MATHEMATICAL NOTE

A NOTE ON SOME THEOREMS FOR ORDINARY DIFFERENTIAL EQUATIONS

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1. In his first stability theorem [(**1**), p. 259], Liapounoff has proved the following fact: Let

$$\frac{dy}{dt} = Y(t, y), \tag{*}$$

where Y is continuous on the region

$$R^*: \quad t \geqq T, \mid y \mid \leqq H,$$

where T and $H(>0)$ are constants, and $Y(t, 0) = 0$ for $t \geqq T$. If for (*) there exists a continuously differentiable positive definite function $V(t, y)$ such that $V(t, 0) = 0$ for $t \geqq T$ and

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial y} Y \leqq 0,$$

then the trivial solution of (*) is stable. Now if we make a transformation $x = 1/t$, then Liapounoff's second method can be used to study the behaviour of the solutions of the equation

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

where f is defined and continuous on the region

$$R: \quad 0 < x \leqq a, \ 0 \leqq y \leqq b,$$

and $f(x, 0) = 0$ for $0 < x \leqq a$. In particular, Theorem 1 below can be obtained in this way. However, since the direct proof is quite short, we give this as well.

2. Some theorems for ordinary differential equations

Theorem 1. *If for* (1) *there exists a function* $g(x) \in C((0, a])$ *satisfying* $0 \leqq g(x) \leqq b$, $g(a) > 0$, *and a function* $V(x, y)$, *defined and continuously differentiable on the region*

$$R_1: \quad 0 < x \leqq a, \ 0 \leqq y \leqq g(x),$$

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such that

$$\inf_{\{0<x\leqq a,\, g(x)\neq 0\}} V(x, g(x))>0, \tag{A}$$

$$V(a, 0)=0, \tag{B}$$

$$\frac{\partial V}{\partial x}+\frac{\partial V}{\partial y}f(x, y)\geqq 0, \tag{C}$$

then there is a $\delta>0$ *such that any solution* $y = y(x)$ *of* (1) *in region* R_1 *satisfying* $y(a) = k$ *with* $0 \leqq k \leqq \delta$ *can be continued to the left as a solution of* (1) in R_1 *defined on* $(0, a]$.

Proof. Let $\gamma = \inf_{\{0<x\leqq a,\, g(x)\neq 0\}} V(x, g(x))$, then by (A) we see that $\gamma>0$. By (B) there is $\delta>0$ ($\delta \leqq g(a)$) such that

$$V(a, k)<\gamma, \tag{2}$$

for all k satisfying $0 \leqq k \leqq \delta$. Take a solution $y = y(x)$ of (1) in region R_1 satisfying $y(a) = k$ with $0 \leqq k \leqq \delta$. Suppose this solution cannot be continued to the left in R_1 at a point x_0 in $(0, a)$; then, by the continuation theorem (for example, see (2), p. 15) and the fact that $y = 0$ is a solution of (1) in R_1 defined on $(0, a]$, we have $y(x_0) = g(x_0) \neq 0$. By (C) we have

$$\frac{dV(x, y(x))}{dx}\geqq 0$$

for all $x \in (x_0, a)$. Therefore,

$$V(a, y(a)) \geqq V(x_0, y(x_0)).$$

But on the other hand by (2) and (A), we have

$$V(a, y(a)) = V(a, k)<\gamma,$$

and

$$V(x_0, y(x_0)) = V(x_0, g(x_0)) \geqq \gamma.$$

This is a contradiction. Thus Theorem 1 is proved.

As an immediate consequence of Theorem 1, we have the following theorem for ordinary differential equations (this can also be obtained directly from the result quoted in § **1**).

Theorem 2. *If for* (1) *there exists a function* $g(x) \in C([0, a])$, *satisfying* $g(0) = 0$ *and* $0<g(x) \leqq b$ *for all* $x \in (0, a]$, *and a function* $V(x, y)$, *defined and continuously differentiable on* R_1, *such that*

$$\inf_{0<x\leqq a} V(x, g(x))>0, \tag{A$'$}$$

$$V(x, 0)=0, \tag{B$'$}$$

$$\frac{\partial V}{\partial x}+\frac{\partial V}{\partial y}f(x, y)\geqq 0, \tag{C$'$}$$

then for any $\alpha>0$ $(\alpha \leqq a)$ *there are infinitely many solutions of* (1) *in* R_1 *defined on* $[0, \alpha]$ *passing through* (0, 0).

In particular, if we take $g(x) = x^\beta$, where β is a positive real number, and take $V(x, y) = y/x^\beta$, then it is obvious that conditions (A′) and (B′) are satisfied. In this case, condition (C′) is equivalent to the condition that $f(x, y) \geqq \beta y/x$. Hence we have the following corollary.

Corollary. *If for* (1), *there exist a* $\beta>0$ *and an* a_1 *satisfying* $0<a_1 \leqq a$ *and* $a_1^\beta \leqq b$ *such that in the region*

$$R_2: \quad 0<x \leqq a_1, \; 0 \leqq y \leqq x^\beta$$

we have $f(x, y) \geqq \beta y/x$ *then for any* $\alpha>0$ $(\alpha \leqq a_1)$ *there are infinitely many solutions of* (1) *in* R_2 *defined on* $[0, a]$ *passing through* (0, 0).

3. An example

Consider the following equation (3)

$$\frac{dy}{dx} = f(x, y) \tag{3}$$

where

$$f(x, y) = \begin{cases} (1+\varepsilon)y/x & \text{for } 0<y<x^{1+\varepsilon}, \\ (1+\varepsilon)x^\varepsilon & \text{for } y \geqq x^{1+\varepsilon}, \\ 0 & \text{for } y \leqq 0, \end{cases}$$

where $x \geqq 0$ and ε is a positive constant. In the region

$$R_3: \quad 0<x \leqq a, \; 0 \leqq y \leqq x^{1+\varepsilon},$$

where a is any positive real number, it is obvious that $f(x, y) \geqq (1+\varepsilon)y/x$. Hence by the Corollary we see for any $a>0$ there are infinitely many solutions of (3) in R_3 defined on $[0, a]$ passing through (0, 0).

REFERENCES

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(2) E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations* (McGraw-Hill, 1955).

(3) O. Perron, Eine Hinreichende Bedingung für die Unität der Lösung von Differentialgleichungen erster Ordnung, *Math. Z.* **28** (1928), 216-219.

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