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On categorical semigroups

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A semigroup S is called categorical if every ideal I of S has the property that $abc \in I$ $(a, b, c \in S)$ implies $ab \in I$ or $bc \in I$. Necessary and sufficient conditions for an orthodox semigroup to be categorical are found and then used to characterize those bands which appear as an isomorphic copy of the band of idempotents of some orthodox categorical semigroup and to simplify the proof of a theorem of Mario Petrich. The structure of commutative categorical semigroups is found modulo the structure of abelian groups and categorical semilattices.

1. Orthodox categorical semigroups

Categorical semigroups have been studied by Petrich [7] and in the commutative case by McMorris and Satyanarayana [5] and the author [6]. We will assume the reader is familiar with Green's relations and with the terminology used in [5].

The following two results were proven in [6]:

PROPOSITION 1. *A semilattice is categorical if and only if it is a tree.*

PROPOSITION 2. *A regular semigroup S is categorical if and only if its set of idempotents E_S satisfies the following property:*

if $e, f, g \in E_S$, $e \leq g$ and $f \leq g$, then $e \in J(ef)$ or $f \in J(ef)$.

An orthodox semigroup is a regular semigroup in which the set of idempotents is a subsemigroup.

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PROPOSITION 3. *An orthodox semigroup* S *is categorical if and only if whenever* $e \leq g$ *and* $f \leq g$ *for some* $e, f, g \in E_S$, *there exists* $t \in E_S$ *such that* $t \leq ef$ *and either* $t\mathcal{D}e$ *or* $t\mathcal{D}f$.

Proof. Suppose that S is an orthodox categorical semigroup. Let $e \leq g$ and $f \leq g$ for some $e, f, g \in E_S$. By Proposition 2, either $e \in J(ef)$ or $f \in J(ef)$. Assume $e \in J(ef)$. Then $e = aefb$ for some $a, b \in S'$. Let $t = efbeaef$. Then t is idempotent, $t \leq ef$, and $t\mathcal{D}e$. Similarly, if $f \in J(ef)$ then there exists an idempotent $t \leq ef$ such that $t\mathcal{D}f$. This proves the necessity of the proposition. Since $\mathcal{D} \subseteq \mathcal{J}$ the sufficiency follows from Proposition 2.

PROPOSITION 4. *If* $e, f \in E_S$ *and* $e\mathcal{D}f$ *then* $eE_Se \cong fE_Sf$.

REMARK. This last result is proven by Baird [*1*] for an arbitrary semigroup S . If E_S is not a band then $\cong$ is to be interpreted as meaning 'partially isomorphic to'. Since there exists a non-categorical semilattice T with the property that $eE_Te \cong fE_Tf$ for any $e, f \in E_T$ (*cf.* [*6*]) one sees that $eE_Se \cong fE_Sf$ does not imply $e\mathcal{D}f$ in general.

THEOREM 1. *A band* B *appears as an isomorphic copy of* E_S *for some orthodox categorical semigroup* S *if and only if whenever* $e \leq g$ *and* $f \leq g$ *for some* $e, f, g \in B$ *then there exists* $t \in B$ *such that* $t \leq ef$ *and either* $eBe \cong tBt$ *or* $fBf \cong tBt$.

Proof. The necessity of the theorem follows from Propositions 3 and 4. To prove sufficiency, assume that B satisfies the property stated in the last part of the theorem. Hall has constructed an orthodox semigroup W such that $B \cong E_W$ and such that $eE_We \cong fE_Wf$ (for some $e, f \in E_W$) if and only if $e\mathcal{D}f$ (*cf.* [*4*]). Identify B with E_W . Suppose that $e \leq g$ and $f \leq g$ for some $e, f, g \in B$. Then there exists $t \in B$ such that $t \leq ef$ and either $tBt \cong eBe$ or $tBt \cong fBf$. Assume that $tBt \cong eBe$. Then $t\mathcal{D}e$ and so $t\mathcal{J}e$. Since $t \leq ef$ it follows that $e \in J(ef)$. Similarly, if $tBt \cong fBf$ then $f \in J(ef)$. By Proposition 2, W is categorical. This completes the proof of the theorem.

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2. A new proof of Petrich's Theorem

A semigroup S is called left-elementary if $\mathcal{L} = \mathcal{D} \cap \lambda$, where $\lambda = \{(x, y) \in S \times S : S^1x \subseteq S^1y\}$.

THEOREM 2. *A left-elementary inverse semigroup is categorical if and only if its semilattice of idempotents is categorical.*

Proof. Suppose that S is a left-elementary inverse semigroup which is also categorical. By Proposition 1, we need only show that E_S is a tree. Let $e \leq g$ and $f \leq g$ for some $e, f, g \in E_S$. Then, by Proposition 3, there exists $t \in E_S$ such that $t \leq ef$ and either $e\mathcal{D}t$ or $f\mathcal{D}t$. If $e\mathcal{D}t$ then $S^1t = S^1tef = S^1tfe$ and so $S^1t \subseteq S^1e$. Since S is left-elementary, $S^1t = S^1e$ and therefore $e \in S^1tef$. This means that $e = ef$. Similarly, if $f\mathcal{D}t$ then $f = ef$. We have thus shown that E_S is a tree. This proves necessity, and sufficiency follows from Propositions 1 and 2.

COROLLARY (Petrich [7]). *A completely semisimple inverse semigroup is categorical if and only if it has a tree of idempotents.*

REMARK. Hall [4] has proved that a regular left-elementary semigroup is completely semisimple.

3. The structure of commutative categorical semigroups

Let S be a semigroup and for each $x \in S$ assign a set A_x such that $x \in A_x$ and $A_x \cap A_y = \emptyset$ if $x \neq y$. Then $\bigcup_{x \in S} A_x$ becomes a semigroup if we define multiplication as:

$$a_x b_y = xy \quad \text{for} \quad a_x \in A_x, \ b_y \in A_y .$$

We write $((S, A_x))$ for the semigroup constructed in the paragraph above. It is called an inflation of S.

A semigroup is called separative if $x^2 = xy = y^2$ implies $x = y$.

THEOREM 3. *Let $S = S^2$, $T = T^2$ and suppose S and T are separative. Then $((S, A_x))$ and $((T, B_y))$ are isomorphic if and only if*

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there is an isomorphism $\phi : S \to T$ *such that* $|A_x| = |B_{\phi(x)}|$ *for all* $x \in S$.

Proof. Let $\psi : ((S, A_x)) \to ((T, B_y))$ be an isomorphism and let ϕ be the restriction of ψ to S . Since $S = S^2$ and $T = T^2$, $\phi(S) = T$ and ϕ is an isomorphism onto T .

Suppose $a \in A_x$ and $\psi(a) \in B_t$ for some $x \in S$, $t \in T$. Then $t^2 = [\psi(a)]^2 = \psi(a^2) = \psi(x^2) = [\psi(x)]^2$ and $\psi(x^2) = \psi(ax) = \psi(a)\psi(x) = t\psi(x)$. Since T is separative, $t = \psi(x)$. We have shown that $\psi(A_x) \subseteq B_{\psi(x)}$. By symmetry, $\psi^{-1}(B_{\psi(x)}) \subseteq A_x$ and so $\psi(A_x) = B_{\psi(x)} = B_{\phi(x)}$ and $|A_x| = |B_{\phi(x)}|$.

This proves the necessity of the theorem.

Now suppose $\phi : S \to T$ is an isomorphism such that for all $x \in S$, $|A_x| = |B_{\phi(x)}|$. For all $x \in S$, let $\psi_x : A_x \to B_{\phi(x)}$ be a bijection such that $\psi_x(x) = \phi(x)$. Define $\psi : ((S, A_x)) \to ((T, B_y))$ as $\psi(a) = \psi_x(a)$ for any $a \in A_x$. Then ψ is an isomorphism.

This proves the sufficiency of the theorem and the proof of Theorem 3 is therefore complete.

Let Y be a semilattice. For each $\alpha \in Y$ assign a group G_α such that $G_\alpha \cap G_\beta = \emptyset$ if $\alpha \neq \beta$. If $\alpha > \beta$ choose a homomorphism $\phi_{\alpha,\beta} : G_\alpha \to G_\beta$ such that if $\gamma < \beta < \alpha$ then $\phi_{\beta,\gamma}\phi_{\alpha,\beta} = \phi_{\alpha,\gamma}$. For $\alpha \in Y$ let $\phi_{\alpha,\alpha}$ be the identity on G_α . Let $S_Y = \bigcup_{\alpha\in Y} G_\alpha$ and define a binary operation $*$ on S_Y as: $a_\alpha * b_\beta = \phi_{\alpha,\alpha\beta}(a_\alpha)\phi_{\beta,\alpha\beta}(b_\beta)$ for $a_\alpha \in G_\alpha$, $b_\beta \in G_\beta$.

Then S_Y with multiplication $*$ is an inverse semigroup and a union of groups. We denote this semigroup by $[Y; G_\alpha\ (\alpha \in Y); \phi_{\alpha,\beta}\ (\alpha > \beta)]$ or simply by $[Y; G_\alpha; \phi_{\alpha,\beta}]$. Any semigroup which is an inverse semigroup and a union of groups can be constructed in this manner (*cf.* Theorem 4.11,

[3]).

REMARK. Henceforth we will identify an element α of Y with the identity element of G_α. This means that Y can be identified with the semilattice of idempotents of $[Y; G_\alpha; \phi_{\alpha,\beta}]$. We will also write $a_\alpha b_\beta$ instead of $a_\alpha * b_\beta$.

It is clear from the above construction that for any $\alpha, \beta \in Y$, $G_\alpha G_\beta \subseteq G_{\alpha\beta}$. The following facts are well known (*cf*. Lemmas 4.8 and 4.9, [3]):

A. if $\alpha \in Y$ then α commutes with every element of S_Y,

B. if $\alpha > \beta$ and $a_\alpha \in G_\alpha$ then $\phi_{\alpha,\beta}(a_\alpha) = a_\alpha \beta$.

The above items will be used repeatedly without mention in the proof of Theorem 4.

THEOREM 4 (Clifford [2]). *There is an isomorphism*

$$i : [Y; G_\alpha; \phi_{\alpha,\beta}] \to [Y'; \hat{G}_\gamma; \hat{\phi}_{\gamma,\sigma}]$$

if and only if there is an isomorphism $\psi : Y \to Y'$ *and isomorphisms* $i_\alpha : G_\alpha \to \hat{G}_{\psi(\alpha)}$ $(\alpha \in Y)$ *such that if* $\alpha > \beta$ $(\alpha, \beta \in Y)$ *then* $\hat{\phi}_{\psi(\alpha),\psi(\beta)} i_\alpha = i_\beta \phi_{\alpha,\beta}$.

Proof of necessity. Let ψ be the restriction of i to Y. Let i_α be the restriction of i to G_α $(\alpha \in Y)$. Then i and i_α are isomorphisms onto Y' and $\hat{G}_{\psi(\alpha)}$ respectively.

If $\alpha > \beta$ in Y and $a_\alpha \in G_\alpha$ then

$$i_\beta[\phi_{\alpha,\beta}(a_\alpha)] = i_\beta(a_\alpha\beta) = i(a_\alpha\beta) = i(a_\alpha)i(\beta) =$$
$$= i_\alpha(a_\alpha)\psi(\beta) = \hat{\phi}_{\psi(\alpha),\psi(\beta)}[i_\alpha(a_\alpha)] .$$

Proof of sufficiency. For $a_\alpha \in G_\alpha$ define $i(a_\alpha) = i_\alpha(a_\alpha)$. Let $a_\alpha \in G_\alpha$ and $a_\beta \in G_\beta$. Then

$$\begin{aligned}
i(a_\alpha b_\beta) &= i_{\alpha\beta}(a_\alpha b_\beta) \\
&= i_{\alpha\beta}(a_\alpha b_\beta)\psi(\alpha\beta) \\
&= i_{\alpha\beta}(a_\alpha b_\beta)i_{\alpha\beta}(\alpha\beta) \\
&= i_{\alpha\beta}(a_\alpha b_\beta \alpha\beta) \\
&= i_{\alpha\beta}(a_\alpha \alpha\beta b_\beta \alpha\beta) \\
&= i_{\alpha\beta}[\phi_{\alpha,\alpha\beta}(a_\alpha)\phi_{\beta,\alpha\beta}(b_\beta)] \\
&= [(i_{\alpha\beta}\phi_{\alpha,\alpha\beta})a_\alpha][(i_{\alpha\beta}\phi_{\beta,\alpha\beta})b_\beta] \\
&= [\hat{\phi}_{\psi(\alpha),\psi(\alpha\beta)}(i_\alpha(a_\alpha))][\hat{\phi}_{\psi(\beta),\psi(\alpha\beta)}(i_\beta(b_\beta))] \\
&= [i_\alpha(a_\alpha)\psi(\alpha\beta)][i_\beta(b_\beta)\psi(\alpha\beta)] \\
&= i_\alpha(a_\alpha)i_\beta(b_\beta) \\
&= i(a_\alpha)i(b_\beta) .
\end{aligned}$$

This completes the proof of Theorem 4.

The proof of the next theorem follows easily from the equivalence of parts 1 and 6 of Theorem 3 [6] and from Theorems 3 and 4 above.

THEOREM 5. *A semigroup* S *is commutative and categorical if and only if there exists a categorical semilattice* Y *and abelian groups* G_α $(\alpha \in Y)$ *such that* $S = (([Y;\, G_\alpha;\, \phi_{\alpha,\beta}];\, A_x))$ *for some collection of sets* A_x, $x \in \bigcup_{\alpha\in Y} G_\alpha$; *furthermore, two commutative categorical semigroups* $(([Y;\, G_\alpha;\, \phi_{\alpha,\beta}];\, A_x))$ *and* $(([Y';\, \hat{G}_\gamma;\, \hat{\phi}_{\gamma,\sigma}];\, B_y))$ *are isomorphic if and only if there exists an isomorphism* $\psi : Y \to Y'$ *and isomorphisms* $i_\alpha : G_\alpha \to \hat{G}_{\psi(\alpha)}$ $(\alpha \in Y)$ *such that if* $\alpha > \beta$ $(\alpha, \beta \in Y)$ *then*

A. $\hat{\phi}_{\psi(\alpha),\psi(\beta)}i_\alpha = i_\beta\phi_{\alpha,\beta}$ *and*

B. *for any* $x \in G_\alpha$, $|A_x| = |B_{i_\alpha(x)}|$.

REMARK. The structure of commutative semigroups which have a regular subsemigroup of products can be obtained modulo the structure of semilattices and abelian groups in a similar manner. Theorems 3 and 4 again give a complete set of invariants.

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4. An open question

It follows from Proposition 2 that the partial groupoid E_S of idempotents of a regular categorical semigroup S satisfies the following property: if $e \leq g$, $f \leq g$ for some $e, f, g \in E_S$ then there exists $t \in E_S$ such that either $t \leq f$ and $tE_St \cong eE_Se$ or $t \leq e$ and $tE_St \cong fE_Sf$.

It is an open question as to whether or not this condition is sufficient for a partial band to appear as an isomorphic copy of the partial groupoid of idempotents of some regular categorical semigroup.

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