline{B}, n = 2). \end{cases}$$

is a Newtonian potential when $n \geq 3$ and a potential on $\mathbb{R}^2 \setminus \overline{B}$ when $n = 2$. We let $\Omega_0 = \mathbb{R}^n$ when $n \geq 3$ and $\Omega_0 = \mathbb{R}^2 \setminus \overline{B}$ when $n = 2$, and define

$$\widehat{R}_v(x) = \begin{cases} \widehat{R}_v^{\Omega_0 \setminus \Omega}(x) & (x \in \Omega_0), \\ 0 & (x \in \overline{B}, n = 2), \end{cases}$$

where $\widehat{R}_v^{\Omega_0 \setminus \Omega}$ denotes the regularized reduced function of v with respect to superharmonic functions on Ω_0 and the set $\Omega_0 \setminus \Omega$. Then $\Delta(\widehat{R}_v - v) = a_n \mu|_{\Omega \setminus \Omega_1}$ on Ω (for $n \geq 3$), since $\widehat{R}_v$ is harmonic on Ω. In the case where $n = 2$, both v and $\widehat{R}_v$ restricted to $\Omega_1^\circ \setminus \overline{B}$ (where Ω_1° denotes the interior of Ω_1) are non-negative harmonic functions with limit 0 at ∂B, so they are subharmonic functions on Ω_1° whose associated Riesz measures have support in ∂B. Let w_1 (respectively w_2) be the potential on Ω of the measure on ∂B associated with v (respectively $\widehat{R}_v$) and let $v_0 = w_2 - w_1$. Then v_0 is harmonic on $\Omega \setminus \partial B$, and $\Delta v_0 = \Delta(v - \widehat{R}_v)$ on Ω_1°, so

$$\Delta(\widehat{R}_v - v + v_0) = a_n \mu|_{\Omega \setminus \Omega_1} \text{ on } \Omega \quad (n = 2).$$

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When $n \geq 2$, both of the functions v and $\widehat{R}_v$ are superharmonic on $\mathbb{R}^n \setminus \overline{B}$ and hence finely continuous there. Also, by [1, Theorem 5.7.3], they are equal quasi-everywhere on $\mathbb{R}^n \setminus \Omega$, which contains $\partial\Omega$. Hence

$$\limsup_{x \to y, x \in \Omega} (\widehat{R}_v - v)(x) \leq 0 = \operatorname{fine} \lim_{x \to y} (\widehat{R}_v - v)(x) \tag{3.3}$$

at quasi-every point $y \in \partial\Omega \subset \mathbb{R}^n \setminus \Omega$, since the set of points in $\partial\Omega$ where $v = +\infty$ is also a polar set. The function v_0 is the difference of two potentials on Ω of measures with compact support, and so

$$\operatorname{fine} \lim_{x \to y} v_0(x) = \lim_{x \to y} v_0(x) = 0 \tag{3.4}$$

at every regular point $y \in \partial\Omega$, that is, at quasi-every point $y \in \partial\Omega$. We define

$$u = \begin{cases} \dfrac{1}{a_n}(\widehat{R}_v - v + s - h_s) & (n \geq 3), \\ \dfrac{1}{a_n}(\widehat{R}_v - v + v_0 + s - h_s) & (n = 2). \end{cases}$$

Then u is subharmonic on Ω, with $\Delta u = \mu|_{\Omega \setminus \Omega_1} + \mu|_{\Omega_1} = \mu$ on Ω. Thus u satisfies conditions (i)(a) and (b), in view of (3.2), (3.3) and (3.4).

Since $\limsup_{x \to y} |s - h_s|(x) < +\infty$ at each $y \in \partial\Omega$, and since v_0 (being the difference of two potentials with compact support) is also bounded near each $y \in \partial\Omega$, it remains to show that $\widehat{R}_v - v$ satisfies conditions (c) and (d) of (i). Condition (c) is immediate, since $\widehat{R}_v - v \leq 0$ on Ω_0. Finally, if $y \in \partial\Omega$ and there exists $\varepsilon > 0$ such that $\int_{B(y,\varepsilon) \cap \Omega} U(\,\cdot\,, z)\, d\mu(z)$ is bounded above, we let

$$v_1 = \begin{cases} \int_{B(y,\varepsilon)} U(\,\cdot\,, z)\, d\mu|_{\Omega \setminus \Omega_1}(z) & (n \geq 3), \\ \int_{B(y,\varepsilon)} G_{\mathbb{R}^2 \setminus \overline{B}}(\,\cdot\,, z)\, d\mu|_{\Omega \setminus \Omega_1}(z) & (n = 2); \end{cases}$$

$$v_2 = \begin{cases} \int_{\mathbb{R}^n \setminus B(y,\varepsilon)} U(\,\cdot\,, z)\, d\mu|_{\Omega \setminus \Omega_1}(z) & (n \geq 3), \\ \int_{(\mathbb{R}^2 \setminus \overline{B}) \setminus B(y,\varepsilon)} G_{\mathbb{R}^2 \setminus \overline{B}}(\,\cdot\,, z)\, d\mu|_{\Omega \setminus \Omega_1}(z) & (n = 2). \end{cases}$$

Then $v = v_1 + v_2$ on $\mathbb{R}^n \setminus \overline{B}$. (We may assume that $\varepsilon < 1$.) The function v_2 is harmonic on $B(y, \varepsilon)$, so it has a finite limit at y. When $n = 2$, for each $z \in \mathbb{R}^2 \setminus \overline{B}$, let h_z denote the greatest harmonic minorant of $U(\cdot, z)$ on $\mathbb{R}^2 \setminus \overline{B}$, so that

$$G_{\mathbb{R}^2 \setminus \overline{B}}(x, z) = U(x, z) - h_z(x).$$

The function $h_x(z)$ is harmonic in (x, z) and hence bounded on $B(y, \varepsilon) \times B(y, \varepsilon)$. Thus

$$\begin{aligned} v_1(x) &= \int_{B(y,\varepsilon)} U(x, z)\, d\mu|_{\Omega \setminus \Omega_1}(z) - \int_{B(y,\varepsilon)} h_x(z)\, d\mu|_{\Omega \setminus \Omega_1}(z) \\ &\leq \int_{B(y,\varepsilon)} U(x, z)\, d\mu|_{\Omega \setminus \Omega_1}(z) + \mu(B(y, \varepsilon)) \sup_{z \in B(y,\varepsilon)} |h_x(z)|, \end{aligned}$$

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and so v_1 is clearly bounded above on $B(y,\varepsilon)$. Hence $\limsup_{x\to y} v(x) < +\infty$ for all $n \geq 2$ whenever $\int_{B(y,\varepsilon)\cap\Omega} U(\cdot, z)\,d\mu(z)$ is bounded above. Since $\widehat{R}_v \geq 0$ on $\mathbb{R}^n$, it follows that $\liminf_{x\to y} u(x) > -\infty$ at any point y where $\int_{B(y,\varepsilon)\cap\Omega} U(\cdot, z)\,d\mu(z)$ is bounded above.

Conversely, suppose that Ω is an unbounded Greenian domain in $\mathbb{R}^n$ for which the topological condition in (ii) fails for some compact subset K of $\mathbb{R}^n$. We define a sequence (Ω_k) of bounded components of $\Omega \setminus K$ and a sequence of points (x_k) which converges to x^* in $\partial\Omega \cap K$ as in the proof of the converse of the previous theorem. We next define f as in (2.1) and let $\mu = f\lambda_n$. Then $f = 0$ on $\mathbb{R}^n \setminus \bigcup_k B_k$. Clearly there exists $\varepsilon > 0$ such that $f = 0$ on $B(x^*, \varepsilon) \cap \Omega$ and thus

$$\int_{B(x^*,\varepsilon)\cap\Omega} U(\cdot, z) f\,d\lambda_n(z) = 0. \tag{3.5}$$

Now suppose that u is a subharmonic function satisfying the four conditions in (i) of the statement of the theorem, with $\mu = f\lambda_n$. Then, by condition (d) and (3.5),

$$\liminf_{x\to x^*} u(x) > -\infty. \tag{3.6}$$

It follows from the upper-semicontinuity of u and condition (c) of (i) that $m = \sup_{K\cap\Omega} u^+$ is finite (see the justification of (2.2)). So $\limsup_{x\to y}(u - m)(x) \leq 0$ at every point y of $\partial\Omega_k \cap K$ and, by condition (b) of (i), $\limsup_{x\to y} u(x) = 0$ at quasi-every point $y \in \partial\Omega_k \cap \partial\Omega$. Thus $\limsup_{x\to y}(u - m)(x) \leq 0$ at quasi-every point y in $\partial\Omega_k$, for each k. Further, by condition (c), the function u is bounded above on Ω_k for each k. Thus, by the maximum principle applied to the bounded set Ω_k, the subharmonic function u and its least harmonic majorant are bounded above by m on Ω_k, for each k. As in Section 2 (see (2.3)), this leads to the conclusion that $u(x_k) \to -\infty$ as $x_k \to x^*$, which contradicts (3.6).

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