THE EQUILIBRIUM OF DEGENERATE CONFIGURATIONS

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1. Great progress has been achieved in recent years in investigations of the dense degenerate matter and in calculations of white dwarf structure. Nevertheless the classical theory by Chandrasekhar (1967) describes the structure of white dwarf stars rather well, with accuracy of few precents. This theory leads to the following second-order differential equation:

$$(x^n y')' = -x^n (y^2 - p)^{3/2} ; \; y(0) = 1, \; y'(0) = 0 \qquad (1)$$

where p is a constant parameter, $0 \leq y < 1$. In Eq. (1) the values of the parameter n = 0,1 and 2 correspond to the plane-symmetric, cylindrical and spherical self-gravitating configurations of degenerate matter.

More generally, Eq. (1) can be treated as the equation of hydrostatic equilibrium for a hypersphere of degenerate matter in the $(n + 1)$ - dimensional space. No analytical solution of Eq. (1) is known for any value of the constants n and p.

The aim of our communication is to describe, by the example of Eq. (1), the effective solution method, which is useful both for the derivation of simple approximate analytical solutions, and for numerical calculations which are faster than the usual Runge-Kutta type methods. The method used is based on the theory of continued fractions and incorporates, in particular, the method by XIX century Russian scientist Viscovaty of the construction of the corresponding fractions for a function, expansion of which in power series is known (see Khovansky (1956)). Besides Eq. (1) we will briefly consider also the problem of critical rotation of cylindrical degenerate configurations.

2. We start with expansion of $y(x)$ in a power series of the form:

$$y(t) = a_i t^i, \quad t = x^2, \; a_0 = 1 \qquad (2)$$

about the origin. For Eq. (1) the following recursion relation is valid (Seidov, Sharma and Kuzakhmedov 1979a - henceforth SSK79a):

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$$a_{i+1} = \frac{1}{2q^2 i(i+1)(2i+1+n)} \sum_{k=1}^{i} ((5k-2i)(i-k+1)$$

$$x(2i-2k+n)\, a_{i-k+1} \sum_{l=0}^{k} (a_l\, a_{k-l})), \; i \geq 1 , \tag{3}$$

$$a_0 = 1, \quad a_1 = \frac{q^3}{2(n+1)} , \quad q = (1-p)^{1/2}.$$

Viskovaty's method leads to the following relation between the power series and the corresponding continued fraction:

$$\sum_{i \geq 0} b_{1i}\, t^i = \frac{b_{10}}{b_{00}} + \frac{b_{20}t}{b_{10}} + \frac{b_{30}t}{b_{20}} + \ldots , \; b_{0i} = \delta_{0i},\; b_{1i} = a_i ;$$

$$b_{k\ell} = b_{k-1,0}\, b_{k-2,\ell+1} - b_{k-2,0}\, b_{k-1,\ell+1}, \; , k \geq 2, \quad \ell \geq 0. \tag{4}$$

From the continued fraction we construct the appropriate fractions of number k:

$$P_k/Q_k = \frac{b_{10}}{b_{00}} + \frac{b_{20}t}{b_{10}} + \ldots + \frac{b_{k+1,0}t}{b_{k,0}} , \quad k \geq 1. \tag{5}$$

For calculation of P_k and Q_k the following recursion relations are useful (where A_k denotes P_k or Q_k):

$$A_k = B_{k,0}A_{k-1} + b_{k+1,0}\, A_{k-2}\, t , \quad k \geq 2 ,$$

$$P_0 = b_{10} = a_0 = 1 , \; Q_0 = 1, \; P_1 = b_{10}^2 = 1, \; Q_1 = 1 + b_{20}\, t. \tag{6}$$

There are the appropriate fractions P_k/Q_k which are required approximations of different order to the function $y(x)$. As it was mentioned before, usage of these fractions is twofold. First, one can use them for derivation of various simple approximate analytical solutions of differential equations (see section 3). Second, a program can be compiled for calculation of the fractions P_k/Q_k and applied for numerical solution of differential equations on electronic computers (section 4).

3. It can be shown that the fraction P_{2k}/Q_{2k} (respectively P_{2k+1}/Q_{2k+1}) coincides with Padé approximations of order (k,k) (respectively of order (k,k+1)). It is also known, Baker (1975), that diagonal Padé approixmations are often more exact than the nearest non-diagonal

approximations. Therefore some authors have derived analytical expressions for Padé approximations of order (2,2) (that is for the fraction P_4/Q_4) for Eq. (1) in the particular case $n = 2$ (Seidov (1978)), and in the general case of arbitrary n(SSK79a), and also for Lane-Emden equation in the particular case $n = 2$ (Pascual (1977)) and in the general case of arbitrary n (Seidov, Sharma and Kuzakhmedov (1979b)).

Degree of accuracy of this approximation in the case of Eq. (1) is seen from Table I. Here the boundary values of $x = x_1$ $(y(x_1) = p^{\frac{1}{2}})$, calculated from Padé approximations (2,2), denoted as Pa in Table I, SSK79a, are compared with the values of x_1 calculated by Runge-Kutta method by various authors. References are abbreviated as follows: KT78 = Karnik and Talwar (1978), Ch67 = Chandrasekhar (1967). In the case $n = 0$ there is no need to use the Runge-Kutta method because x_1 is expressed through a definite integral, which is calculated with Gauss' quadrature formula in SSK79a (denoted as Ga). One can conclude that Padé approximations have an accuracy quite sufficient for astrophysical applications in whole regions of values of n, p and x, in the case of a such low order as (2,2).

Table I

p	$n = 0$	$n = 1$	$n = 2$
.1	1.5613 Pa 1.5612 Pa	2.6897 RK KT78 2.6824 Pa	4.0690 RK Ch67 4.0797 Pa
.4	1.4789 Ga 1.4951 Pa	2.4747 RK KT78 2.4864 Pa	3.5245 RK Ch67 3.5447 Pa
.8	1.7982 Ga 1.8013 Pa	2.9096 RK KT78 2.9224 Pa	4.0446 RK Ch67 4.0773 Pa

4. To get higher accuracy, the appropriate fractions of larger numbers can be calculated on electronic computers. In many cases this method is more effective than usual Runge-Kutta method due to simplicity of procedure and easiness of accuracy control. In Table II the values of function $y(x)$ from Eq. (1) are shown for the case $n = 2$ and $p = .2$, calculated from the appropriate fractions of decreasing numbers.

The convergence of subsequent approximations is rather rapid for values of x up to boundary value $x_1 = 3.7271$. At least for a given value of $x \lesssim 3.7$ desired accuracy is achieved by this method with less number of arithmetical operations, and consequently in less computer time, than by the Runge-Kutta method.

Table II

k	$x = 1$	$x = 2$	$x = 3$	$x = 4$
2	.89485035	.68956765	.513783	.39360
4	.89492873	.69217842	.527092	.42667
6	.89492872	.69217000	.526843	.42410
8	.89492872	.69217011	.526859	.42460
10	-"-	.69217011	.526858	.42509
12	-"-	-"-	.526858	.42450

5. Now we briefly consider a problem of rotating degenerate cylinders, which is reduced to the following equation, Karnik and Talwar (1978)

$$(xy')' = -x(y^2 - p)^{3/2} + eq^3 x, \quad q = (1 - p)^{1/2}, \quad e = \frac{\Omega^2}{2\pi G\rho_a} . \qquad (7)$$

here Ω is an angular velocity, ρ_a is material density at the axis of cylinder. Eq. (7) has no analytical solution. Some authors, Karnik and Talwar (1978), Gupta and Talwar (1973), have solved Eq. (7) by the Runge-Kutta method for a few values of parameters p and e.

By the method described in Section 2, we found analytical expression for Padé approximation of order (2,2) to solution of Eq. (7) for arbitrary values of p and e. Using condition $y(x_1) = p^{1/2}$ we found the boundary values of $x = x_1$ (radius of cylinder) for given p and e. At critical rotation, when centrifugal force just balances gravitational force at the outer boundary of the cylinder, we have condition $y'(x_1) = 0$. From conditions we found the critical value of rotational parameter e as function of p (see Table III).

Table III

p	x_1	e_{Max}	p	x_1	e_{Max}
.1	3.905	.09662	.5	3.882	.1544
.2	3.851	.1108	.6	3.993	.1613
.3	3.816	.1354	.7	4.183	.1671
.4	3.825	.1460	.8	4.535	.1722

Also presented are radii of critically rotating cylinders. The maximal value of rotational parameter $e = \Omega^2/2\pi G\rho_a$ gradually increases with

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increasing p, that is with transition from relativistic cylinders to non-reativistic ones. The dimensionless radius of critically rotating cylinders has minimal value at $p \simeq .35$ while the corresponding minimum for degenerate non-rotating spheres occurs at $p \approx .45$.

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