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Two fixed point theorems and invariant integrals

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Two fixed point theorems for a subset C of a normed vector space X are established by using the concept of centre. These results differ from previous fixed point theorems in that X is assumed to have a topology T as well as a norm. The norm is required to be lower semi-continuous with respect to T and C is required to be convex, bounded with respect to the norm and compact with respect to T.

The first theorem shows that if the norm is locally uniformly convex, then the semigroup of all non-expansive mappings of C onto C has a common fixed point in C. It is shown how this theorem can be used to prove the existence of a right invariant integral on a compact metrizable semigroup with a unique minimal left ideal.

The second theorem shows that, if the norm is again locally uniformly convex and if H is a semigroup of continuous (with respect to T), non-expansive, affine mappings of C into C such that H is left reversible; that is, $TH \cap T'H \neq \emptyset$ for all $T, T' \in H$; then the mappings of H have a common fixed point in C.

1. Introduction

Throughout the paper, X will denote a normed vector space over the real numbers, which is also endowed with a locally convex Hausdorff

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topology T, such that

(i) the vector space operations are continuous with respect to T in the usual way, and

(ii) the norm is lower semicontinuous with respect to T in the following sense:

for every $x \in X$ and $\varepsilon > 0$, there exists a $U \in T$, such that $x \in U$ and

$$|y| > |x| - \varepsilon$$

for all $y \in U$.

C is a non-empty convex subset of X, which is bounded with respect to the norm and compact with respect to T. For each $x \in C$, define

$$r(x) = \sup_{y \in C} |x-y| . \tag{1}$$

Put

$$\alpha(C) = \inf_{x \in C} r(x) \tag{2}$$

and let $\gamma(C)$ denote the set

$$\{x \in C;\ r(x) = \alpha(C)\} . \tag{3}$$

Similarly to [6], $\gamma(C)$ will be called the centre of C.

The concept of centre has been used to show the existence of fixed points by Edelstein [6] and [7] and Belluce and Kirk [1]. The concept of centre has also been used by Brodskiĭ and Mil'man [2].

It will be shown that $\gamma(C)$ is non-empty, convex and compact with respect to T. It will also be shown that every mapping of C onto C, which is non-expansive with respect to the norm, takes $\gamma(C)$ into $\gamma(C)$ and that when the norm satisfies a special convexity condition, $\gamma(C)$ has exactly one point. In this case $\gamma(C)$ is therefore a common fixed point for all non-expansive mappings of C onto C.

The existence of a common fixed point is then used to prove the existence of a right invariant integral on a compact metrizable semigroup with a unique minimal left ideal. The existence of such an integral has of course been known since 1956, when it was established by Rosen in [11].

It is given here as an application of the fixed point theorem. In [12] Šneperman has given a similar application using a somewhat different fixed point theorem.

The existence of the common fixed point for onto mappings is also used to show the existence of a common fixed point for a semigroup of non-expansive mappings of C into C which satisfies the special intersection property of left reversibility.

2. The first fixed point theorem

The structure of the centre and the first of the fixed point theorems will now be discussed.

THEOREM 2.1. *r is lower semicontinuous on C with respect to T.*

Proof. Let $x \in C$ and $\varepsilon > 0$. There exists $z \in C$ such that

(4) $$|x-z| > r(x) - \tfrac{1}{2}\varepsilon .$$

Since the norm is lower semicontinuous, there exists a $V \in T$ such that $x - z \in V$ and

(5) $$|w| > |x-z| - \tfrac{1}{2}\varepsilon$$

for all $w \in V$. Put $U = V + z$. Then $U \in T$ and $x \in U$. When $y \in U \cap C$, we have $y - z \in V$, so that by (5), $|y-z| > |x-z| - \frac{1}{2}\varepsilon$. Since $r(y) \geq |y-z|$, it follows that $r(y) > |x-z| - \frac{1}{2}\varepsilon$, so that by (4), $r(y) > r(x) - \varepsilon$.

THEOREM 2.2. *$\gamma(C)$ is a non-empty convex subset of C, which is bounded with respect to the norm and compact with respect to T.*

Proof. For each positive integer n, let

$$\gamma_n(C) = \left\{x;\ x \in C \text{ and } r(x) \leq \alpha(C) + \frac{1}{n}\right\} .$$

Since r is lower semicontinuous and C is compact, $\gamma_n(C)$ is compact with respect to T for all n. From (2), $\gamma_n(C)$ is non-empty for all n. Therefore

$$\gamma(C) = \bigcap_{n=1}^{\infty} \gamma_n(C)$$

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is a non-empty compact subset of C . The boundedness of $\gamma(C)$ is trivial and the convexity is easily verified.

We define a mapping T of C into C to be non-expansive if, for all x and $y \in C$,

$$|T(x)-T(y)| \le |x-y| . \tag{6}$$

THEOREM 2.3. *Every non-expansive mapping* T *of* C *onto* C *maps* $\gamma(C)$ *into* $\gamma(C)$ *.*

Proof. Consider any $x \in \gamma(C)$ and $y \in C$. There exists $\xi \in C$ such that $T(\xi) = y$. Now

$$|x-\xi| \le \alpha(C) .$$

Hence, from (6),

$$|T(x)-T(\xi)| \le \alpha(C) ;$$

that is,

$$|T(x)-y| \le \alpha(C) .$$

Since this holds for all $y \in C$, then $r(T(x)) = \alpha(C)$; hence $T(x) \in \gamma(C)$. Thus T maps $\gamma(C)$ into $\gamma(C)$.

The norm is said to be locally uniformly convex if for every x and $y \in X$, with $x \ne y$, and every $D \ge \frac{1}{2}|x-y|$,

$$\delta(x, y, D) = \inf\{D-|\tfrac{1}{2}(x+y)-\xi| ; \xi \in X, |x-\xi| \le D, |y-\xi| \le D\}$$

is a positive number.

THEOREM 2.4. *If the norm is locally uniformly convex, then* $\gamma(C)$ *contains only one point.*

REMARK. The norm need only be locally uniformly convex on C .

Proof. Suppose $\gamma(C)$ contains two distinct points x, y . Consider any $\xi \in C$. Then $|x-\xi| \le \alpha(C)$, $|y-\xi| \le \alpha(C)$ and

$$\delta(x, y, \alpha(C)) \le \alpha(C) - |\tfrac{1}{2}(x+y)-\xi| .$$

Hence

$$|\tfrac{1}{2}(x+y)-\xi| \le \alpha(C) - \delta(x, y, \alpha(C)) .$$

This holds for all $\xi \in C$ and therefore

$$r(\tfrac{1}{2}(x+y)) \le \alpha(C) - \delta(x, y, \alpha(C)) .$$

But since $\alpha(C) \geq \frac{1}{2}|x-y|$, then $\delta(x, y, \alpha(C))$ is a positive number and so $r(\frac{1}{2}(x+y)) < \alpha(C)$, a contradiction.

The following fixed point theorem now follows from Theorems 2.2, 2.3 and 2.4.

THEOREM 2.5. *If the norm is locally uniformly convex and if* H *is any set of non-expansive mappings of* C *onto* C , *then the mappings of* H *have a common fixed point in* C .

REMARK. The members of H do not have to be linear.

COROLLARY 2.6. *If* X *is a Hilbert space,* C *is a non-empty convex bounded weakly compact subset of* X *and* H *is any set of non-expansive mappings of* C *onto* C , *then the mappings of* H *have a common fixed point in* C .

3. Invariant integrals

We now show how Theorem 2.5 can be used to prove the existence of a right invariant integral on a compact metric semigroup with a unique minimal left ideal.

Let G be a compact metric semigroup with metric d . $C(G)$ is the Banach space of all real valued continuous functions on G with the supremum norm. $\{f^{(n)}\}$ is a sequence of members of $C(G)$ such that the linear manifold M spanned by them is dense in $C(G)$ and

$$|f^{(n)}| = 1$$

for all n . (See page 246 of [8] for a proof of the separability of $C(G)$.) For each $f \in C(G)$ and $a \in G$, f_a is the member of $C(G)$ defined by

$$f_a(x) = f(xa) ,$$

for all $x \in G$. Λ is the vector space of all bounded linear functionals λ on $C(G)$. For each $\lambda \in \Lambda$ and $a \in G$, define

$$\phi(\lambda, a) = \left[\sum_{n=1}^{\infty} 2^{-n}\left(\lambda\left(f_a^{(n)}\right)\right)^2\right]^{\frac{1}{2}} . \tag{8}$$

Define a norm on Λ , by

(9) $$|\lambda| = \sup_{a \in G} \phi(\lambda, a) .$$

The following lemma is a well known result.

LEMMA 3.1. *If* $f \in C(G)$ *and* $\{a_k\}$ *is a sequence in* G *which converges to an element* a *of* G *, then*

$$|f_{a_k} - f_a| \to 0$$

as $k \to \infty$.

THEOREM 3.2. *For all* $\lambda \in \Lambda$ *there exists an* $a \in G$ *such that*

(10) $$|\lambda| = \phi(\lambda, a) .$$

Proof. Consider any $\lambda \in \Lambda$. Choose a sequence $\{a_k\}$ in G such that

(11) $$|\lambda| \geq \phi(\lambda, a_k) > |\lambda| - \frac{1}{k}$$

for all k . Since G is compact metric, there exists a subsequence $\{b_r\}$ of $\{a_k\}$ converging to an element a of G . By (11),

(12) $$\phi(\lambda, b_r) \to |\lambda|$$

as $r \to \infty$. Let K be a constant such that

(13) $$|\lambda(f)| \leq K|f|$$

for all $f \in C(G)$. Therefore, by (7),

(14) $$\left|\lambda\left(f_{b_r}^{(n)}\right)\right| \leq K$$

for all r and n .

By Lemma 3.1, $\left|f_{b_r}^{(n)} - f_a^{(n)}\right| \to 0$ as $r \to \infty$, hence by (13), $\lambda\left(f_{b_r}^{(n)}\right) \to \lambda\left(f_a^{(n)}\right)$ as $r \to \infty$. By (14), the series concerned is uniformly convergent and hence

$$\phi(\lambda, b_r) \to \phi(\lambda, a)$$

as $r \to \infty$, so that, by (12),

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$$\phi(\lambda, a) = |\lambda| .$$

We define a functional β on Λ, by

(15) $$\beta(\lambda) = \inf_{a \in G} \phi(\lambda, a) .$$

THEOREM 3.3. *For all* $\lambda \in \Lambda$, *there exists an* $a \in G$, *such that*

$$\beta(\lambda) = \phi(\lambda, a) .$$

This can be proved in a similar manner to Theorem 3.2.

THEOREM 3.4. *If* G *is left simple, that is,* $Gx = G$ *for all* $x \in G$, *then*

(16) $$\phi(\lambda, b) > 0$$

for all $\lambda \in \Lambda$, *with* $\lambda \neq 0$, *and all* $b \in G$. *Hence* $\beta(\lambda)$ *is a positive number for all* $\lambda \in \Lambda$, *with* $\lambda \neq 0$.

Proof. Let $\lambda \in \Lambda$, with $\lambda \neq 0$ and let $b \in G$. Since G is compact it contains at least one idempotent (see [14]). By Theorem 1-27 on page 38 of [3], G is a left group. Then the mapping χ of G onto G defined by

$$\chi(x) = xb ,$$

for all $x \in G$, is one-to-one. Hence χ is a homeomorphism of G onto G. It follows that the linear manifold spanned by the set of functions

$$\left\{f_b^{(n)};\ n = 1, 2, \ldots\right\} \text{ is dense in } C(G) .$$

Then $\lambda\left(f^{(n)}\right) \neq 0$ for some n and hence $\phi(\lambda, b) > 0$.

THEOREM 3.5. *If* G *is left simple, then the norm for* Λ *is locally uniformly convex.*

Proof. Consider any λ and $\mu \in \Lambda$ with $\lambda \neq \mu$, and any real number D, with $D \geq \frac{1}{2}|\lambda-\mu|$. We have to show that

$$\delta(\lambda, \mu, D) = \inf\{D-|\tfrac{1}{2}(\lambda+\mu)-\xi|;\ \xi \in \Lambda,\ |\lambda-\xi| \leq D,\ |\mu-\xi| \leq D\}$$

is a positive number. Consider any $\xi \in \Lambda$ with $|\lambda-\xi| \leq D$ and $|\mu-\xi| \leq D$.

There exists an element $b \in G$ such that

$$\begin{aligned}|\tfrac{1}{2}(\lambda+\mu)-\xi|^2 &= [\phi(\tfrac{1}{2}\lambda+\tfrac{1}{2}\mu-\xi,\ b)]^2\\ &= \tfrac{1}{2}[\phi(\lambda-\xi,\ b)]^2 + \tfrac{1}{2}[\phi(\mu-\xi,\ b)]^2 - \tfrac{1}{4}[\phi(\lambda-\mu,\ b)]^2\\ &\le \tfrac{1}{2}\Big[\sup_{a\in G}\phi(\lambda-\xi,\ a)\Big]^2 + \tfrac{1}{2}\Big[\sup_{a\in G}\phi(\mu-\xi,\ a)\Big]^2 - \tfrac{1}{4}\Big[\inf_{a\in G}\phi(\lambda-\mu,\ a)\Big]^2\\ &= \tfrac{1}{2}|\lambda-\xi|^2 + \tfrac{1}{2}|\mu-\xi|^2 - \tfrac{1}{4}[\beta(\lambda-\mu)]^2\\ &\le D^2 - \tfrac{1}{4}[\beta(\lambda-\mu)]^2 .\end{aligned}$$

Since $\lambda \ne \mu$, then, from Theorem 3.4, $\beta(\lambda-\mu)$ is a positive number. Hence there exists a positive number δ such that, for all $\xi \in \Lambda$ with $|\lambda-\xi| \le D$ and $|\mu-\xi| \le D$,

$$|\tfrac{1}{2}(\lambda+\mu)-\xi| \le D - \delta , \tag{17}$$

where δ depends only on $\lambda,\ \mu$ and D . Hence $\delta(\lambda,\ \mu,\ D) > 0$.

Let $\mathcal{T}$ denote the weak topology for Λ .

THEOREM 3.6. *The norm for* Λ *is lower semicontinuous with respect to* $\mathcal{T}$.

Proof. It is easily verified that, when a is kept fixed and $\phi(\lambda,\ a)$ is regarded as a function of λ , then ϕ is lower semicontinuous with respect to $\mathcal{T}$. It now follows that the norm is lower semicontinuous.

We now let Γ be the set of all $\lambda \in \Lambda$ such that, λ is positive,

$$\lambda(1) = 1 , \tag{18}$$

and

$$|\lambda(f)| \le |f| \tag{19}$$

for all $f \in C(G)$. (On the left hand side of (18), 1 denotes the function with constant value 1 .)

THEOREM 3.7. Γ *is a non empty convex subset of* Λ *which is bounded with respect to the norm and weakly compact.*

Proof. The convexity and boundedness of Γ are straightforward. The existence of a positive linear functional with $\lambda(1) = 1$ and $|\lambda(f)| \le |f|$, for all $f \in C(G)$, can be shown by letting $a \in G$ and defining

$$\lambda(f) = f(a)$$

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for all $f \in C(G)$.

Clearly Γ is weakly closed, so that by Theorem 4-61-A on page 228 of [13], Γ is weakly compact.

We note that Γ does not contain the zero functional. For each $a \in G$, let T_a denote the transformation of Λ into Λ , defined by

(20) $$\left(T_a\lambda\right)(f) = \lambda\left(f_a\right)$$

for all $f \in C(G)$ and $\lambda \in \Gamma$. Let H denote the semigroup of all the transformations T_a .

THEOREM 3.8. *For all $a \in G$, T_a is a non-expansive map of Γ into Γ .*

If G is left simple, then for all $a \in G$, T_a maps Γ onto Γ .

Proof. Let $a \in G$. It is easily verified that T_a maps Γ into Γ . Let $\lambda, \mu \in \Gamma$. We have to show that $|T_a(\lambda-\mu)| \leq |\lambda-\mu|$. Since

$$\left(g_b\right)_a = g_{ab}$$

for all $b \in G$ and $g \in C(G)$, it follows that

$$\begin{aligned} |T_a(\lambda-\mu)| &= \sup_{b\in G}\left[\sum_{n=1}^{\infty} 2^{-n}\left\{(\lambda-\mu)\left(f_{ab}^{(n)}\right)\right\}^2\right]^{\frac{1}{2}} \\ &\leq \sup_{c\in G}\left[\sum_{n=1}^{\infty} 2^{-n}\left\{(\lambda-\mu)\left(f_{c}^{(n)}\right)\right\}^2\right]^{\frac{1}{2}} \\ &= |\lambda-\mu| . \end{aligned}$$

Now suppose that G is left simple and let $a \in G$. Now G has an idempotent e and $xe = x$ for all $x \in G$. Hence $f_e = f$ for all $f \in C(G)$, so that $T_e(\xi) = \xi$ for all $\xi \in \Gamma$. Let $c \in G$ be such that $ca = e$. Consider an arbitrary $\lambda \in \Gamma$ and put $\eta = T_c(\lambda)$. Then, for all $f \in C(G)$,

$$\begin{aligned} \left(T_a(\eta)\right)(f) &= \eta\left(f_a\right) = \left(T_c(\lambda)\right)\left(f_a\right) = \lambda\left(f_{ca}\right) \\ &= \lambda\left(f_e\right) = \lambda(f) , \end{aligned}$$

so that $T_a(\eta) = \lambda$. Thus T_a is onto.

THEOREM 3.9. *If G is left simple, then there exists a non-trivial positive right invariant integral on G .*

Proof. Since the conditions of Theorem 2.5 are satisfied, the mappings of H have a common fixed point λ_0 in Γ . Hence, for all $a \in G$ and $f \in C(G)$,

$$\lambda_0(f_a) = \lambda_0(f) .$$

Thus λ_0 is a positive right invariant integral on G .

We now assume that G has a unique minimal left ideal. One of the results of Rosen in [11] is that this is a necessary and sufficient condition for the existence of a right invariant integral. Šneperman has shown in [12] that the right reversibility of G (that is, $Gx \cap Gy$ is non-empty for all $x, y \in G$) is equivalent to the existence of a unique minimal left ideal.

Let K denote the unique minimal left ideal. Then K is a compact subsemigroup and it can be shown that K is left simple (see Michael [9], Theorem 5.1).

By Theorem 3.9 there exists a positive right invariant integral λ_0 on K . For each $f \in C(G)$, let f^* denote the restriction of f to K . Define a positive linear functional λ_1 on $C(G)$ by putting

(21) $$\lambda_1(f) = \lambda_0(f^*)$$

for all $f \in C(G)$. By considering the constant functions one can easily see that λ_1 is non-trivial.

We show that λ_1 is right invariant. Now K has the properties

(22) $$Kx \supset K$$

for all $x \in G$ and

(23) $$Kx = K$$

for all $x \in K$. Suppose that $a \in K$ and $f \in C(G)$. One can easily verify that $(f_a)^* = (f^*)_a$, hence

$$\lambda_1(f_a) = \lambda_0((f_a)^*) = \lambda_0((f^*)_a) = \lambda_0(f^*) ,$$

so that

(24) $$\lambda_1(f_a) = \lambda_1(f) .$$

Now suppose $a \in G \sim K$ and $f \in C(G)$. Let $c \in K$. By (22) there exists an element $b \in K$, such that $ba = c$. By (24), $\lambda_1(f) = \lambda_1(f_c)$, so that

$$\lambda_1(f) = \lambda_1(f_{ba}) = \lambda_1((f_a)_b)$$

and by (24),

$$\lambda_1(f) = \lambda_1(f_a) .$$

Thus λ_1 is right invariant.

We have shown that there exists a right invariant integral on G.

4. Some further fixed point theory

Some additional fixed point theory will now be discussed. The first theorem is similar to Šneperman's Theorem in both the statement and the proof. We again assume that X is a normed vector space, with a topology T, satisfying (i) and (ii) of Section 1 and that C is as described in Section 1.

THEOREM 4.1. *If the norm is locally uniformly convex and if* H *is any semigroup of continuous (with respect to* T*) non-expansive affine mappings of* C *into* C *such that* H *is left reversible; that is,*

(25) $$TH \cap T'H \neq \emptyset$$

for all T *and* $T' \in H$*; then the mappings of* H *have a common fixed point in* C.

Proof. If C contains only one point, there is nothing to prove. Therefore assume C contains more than one point. Let K be the collection of all subsets K of C which are non-empty convex and compact, with respect to T, and for which HK is a subset of K. Order K by inclusion. Then $(K, \subset)$ is a pre-ordering. Consider a chain K_0 in K. Then, since any two members of K_0 are related,

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$A_1 \cap A_2 \cap \ldots \cap A_r$ is nonempty for any finite sequence $A_1, A_2, \ldots, A_r$ in K_0. Hence

$$K_1 = \bigcap\{A ; A \in K_0\}$$

is nonempty. It is straightforward to show that K_1 is convex and compact, with respect to T, and contains HK_1. Hence K_1 is a member of K. It is obviously also a lower bound of the chain K_0. Therefore by Zorn's Lemma, K has a minimal element, K_0 say. If K_0 contains only one point there is nothing further to prove, therefore assume K_0 contains more than one point.

We now show that $TK_0 = K_0$ for all $T \in H$. We first show that, for all n and $T_1, T_2, \ldots, T_n \in H$, there exists $H_1, H_2, \ldots, H_n \in H$ such that

(26) $$T_1H_1 = T_2H_2 = \ldots = T_nH_n .$$

From (25), there exist H_1 and H_2 such that (26) holds for $n = 2$. Suppose there exists $H_1, H_2, \ldots, H_{k-1} \in H$ such that (26) holds for $n = k - 1$. Then, from (25), there exists H and $H_k \in H$ such that $T_1H_1H = T_kH_k$. Hence

$$T_1H_1H = T_2H_2H = \ldots = T_{k-1}H_{k-1}H = T_kH_k$$

and so (26) holds for $n = k$. By induction, (26) holds for all n.

Let $x \in K_0$. Then (26) gives, for all n,

$$T_1H_1(x) = T_2H_2(x) = \ldots = T_nH_n(x) ;$$

that is, for all n and all $T_1, T_2, \ldots, T_n \in H$, there exists $x_1, x_2, \ldots, x_n \in K_0$ such that

$$T_1x_1 = T_2x_2 = \ldots = T_nx_n .$$

Hence, for every finite sequence $T_1, T_2, \ldots, T_n$ in H,

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$$\bigcap_{i=1}^{n} T_i\left(K_0\right) \text{ is non empty.}$$

Since K_0 is compact, with respect to $\mathcal{T}$, and for each $T \in H$, $T\left(K_0\right)$ is closed, it follows that

(27) $$K_0' = \bigcap_{T\in H} T\left(K_0\right)$$

is non empty and compact, with respect to $\mathcal{T}$. Since each T is affine, K_0' is convex.

If now T_0 and T_1 are arbitrary transformations in H, then by (25) there exist H_0 and H_1 such that $T_0H_0 = T_1H_1$ and therefore, since $K_0' \subset H_0\left(K_0\right)$, it follows that

$$\begin{aligned} T_0\left(K_0'\right) &\subset T_0\left[H_0\left(K_0\right)\right] \\ &= T_1\left[H_1\left(K_0\right)\right] \\ &\subset T_1\left(K_0\right) . \end{aligned}$$

Hence $T_0\left(K_0'\right) \subset T\left(K_0\right)$, for all $T \in H$, and so, from (27),

(28) $$T_0\left(K_0'\right) \subset K_0' .$$

Therefore $K_0' = K_0$, since K_0 is minimal. Hence by (27) and (28), $T\left(K_0\right) = K_0$ for all $T \in H$.

Then K_0 and H satisfy the conditions of Theorem 2.5 and therefore K_0 contains a fixed point under H. This is also a fixed point in C.

If the norm topology and $\mathcal{T}$ are the same, if X is a Banach space and if the diameter of C is positive, then it follows from Lemma 1 of [4] that there exists a $u \in C$ such that

$$r(u) < \operatorname{diam}(C) .$$

It can be easily shown that $\gamma(C) = C$ iff

$$r(x) = \operatorname{diam}(C)$$

for all $x \in C$. Therefore in the above case $\gamma(C)$ is a proper subset of

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C if C contains more than one point. Hence the existence of a fixed point in C, under a left reversible semigroup of non-expansive affine mappings of C into C, can be shown by using a Zorn's Lemma Argument similar to the one used in the proof of Kakutani's Fixed Point Theorem on page 457 of [5]. This will be valid without the norm property of local uniform convexity. This is a slightly less general version of the fixed point theorem established by Mitchell in [10].

If the norm and the topology T are not the same, but C has normal structure or the stronger condition of completely normal structure (see Brodskiĭ and Mil'man [2] and Belluce and Kirk [1]), then $\gamma(K)$ can be shown to be a proper subset of any convex subset K of C which is compact with respect to T and contains more than one point. Similarly to the above, a Zorn's Lemma Argument shows the existence of a fixed point in C under a left reversible semigroup of non-expansive affine mappings of C into C. Local uniform convexity of the norm is again not required. This is similar to the fixed point theorem of Belluce and Kirk [1]. It weakens many of their assumptions, but, of course, it requires the norm to be lower semicontinuous with respect to the topology T.

We conclude with an example which shows that in the general case, where both the norm and the topology T are being considered and neither the norm nor C has special properties, $\gamma(C)$ need not be a proper subset of C.

COUNTER EXAMPLE 4.2. Let m be the space of all bounded real sequences $a = \{a_n\}_{n=1}^{\infty}$ with the usual norm; that is,

$$|a| = \sup_n |a_n| .$$

Let T be the Tychonoff product topology. Then the norm is lower semicontinuous with respect to T.

Let C be the closed (with respect to T) convex hull of

$$a^{(0)} : 2, 2, 2, \ldots$$
$$a^{(1)} : 1, 2, 2, \ldots$$
$$a^{(2)} : 2, 1, 2, \ldots$$
$$a^{(3)} : 2, 2, 1, \ldots$$
$$\cdots\cdots\cdots$$

C is clearly bounded with respect to the norm. Since any set of the form

$$\{a;\ a \in m \ \text{ and } \ |a| \leq \rho\}$$

is compact with respect to T and C is a closed subset of such a set, it follows that C is compact with respect to T.

It is not difficult to show that the diameter of C is 1. With greater difficulty it can be shown that for all $x \in C$, $r(x) = 1$. Then $\alpha(C) = 1$ and hence $\gamma(C) = C$.

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