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ON ω–LIMIT SETS OF AUTONOMOUS SYSTEMS IN INFINITE DIMENSIONAL BANACH SPACES

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We prove that each Polish space is homeomorphic to the ω–limit set of a bounded solution of an autonomous equation $x' = f(x)$ in $l^2 \times c_0$, in which f is Lipschitz continuous on bounded sets.

1. INTRODUCTION

Let $(E, \|\cdot\|)$ be a real Banach space, let $f : E \to E$ be a continuous function and let $x : [t_0, \infty) \to E$ be a solution of $x' = f(x)$. A point $\xi \in E$ is called ω–limit point of x if there is a sequence $(t_k)_{k=1}^{\infty}$ in $[t_0, \infty)$ with $t_k \to \infty$ as $k \to \infty$ such that $x(t_k) \to \xi$ as $k \to \infty$. The ω–limit set of x is defined by

$$\omega(x) := \{\xi \in E : \ \xi \text{ is } \omega\text{–limit point of } x\}.$$

The set $\omega(x)$ is closed. If $\dim E < \infty$ and if x is bounded then $\omega(x)$ is compact and connected (see for example [**3**, p.145]). If x is unbounded, then $\omega(x)$ can be disconnected (see for example [**8**, p.343]). The topological structure of ω–limit sets depends on the space dimension. For $\dim E = 1$ any solution of $x' = f(x)$ is monotone. Hence we have $\omega(x) = \emptyset$ or $\omega(x) = \left\{\lim\limits_{t\to\infty} x(t)\right\}$. For $\dim E = 2$ the Poincaré–Bendixson Theory (see for example [**3**, p.144]) gives some information on what $\omega(x)$ can look like. For $\dim E \geqslant 3$ less is known. Numerical plots of bounded solutions, for example of the Lorentz–equation (see for example [**9**, p.205]), indicate that ω–limit sets can look quite weird. Moreover ω–limit sets with topologically complicated structure are known (see for example [**5**] and the references given there). In infinite dimensions the ω–limit set of a bounded solution can be empty or can be disconnected and/or not compact [**6**]. In [**2**] it is proved that given a closed separable set $P \subset E$ then there exists an autonomous system with continuous right hand side in $\mathbb{R} \times E$ and a solution x of this system such that $\omega(x) = \{1\} \times P$ (in [**2**] limit sets are defined for $t \to 1-$ instead of $t \to \infty$ which is essentially equivalent; the right hand sides of the differential equations constructed in [**2**] are in general not Lipschitz continuous on bounded subsets of $\mathbb{R} \times E$).

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In this paper we shall prove that there is almost no topological restriction for ω–limit sets in a certain infinite dimensional Banach space and for right hand sides which are Lipschitz continuous on bounded sets. If E is separable, since $\omega(x)$ is closed we have that $\omega(x)$ is a Polish space. On the other hand we shall prove:

Theorem 1. *Let $E = l^2(\mathbb{N}) \times c_0(\mathbb{N})$ and let P be a Polish space. Then there exists a function $f : E \to E$ with the following properties:*

1. *f is Lipschitz continuous on each bounded subset of E.*
2. *There is a bounded solution $x : [t_0, \infty) \to E$ of $x' = f(x)$ such that $\omega(x)$ is homeomorphic to P.*

2. Proof of Theorem 1

In the sequel $\|\cdot\|_2$ denotes the Euclidean norm on $l^2(\mathbb{N})$ and $\|\cdot\|_\infty$ denotes the supremum norm on $c_0(\mathbb{N})$. Moreover let $(\cdot,\cdot)$ denote the standard inner product in $l^2(\mathbb{N})$.

Step 1. (Reduction of the problem). We shall only consider the case $P \neq \emptyset$. It is known [7] that each Polish space is homeomorphic to a closed subset of the unit sphere $\left\{\xi \in c_0(\mathbb{N}) : \|\xi\|_\infty = 1\right\}$. Since the unit spheres of two separable infinite dimensional Banach spaces are homeomorphic (see [1, p.188]) we can assume without loss of generality that P is a closed subset of $\left\{\xi \in l^2(\mathbb{N}) : \|\xi\|_2 = 1\right\}$.

The differential equation will be constructed in the space

$$F = \mathbb{R}^6 \times \left(l^2(\mathbb{N})\right)^2 \times \left(c_0(\mathbb{N})\right)^2$$

which is isomorphic to E.

Step 2. (The unit sphere of $l^2(\mathbb{N})$ is a limit set). Let $S : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ denote the shift operator $S\big((x_n)\big) = (x_{n+1})$.

The operator $\exp(S)$ is hypercyclic (that is, it has dense orbits, see for example [4]) hence there exists $\xi_0 \in l^2(\mathbb{N})$ such that $\left\{\exp(tS)\xi_0 : t \geqslant 0\right\}$ is dense in $l^2(\mathbb{N})$. Now $h(t) := \big(\exp(tS)\xi_0\big)/\big\|\exp(tS)\xi_0\big\|_2$, $t \geqslant 0$ solves

$$h' = Sh - (h, Sh)h,$$

and $\omega(h) = \left\{\xi \in l^2(\mathbb{N}) : \|\xi\|_2 = 1\right\}$. Moreover there exists $L > 0$ such that $\big\|h(t) - h(s)\big\|_2 \leqslant L|t - s|$, $t, s \geqslant 0$.

Step 3. (The system and the solution). In the sequel let $A : c_0(\mathbb{N}) \to c_0(\mathbb{N})$ denote the linear operator $A\big((x_n)\big) = (x_n/n)$ and let $\eta \in c_0(\mathbb{N})$ denote the sequence $\eta = (1/n)$.

We consider the following differential equation (D) in F.

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$$\begin{aligned}
u_1' &= -u_1^2 \\
u_2' &= -u_1 u_2^2 \\
u_3' &= -u_1 u_3^2 \\
u_4' &= u_1\big(\operatorname{dist}(P,w) - \operatorname{dist}(P,v)\big) \\
u_5' &= -u_1 u_6\big(\operatorname{dist}(P,w) - \operatorname{dist}(P,v)\big) \\
u_6' &= u_1 u_5\big(\operatorname{dist}(P,w) - \operatorname{dist}(P,v)\big) \\
v' &= u_1 u_2\big(Sv - (v,Sv)v\big) \\
w' &= u_1 u_3\big(Sw - (w,Sw)w\big) \\
x' &= \big(\operatorname{dist}(P,w) - \operatorname{dist}(P,v)\big)y + u_6\eta - Ax \\
y' &= -u_1^2\big(\operatorname{dist}(P,w) - \operatorname{dist}(P,v)\big)x + u_1 u_5\eta - u_1 y - Ay\,.
\end{aligned}$$

Obviously the right hand side of (D) is Lipschitz continuous on bounded subsets of F, and for $t_0 \geqslant e$ the following function $z = (u,v,w,x,y) : [t_0,\infty) \to F$ is a bounded solution of (D), as can be verified by differentiation.

$$\begin{aligned}
&u_1(t) = 1/t, \qquad u_2(t) = 1/\log(t), \qquad u_3(t) = 1/\big(1+\log(t)\big), \\
&u_4(t) = \int_{\log(t)}^{1+\log(t)} \operatorname{dist}\Big(P, h\big(\log(s)\big)\Big)\, ds, \qquad u_5(t) = \cos\big(u_4(t)\big), \\
&u_6(t) = \sin\big(u_4(t)\big), \qquad v(t) = h\Big(\log\big(\log(t)\big)\Big), \qquad w(t) = h\Big(\log\big(1+\log(t)\big)\Big), \\
&x(t) = \big(x_n(t)\big) = \sin\big(u_4(t)\big)\Big(\big(1-\exp(-t/n)\big)\Big)_{n=1}^{\infty}, \\
&y(t) = \big(y_n(t)\big) = \frac{\cos\big(u_4(t)\big)}{t}\Big(\big(1-\exp(-t/n)\big)\Big)_{n=1}^{\infty}.
\end{aligned}$$

STEP 4. (THE ω-LIMIT SET). Let $c = (c_1,c_2,c_3,c_4,c_5,c_6,c_v,c_w,c_x,c_y)$ be a limit point of z, and $(t_k)_{k=1}^{\infty}$ a corresponding sequence such that $z(t_k) \to c$ as $k \to \infty$. Note that $\|c_w\|_2 = \|c_v\|_2 = 1$. Obviously $c_1 = c_2 = c_3 = 0$ and $c_y = 0$. Moreover, by the mean value theorem

$$\big\|w(t) - v(t)\big\|_2 \leqslant L\big|\log\big(1+\log(t)\big) - \log\big(\log(t)\big)\big| \leqslant \frac{L}{\log(t)}.$$

Hence $c_w = c_v$. Since $\operatorname{dist}(P,\cdot)$ is Lipschitz continuous with constant 1 we have

$$\begin{aligned}
&\big|u_4(t_k) - \operatorname{dist}(P,c_v)\big| \\
&\quad = \Big|\int_{\log(t_k)}^{1+\log(t_k)} \Big(\operatorname{dist}\big(P,h\big(\log(s)\big)\big) - \operatorname{dist}(P,c_v)\Big)\, ds\Big| \\
&\quad \leqslant \int_{\log(t_k)}^{1+\log(t_k)} \Big\|h\big(\log(s)\big) - h\Big(\log\big(\log(t_k)\big)\Big) + h\Big(\log\big(\log(t_k)\big)\Big) - c_v\Big\|_2\, ds
\end{aligned}$$

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$$\leqslant \int_{\log(t_k)}^{1+\log(t_k)} L\Big(\log(s) - \log\big(\log(t_k)\big)\Big)\, ds + \big\|v(t_k) - c_v\big\|_2$$
$$\leqslant L\Big(\log\big(1+\log(t_k)\big) - \log\big(\log(t_k)\big)\Big) + \big\|v(t_k) - c_v\big\|_2 \to 0$$

as $k \to \infty$. Therefore $c_4 = \operatorname{dist}(P, c_v)$. Next, $\operatorname{dist}(P,\xi) \in [0,2]$ for $\|\xi\|_2 = 1$, hence $u_4(t) \in [0,2]$. Therefore $\big(x(t_k)\big)$ cannot converge in c_0 in the case that $c_4 \neq 0$. For this reason we have $c_4 = 0$ (in particular $c_v \in P$) and from this we obtain $c_5 = 1$, $c_6 = 0$ and $c_x = 0$. All together $c = (0,0,0,0,1,0,c_v,c_v,0,0)$ which means that

$$\omega(z) \subset Q := \Big\{(0,0,0,0,1,0,p,p,0,0) : p \in P\Big\}.$$

Now, let $p \in P$. According to Step 2. there is a sequence $(t_k)_{k=1}^{\infty}$ such that $h\Big(\log\big(\log(t_k)\big)\Big) \to p$ as $k \to \infty$. Then $v(t_k), w(t_k) \to p$ and $u_4(t_k) \to 0$ as $k \to \infty$. Hence $z(t_k) \to (0,0,0,0,1,0,p,p,0,0)$ as $k \to \infty$.

Thus, we have $\omega(z) = Q$ and Q is obviously homeomorphic to P. ∎

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