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FACTORING IDEALS INTO SEMIPRIME IDEALS

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Let D be an integral domain with $1 \neq 0$. We consider "property SP" in D, which is that every ideal is a product of semiprime ideals. (A semiprime ideal is equal to its radical.) It is natural to consider property SP after studying Dedekind domains, which involve factoring ideals into prime ideals. We prove that a domain D with property SP is almost Dedekind, and we give an example of a nonnoetherian almost Dedekind domain with property SP.

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1. Introduction.

In general we use the notation and terminology of [**11**; **12**]. In particular $\subset$ denotes containment, while $<$ denotes proper containment. To say that A is a proper ideal of D means $(0) < A < D$.

A domain is called *Prüfer* if the quotient ring D_P is a valuation ring for each proper prime ideal P. See [**1**; **7**]. Also D is an *almost Dedekind* domain provided each D_P is a rank one discrete valuation ring (i.e., a valuation ring which is a Dedekind domain). See [**4**; **5**]. The domain D is said to have *dimension* n if there is a strictly increasing chain of n proper prime ideals but no such chain of $n + 1$ proper prime ideals. In this case, we write $\dim D = n$.

LEMMA 1.1. *If domain D has property SP, then so do the domains D_P and D/P for every proper prime ideal P of D.*

Proof. First consider D_P. If S is a semiprime ideal of D, then $SD_P = \sqrt{S}P_P = \sqrt{SD_P}$ [**6**, p. 34, Theorem 3.4(6)], so SD_P is a semiprime ideal of D_P. If B is an ideal of D_P, then $B = AD_P$ for some ideal A of D. Since $A = \prod_{i=1}^{n} S_i$ where $S_i = \sqrt{S_i}$ for each i, we have $B = \prod_{i=1}^{n} (S_iD_P)$, a product of semiprime ideals.

Now consider $\bar{D} = D/P$, and let S be a semiprime ideal of D containing P. Then $\sqrt{S/P} = \sqrt{S}/P = S/P$ by [**11**, p. 148, (16)]. If B is an ideal of D/P, then $B = A/P$ for some ideal A of D containing P. When $A = \prod_{i=1}^{n} S_i$ where each S_i is a semiprime ideal of D containing P, and $B = \prod_{i=1}^{n} (S_i/P)$ [**11**, p. 148, (13)], a product of semiprime ideals.

2. Domains with property SP.

LEMMA 2.1. *If D is a domain with property SP and if the ascending chain condition for prime ideals holds in D, then D is almost Dedekind.*

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Proof. We first show that primary ideals of D are prime powers without using the ascending chain condition for prime ideals. Suppose Q is a primary ideal with radical P and $Q < P$. Then $Q = \prod_{i=1}^n S_i$ with $\sqrt{S_i} = S_i$ for each i. Also by [**11**, p. 147, (8)], we have

$$P = \sqrt{Q} = \sqrt{\prod_{i=1}^{n} S_i} = \bigcap_{i=1}^{n} \sqrt{S_i} \subset S_i, \quad \text{for each } i.$$

However from $P > Q = \prod_{i=1}^n S_i$, we conclude that $P \supset S_j$ and hence $P = S_j$ for some j. We may suppose that $S_1, \ldots, S_n$ are arranged so that $P = S_i$ for $1 \leqq i \leqq k$ and $P < S_i$ for $i > k$. Then $Q = P^k \prod_{i>k} S_i$ and $Q \subset P^k$. If $P^k \not\subset Q$, then $\prod_{i>k} S_i \subset P$, since Q is P-primary. Hence $S_j \subset P$ for some $j > k$, implying $P = S_j$, a contradiction. Thus $P^k \subset Q$, so $P^k = Q$.

Since primary ideals are prime powers and since the ascending chain condition for prime ideals holds in D, it follows from [**3**, Corollary 4] that D is a Prüfer domain. Let P be a proper prime ideal of D. We will show that D_P is a Dedekind domain. By Lemma 1.1 every ideal of D_P is a product of semiprime ideals. However D_P is a valuation ring, so semiprime ideals of D_P are prime [**9**, p. 135, 5.10(1)]. Thus D_P is a Dedekind domain, so D is almost Dedekind.

LEMMA 2.2. *Suppose D has property SP and a unique invertible maximal ideal M.* Then dim $D = 1$.

Proof. Suppose the conclusion is false. Then there would exist a nonzero prime ideal $P < M$. Let $x \in P\backslash\{0\}$. We have $(x) = \prod_{i=1}^n S_i$ where $\sqrt{S_i} = S_i$ for each i. Then $P \supset \prod_{i=1}^n S_i$, so P contains S_i for some i. Say $P \supset S_1$. If $A = M^{-1}S_1$, an ideal of D, then $S_1 = AM$. If $A = D$, then $S_1 = M > P$, a contradiction. Therefore $A \subset M$, so $S_1 = AM \supset A^2$. Since S_1 is semiprime, we have $S_1 \supset A$. Hence $S_1M \supset AM = S_1$, so $S_1 = S_1M$, and

$$(x)M = MS_1 \ldots S_n = S_1 \ldots S_n = (x).$$

Choose $m \in M$ such that $xm = x$. Then $x(1 - m) = 0$, but $1 - m$ is a unit of D, so $x = 0$, a contradiction. Therefore dim $D = 1$.

LEMMA 2.3. *If D has property SP, then a minimal prime of a nonzero principal ideal is minimal in D.*

Proof. Let $d \in D\backslash\{0\}$ and let P be a minimal prime of (d). Since $\sqrt{dD_P}$ is the intersection of all primes of D_P which contain dD_P [**9**, p. 43, 2.14], it follows that $\sqrt{dD_P} = PD_P$. Then dD_P is a primary ideal of D_P by [**11**, p. 153, Corollary 1]. Since D_P has property SP by Lemma 1.1, we conclude that dD_P is a power of PD_P, since we showed in the proof of Lemma 2.1 that primary ideals are prime powers in a domain with property SP. Then PD_P is invertible by [**11**, p. 272, Lemma 4], so dim $D_P = 1$ by Lemma 2.2. Thus P is minimal in D.

THEOREM 2.4. *A domain D with property SP is almost Dedekind.*

Proof. By Lemma 2.1 we need only show that D has dimension one. Suppose

$\dim D > 1$. Then D has a maximal ideal M which is not minimal. Let P_1 be a nonzero prime ideal properly contained in M and choose $x \in P_1 \backslash \{0\}$. We know that P_1 contains a minimal prime P of the ideal (x), [**6**, pp. 43, 44]. Pick $m \in M \backslash P_1$ and let Q be a minimal prime in M of the ideal $P + (m)$. For an ideal A of D, let A^e denote the extension of A to the quotient ring D_Q. Since P is a minimal prime of D by Lemma 2.3, it follows that P^e is a minimal prime of D_Q. Also Q^e is a minimal prime of the ideal $P^e + mD_Q$.

It follows from Lemma 1.1 that D_Q and $\bar{D}_Q = D_Q/P^e$ have property SP. Thus by Lemma 2.3, $\bar{Q} = Q^e/P^e$ is a minimal prime of $\bar{D}$, so $\dim \bar{D} = 1$. Then Lemma 2.1 tells us that $\bar{D}$ is an almost Dedekind domain. Since $\bar{D}$ has a unique maximal ideal $\bar{Q}$, we conclude that $\bar{D}$ is a rank one discrete valuation ring. Thus in $\bar{D}$, $(\bar{O}) = \bigcap_{n=1}^{\infty} \bar{Q}^n$. From [**3**, Theorem 1] we know that $P^e \subset \bigcap_{n=1}^{\infty} (Q^e)^n$, and hence $P^e = \bigcap_{n=1}^{\infty} (Q^e)^n$. Then by [**3**, Theorem 3] it follows that each prime ideal of D_Q is contained in P^e. However P^e is a minimal prime of D_Q, so $\dim D_Q = 2$. But then by Lemma 2.1, D_Q is almost Dedekind and $\dim D_Q = 1$, a contradiction. Therefore $\dim D = 1$.

3. An example. In Section 2 we showed that a domain must be almost Dedekind if every proper ideal is a product of semiprime ideals. Clearly every Dedekind domain has property SP. In this section we study an example of a domain which has property SP but is not Dedekind. We let N, Z denote the sets of natural numbers and integers, respectively.

First let D be any almost Dedekind domain with quotient field K, and let $\mathscr{M}(D)$ denote the set of maximal ideals of D. If $A \neq (0)$ is a fractional ideal of D, and if $P \in \mathscr{M}(D)$, then for some $q \in Z$ we have $AD_P = (PD_P)^q$. We indicate this fact by the notation $v(P, A) = q$. In case $x \in K \backslash \{0\}$, we write $v(P, x)$ instead of $v(P, (x))$. Then $v(P, \cdot\,)$ is the P-adic valuation on K.

LEMMA 3.1. *Let A, B be nonzero fractional ideals of D.*
(i) $v(P, A) = \min \{v(P, a) | a \in A \backslash \{0\}\}$.
(ii) $v(P, AB) = v(P, A) + v(P, B)$ *and* $v(P, A^q) = q \cdot v(P, A)$, $q \in Z$.
(iii) $A = \bigcap \{P^{v(P,A)} | P \in \mathscr{M}(D)\}$.

Proof. These assertions are evident. Part (iii) follows since

$$A = \bigcap_{P \in \mathscr{M}(D)} AD_P$$

[**6**, p. 42, 3.10(3)].

LEMMA 3.2. *Let D be an almost Dedekind domain. Then an ideal $S \neq (0)$ is semiprime if and only if $v(P, S) = 0$ or 1 for all $P \in \mathscr{M}(D)$.*

Proof. If $S \neq (0)$ is semiprime, then so is SD_P for every $P \in \mathscr{M}(D)$, as we showed in the proof of Lemma 1.1. Since D_P is a rank one discrete valuation ring, we either have $SD_P = D_P$ and $v(P, S) = 0$, or else $SD_P = PD_P$ and $v(P, S) = 1$.

If $v(P, S) = 0$ or 1 for every $P \in \mathcal{M}(D)$, then $S = \bigcap \{P^{v(P,S)} | P \in \mathcal{M}(D)\}$, an intersection of prime ideals. Thus $\sqrt{S} = S$, so S is semiprime.

The example to be studied is given by Heinzer and Ohm in [**8**, pp. 276–278]. We will review the construction. Let G denote the set of all integer-valued infinite sequences which are eventually constant; i.e.,

$$G = \{f = (f(1), f(2), \ldots) | f : N \to Z \text{ and } \exists J \in N \ni j > J \Rightarrow f(j) = f(j+1)\}.$$

For $f \in G$, let $f(\infty)$ denote the eventual constant value of f and let $N_\infty = N \cup \{\infty\}$. The following definitions make G into a lattice ordered group. The operation is component-wise addition, and order is defined thus: $f \leqq g$ if $f(i) \leqq g(i)$ for all $i \in N$. A proper subset Σ of $G^+ = \{g \in G | g \geqq 0\}$ is called a *segment* of G if $a \in \Sigma$ and $b \geqq a$ implies $b \in \Sigma$ and if $a, b \in \Sigma$ implies $\inf \{a, b\} \in \Sigma$. A segment Σ is called *prime* if the complement $G^+ \backslash \Sigma$ is closed under addition. For each $i \in N_\infty$, the set $\Sigma_i = \{f | f \in G, f(i) > 0\}$ is a prime segment of G, and there are no others.

Using the method of Jaffard [**10**], Heinzer and Ohm construct a domain D with quotient field K which has G as its group of divisibility. Thus there exists a function ϕ from $K \backslash \{0\}$ onto G such that $\phi(ab) = \phi(a) + \phi(b)$ for all a, $b \in K \backslash \{0\}$, and $\phi(D \backslash \{0\}) = G^+$. The proper prime ideals of D are the sets $P_i = \phi^{-1}(\Sigma_i)$, $i \in N_\infty$. For each $i \in N_\infty$, if H_i is the subgroup of G generated by $G^+ \backslash \Sigma_i$, then there is an isomorphism $\psi_i : G/H_i \to Z$. If $\eta_i : G \to G/H_i$ is the canonical homomorphism then the function $\psi_i \eta_i \phi : K \backslash \{0\} \to Z$ is the P_i-adic valuation $v(P_i, \cdot)$. Thus D is almost Dedekind. Furthermore for each $a \in K \backslash \{0\}$, the sequence $v(P_1, a), v(P_2, a), \ldots$ is eventually constant, and the eventual constant value is $v(P_\infty, a)$.

LEMMA 3.3. *Let $u_1, u_2, \ldots$ be a bounded sequence of nonnegative integers and set $B = \bigcap_{i \in N} P_i^{u_i}$ (in the above domain). Then for each $j \in N$, $v(P_j, B) = u_j$ and $v(P_\infty, B) = \limsup_{i \in N} u_i$.*

Proof. Choose $j \in N$. If $u = \limsup_{i \in N} u_i$, then there exists $I \in N$ such that $i > I$ implies $u_i \leqq u$. We know that there exists $x \in D$ such that

$$\phi(x) = (u_1, u_2, \ldots, u_j, \ldots, u, u, u, \ldots);$$

i.e., $\phi(x)(i) = u_i$ if $i \leqq \max \{j, I\}$ and $\phi(x)(i) = u$ if $i > \max \{j, I\}$. Then $x \in P_i^{u_i}$ for each $i \in N$, so $x \in B$. Since $v(P_j, x) = u_j$, we have $v(P_j, B) \leqq u_j$ by Lemma 3.1, i. However $B = \bigcap_{i \in N} P_i^{u_i} \subset P_j^{u_j}$, so $v(P_j, B) \geqq u_j$. Combining the inequalities yields $v(P_j, B) = u_j$.

Now we consider $v(P_\infty, B)$. For any j, if x is chosen as above, we have $v(P_\infty, x) = u$, so $v(P_\infty, B) \leqq u$, by Lemma 3.1, i. For the reverse inequality, take $b \in B \backslash \{0\}$. Since $u = \limsup_{i \in N} u_i = \limsup_{i \in N} v(P_i, B)$, we know that there must be an infinite sequence $i_1, i_2, \ldots$ from N such that $v(P_{i_k}, B) = u$ for every $k \in N$. Thus $v(P_{i_k}, b) \geqq u$ for each k, and so the eventual constant value of $v(P_i, b)$, $i \in N$, cannot be less than u. Thus $v(P_\infty, b) \geqq u$, and $v(P_\infty, B) \geqq u$ by Lemma 3.1, i. Therefore $v(P_\infty, B) = u$.

THEOREM 3.4. *If D is the almost Dedekind domain of Heinzer and Ohm, then every proper ideal of D is a product of semiprime ideals.*

Proof. Let A be a proper ideal of D. By Lemma 3.1, i, we see that the set $\{v(P_i, A)|i \in N_\infty\}$ is bounded above by $\max\{v(P_i, a)|i \in N_\infty\}$ for any $a \in A\backslash\{0\}$, and so the set $\{v(P_i, A)|i \in N_\infty\}$ is finite. Let $w_1 < w_2 < \ldots < w_n$ be its elements, and let $w_0 = 0$. For $j \in \{1, 2, \ldots, n\}$ let

$$X_j = \{i|v(P_i, A) \geqq w_j\}.$$

Observe that $X_1 \supset X_2 \supset \ldots \supset X_n$. Now consider the product of semiprime ideals

$$A_1 = \prod_{j=1}^{n} \left(\bigcap_{i \in X_j} P_i \right)^{w_j - w_{j-1}}.$$

By Lemma 3.1, iii, we can prove $A = A_1$ by showing that $v(P_k, A) = v(P_k, A_1)$ for each $k \in N_\infty$.

Suppose $k \in N$. We have $v(P_k, A) = w_{j(0)}$ for some $j(0)$. Suppose $j > j(0)$. Then $w_j > w_{j(0)}$ and $k \notin X_j$, so $v(P_k, \bigcap_{i \in X_j} P_i) = 0$ by Lemma 3.3. On the other hand, $j \geqq j(0)$ implies $k \in X_j$ and $v(P_k, \bigcap_{i \in X_j} P_i) = 1$. Thus by Lemma 3.1, ii,

$$v(P_k, A_1) = \sum_{j=1}^{j(0)} (w_j - w_{j-1}) = w_{j(0)} = v(P_k, A).$$

Consider $v(P_\infty, A_1)$. We have $\limsup_{i \in N} v(P_i, A) = w_{j(1)}$ for some $j(1)$, and so there exists $I \in N$ such that $i > I$ implies $v(P_i, A) \leqq w_{j(1)}$. So, if $j > j(1)$ and $i \in X_j$, then $v(P_i, A) \geqq w_j > w_{j(1)}$ and $i \leqq I$. Therefore if $j > j(1)$, then X_j is a finite set and in this case $v(P_\infty, \bigcap_{i \in X_j} P_i) = 0$ by Lemma 3.3. On the other hand $X_{j(1)}$ is an infinite set since $w_{j(1)}$ must occur infinitely often in the sequence $v(P_1, A), v(P_2, A), \ldots$. Since

$$X_1 \supset X_2 \supset \ldots \supset X_{j(1)},$$

all these sets are infinite, and so $v(P_\infty, \bigcap_{i \in X_j} P_i) = 1$ if $j \leqq j(1)$ by Lemma 3.3. Therefore

$$v(P_\infty, A_1) = \sum_{j=1}^{j(1)} (w_j - w_{j-1}) = w_{j(1)} = \limsup_{i \in N} v(P_i, A) = v(P_\infty, A),$$

by Lemma 3.3. Thus $A = A_1$, and so A is a product of semiprime ideals.

One might wonder if there exist almost Dedekind domains without property SP. In [**13**] there is a necessary and sufficient condition for a domain which is the union of a tower of Dedekind domains to have this property. With this condition the author of [**13**] exhibits an almost Dedekind domain which does not have property SP.

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