J. Austral. Math. Soc. **25** (Series A) (1978), 19–24

FINITE GROUPS AS AUTOMORPHISM GROUPS OF ORTHOCOMPLEMENTED PROJECTIVE PLANES

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(Received 18 December 1975; revised 8 March 1977)
Communicated by J. N. Crossley

Abstract

A construction is given for a non-desarguesian projective plane P and an absolute-point free polarity on P such that the group of collineations of P which commute with the polarity is isomorphic to an arbitrary preassigned finite group.

Sabidussi (1957), in extending the fundamental work of Frucht (1949), proved that every finite group G (of order at least 2) may be represented as the automorphism group of a regular graph of degree 5. Schrag (1971, 1976) showed how to parlay this result into a representation of G as the automorphism group of some orthomodular lattice L. In this paper we show that L may be chosen to be an irreducible orthocomplemented modular lattice of height 3, herein called an orthocomplemented projective plane. In fact we shall prove the following result.

THEOREM. *Let G be a finite group. There exists a non-desarguesian projective plane P and an absolute-point free polarity $'$ on P and an injection $\varphi: G \to \mathrm{coll}(P)$ of G into the collineation group of P such that* $\mathrm{image}(\varphi) = \{\alpha \in \mathrm{coll}(P) \mid \alpha' = {}'\alpha\}$.

The idea of the proof is to look at Schrag's orthomodular lattice L with $\mathrm{Aut}_\perp(L) \cong G$ and observe that $\mathrm{Aut}_\perp(L) = \mathrm{Aut}(L)$, that is, every lattice automorphism of L preserves the orthocomplementation. We then construct a wide cubic structure space (see Greechie (1974)) from the elements of L of height 1 or 2. We extend this to a larger such structure which is also a confined configuration (see Greechie (1974)) by adjoining a wide cubic structure space with no automorphisms. By embedding the final result in a tight cubic structure space (see Greechie (1974)) and passing to the lattice we obtain the desired result. Until the last paragraph we shall assume that $|G| > 1$.

This paper was presented at the Lattice Theory Conference, Ulm, Germany, July 1975, under the title, "Every Finite Group is a Subgroup of the Collineation Group of some Projective Plane".

LEMMA 1. *Let G be a finite group of order at least* 2 *and let* L_G *be the lattice constructed in* Schrag (1971, 1976) *such that* $\mathrm{Aut}_\perp(L_G)\cong G$. *Then*

$$\mathrm{Aut}_\perp(L_G)=\mathrm{Aut}(L_G).$$

PROOF. We begin by recalling the construction of L_G. Schrag begins with the finite regular graph H_0 of degree 5, given by Sabidussi (1957), such that $\mathrm{Aut}(H_0)\cong G$. He then takes the subdivision graph H_1 of H_0 obtained by splitting each edge in H_0 in two and for each pair of new edges adding one new point which is the common vertex of the two new edges which replace the old one (for example, a triangle becomes a hexagon). (The vertices of H_1 have degree 5 or degree 2.) Schrag then considers the graph H_2 given in Fig. 1. He manufactures a family of disjoint copies of H_2—one for each element of H_1 of degree 2. Let H_y be the copy H_2 corresponding to y in H_1 of degree 2. Now he identifies the unique element (called x in Fig. 1) of each H_y of degree 2 with the corresponding y in H_1 forming a

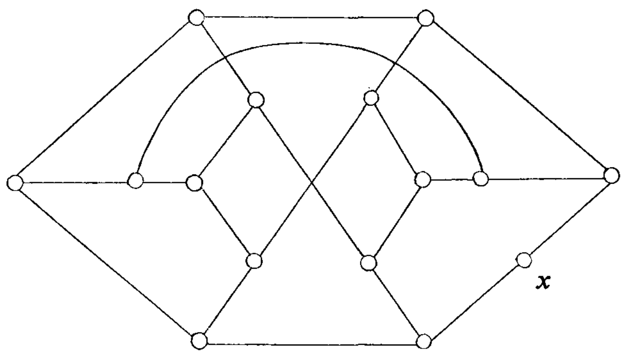

FIG. 1. Schrag's graph H_2 having a one-element automorphism group.

larger graph having elements of degrees 3, 4 and 5. Call this new graph H_3. Then $\mathrm{Aut}(H_3)\cong G$. The dual incidence structure H_3^* has the same automorphism group. So does its logic $\mathscr{L}(H_3^*)$ which is Schrag's L_G.

Figure 2 represents the graph H_2 of Fig. 1. The point x has degree 2; all other points have degree 3 and are represented by triangles.

In Fig. 3 we depict the graph H_3. The circles represent the elements of degree 5 from H_0. The squares represent the elements of degree 4 obtained by identifying the element x of degree 2 of H_y with the element y of degree 2 of H_1. The triangles represent the elements of $H_2-\{x\}$ of degree 3.

The edges of H_3 are the points of H_3^* and the atoms of $\mathscr{L}(H_3^*)$. The five edges on a point of degree 5 in H_3 determine a copy of 2^5 in $\mathscr{L}(H_3^*)$. These are the only sublattices of $\mathscr{L}(H_3^*)$ isomorphic to 2^5. Hence, a lattice automorphism φ of $\mathscr{L}(H_3^*)$ permutes these copies of 2^5 among themselves. Similarly, such φ's permute the copies of 2^4 in $\mathscr{L}(H_3^*)$, corresponding to the vertices of degree 4 in H_3, among themselves. However, as we shall see, there are copies of 2^3 in $\mathscr{L}(H_3^*)$ which do

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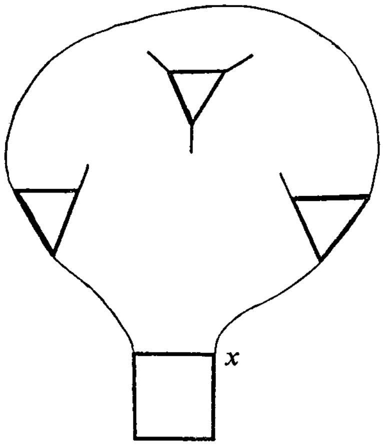

FIG. 2. An abstract version of the graph H_2.

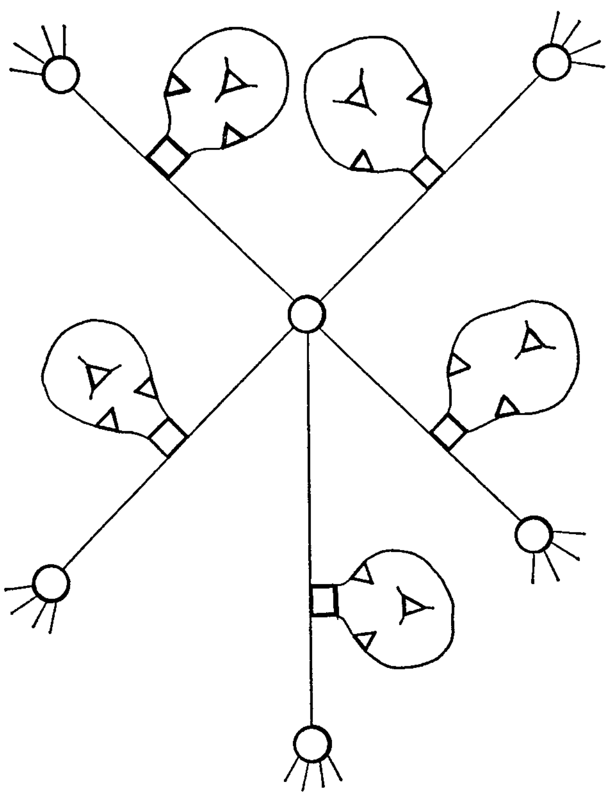

FIG. 3. A local view of the graph H_3.

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not correspond to sublattices obtained from the dual of points of degree 3. We must therefore prove separately that there are no non-trivial lattice automorphisms of each of the sublattices determined by a single H_y which fix the two edges adjacent to y in H_3. The orthogonality space diagram for this lattice L_0 is given in Fig. 4.

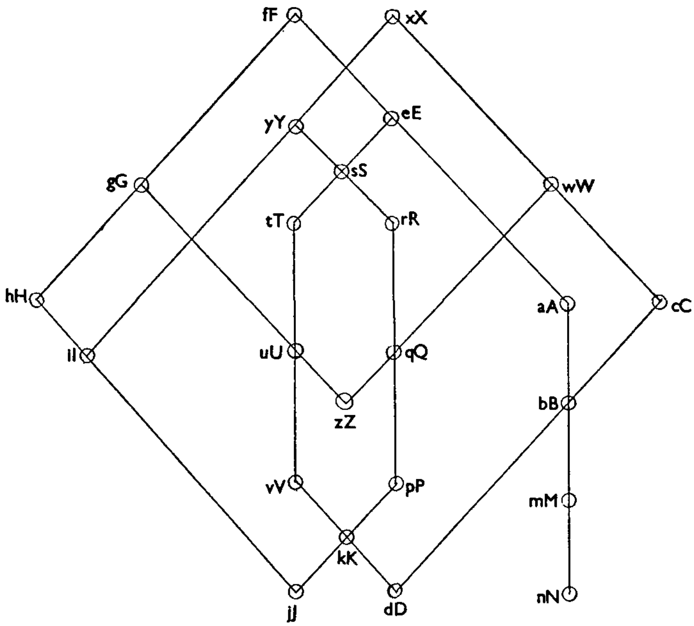

FIG. 4. The orthogonality space determined by the atoms of L_0.

L_0 has one sublattice β isomorphic to 2^4; all sublattices of L_0 of height 4 are subsets of the lattice β. We must show that L_0 contains no lattice automorphisms which fix n and m. Since the lattice automorphisms need not preserve the orthocomplementation *a priori*, we must use the full force of the diagram. The reader should visualize two copies of the diagram one on top of the other. Think of the small letters as representing the elements of the bottom copy X_0; these are the atoms of L_0. Think of the capital letters as representing elements of the top copy X_1; these are the coatoms of L_0. (To complete the picture the reader should mentally visualize the six 2-element subsets of the set $\{a, b, m, n\}$ but this will not be necessary for our argument.) L_0 is an orthomodular lattice with $A = a'$, $B = b'$, etc. However, we shall ignore this fact and treat it simply as a lattice. Note that r, S, t, U, z, Q, together with 0 and 1 is lattice isomorphic to 2^3 but not a "block" of L_0. Herein lies the problem. There may exist lattice isomorphisms which are not ortho-lattice isomorphisms. The following remark states that this is not the case.

REMARK. *The only lattice automorphism of* L_0 *which fixes* m *and* n *is the identity map.*

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PROOF. Let $\varphi: L_0 \to L_0$ (and hence φ^{-1}) be a bijection preserving $\vee$ and $\wedge$ with $\varphi(m) = m$ and $\varphi(n) = n$.

We first note that (i) $t \vee d = V < 1$ and (ii) $t \vee g = U < 1$. By considering the unique copy of 2^4 in L_0 we may infer that $\varphi(a) \in \{a, b\}$. If $\varphi(a) = b$ then $\{\varphi(c), \varphi(d)\} = \{e, f\}$ and $\{\varphi(e), \varphi(f)\} = \{c, d\}$ by considering what happens to the elements under A and B, respectively. Since $t < E$, $\varphi(t) \in \{x, w, k, v\}$; but (i) above is violated unless $\varphi(d) = e$ and $\varphi(t) = v$; it follows that $\varphi(E) = D$ so that $\varphi(F) = C$ and hence $\varphi(f) = c$ and $\varphi(e) = d$. Since $h, g < F$, $\{\varphi(h), \varphi(g)\} = \{w, x\}$; both possible choices of $\varphi(g)$ violate (ii), since $\varphi(t) = v$. Therefore $\varphi(a) \neq b$.

It follows that $\varphi(a) = a$, $\varphi(b) = b$, $\varphi(A) = A$ and $\varphi(B) = B$. By (i) and the fact that $t \leqslant E$ we conclude that $\varphi(E) \neq F$ so that $\varphi(E) = E$; it follows that $\varphi(F) = F$ and, taking meets, $\varphi(e) = e$ and $\varphi(f) = f$. Therefore $\varphi(t) \in \{s, t\}$; by (i) again the only consistent choice of $\varphi(t)$ and $\varphi(d)$ are $\varphi(t) = t$ and $\varphi(d) = d$; hence $\varphi(C) = C$, $\varphi(D) = D$, $\varphi(c) = c$, $\varphi(s) = s$, $\varphi(T) = T$, $\varphi(S) = S$. It now readily follows that φ is the identity function. The remark is proved.

We return to the proof of Lemma 1. Let $\varphi: L_G \to L_G$ be a lattice automorphism. Let x be a vertex of $H_0 \subset H_3$ so that x corresponds to a copy of $2^5 = B_x$ in $L_G = \mathscr{L}(H_3^*)$; the elements of B_x of height 2, 3 and 4 are in exactly one copy of 2^5 in L_G; hence $\varphi | B_x = B_y$ for some vertex y of H_0. Similarly, each vertex $u \in H_1 \setminus H_0$ corresponds to a copy of $2^4 = B_u$ in L_G and $\varphi | B_u = B_v$ for some vertex v in $H_1 \setminus H_0$. For each vertex $w \in H_3 \setminus (H_1 \cup H_2)$, $w \in H_y$ for some vertex $y \in H_2 \setminus H_1$; and φ maps the two atoms of B_y which are also in some 2^5 to two such atoms in some B_z, $z \in H_2 \setminus H_1$. Let L_0^y and L_0^z be the copy of L_0 corresponding to H_y and H_z. Then $\varphi | L_0^y : L_0^y \to L_0^y$ is an ortho-automorphism, by the remark, and therefore φ induces a graph isomorphism of H_y onto H_z. It follows that φ induces a graph automorphism $\varphi^\sharp$ of H_3. $\varphi^\sharp$ induces an ortho-automorphism $\varphi^{\sharp *}$ on $\mathscr{L}(H_3^*) = L_G$ and clearly $\varphi = \varphi^{\sharp *}$. Thus φ is an ortho-automorphism and so we have $\mathrm{Aut}(L_G) \subseteq \mathrm{Aut}_\perp(L_G)$. Since containment certainly goes in the other direction, we have equality.

We shall now prove the theorem. For $i = 1, 2$, let

$$X_i = \{x \in L_G \mid \text{there exists a covering chain from 0 to } x \text{ of length } i\}$$

so that X_1 is the set of atoms of L_G and X_2 is the set of elements which cover atoms. Let $X_0 = X_1 \cup X_2$ and let $\mathscr{E}_0 = \{\{x, y\} \mid x \in X_1,\ y \in X_2 \text{ and } x \leqslant y \text{ in } L_G\}$. Let $(X_3, \mathscr{E}_3) = \delta(X_0, \mathscr{E}_0)$ be the Dacification of $(X_0, \mathscr{E}_0)$, so that

$$X_3 = X_0 \cup \mathscr{E}_0 \quad \text{and} \quad \mathscr{E}_3 = \{\{x, y, \{x, y\}\} \mid \{x, y\} \in \mathscr{E}_0\}.$$

Since L_G is a lattice, $(X_3, \mathscr{E}_3)$ is a wide cubic structure space (see Greechie (1974)) (in other words, a point closed complete Dacey space in which all maximal orthogonal sets have three elements).

Henceforth we shall utilize the definitions and notation of Greechie (1974). Since $|X_2| > |X_1|$, $\mathrm{Aut}(X_3, \mathscr{E}_3) \cong G$.

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Let $(Y, \mathscr{F})$ be the wide cubic structure space obtained by deleting the point m (and M) from Fig. 4. Note that $(Y, \mathscr{F})$ has a trivial automorphism group. Take finitely many disjoint copies of this space, one (call it $Y_{\{x,y\}}$) for each $\{x, y\} \in X_3 \setminus X_0$, and identify the unique point of $Y_{\{x,y\}}$ which is in only one block with $\{x, y\}$ to obtain a finite wide cubic structure space $(X, \mathscr{E})$. Observe that (1) $x \in X$ implies x is in at least two blocks of $\mathscr{E}$ and (2) $\mathrm{Aut}\,(X, \mathscr{E}) \cong G$. Apply Theorem 2 of Greechie (1974) to embed $(X, \mathscr{E})$ in an on-trivial tight cubic structure space $\pi(X, \mathscr{E})$ so that $\mathscr{C}(\pi(X, \mathscr{E}))$ is an orthocomplemented projective plane. From (1), (2) and Proposition 3 of Greechie (1974) it follows that

$$\mathrm{Aut}_\perp(\mathscr{C}(\pi(X, \mathscr{E}))) \cong G.$$

The orthocomplementation of $\mathscr{C}(\pi(X, \mathscr{E}))$ is the absolute-point free polarity of $P = \mathscr{C}(\pi(X, \mathscr{E}))$ mentioned in the theorem. Moreover, $\mathrm{Aut}_\perp(\mathscr{C}(\pi(X, \mathscr{E})))$ is precisely the group of collineations of P which "commute" with this polarity. P is a non-Desarguesian projective plane by Corollary 2 of Greechie (1974).

The case in which $|G| = 1$ is easily handled by ignoring Schrag's L_G (which is 2^1 in this case) and constructing directly a wide cubic structure space $(Y_0, \mathscr{F}_0)$ with a trivial automorphism group in which each element is in at least two blocks and then passing to $\pi(Y_0, \mathscr{F}_0)$ and $\mathscr{C}(\pi(Y_0, \mathscr{F}_0))$. Such a space $(Y_0, \mathscr{F}_0)$ is as easy to construct as our $(Y, \mathscr{F})$ above. The theorem is proved.

The corresponding result for infinite groups may yield to the techniques of E. Mendelsohn. See, for example, "Every group is the collineation group of some projective plane", *J. Geom.*, (1972) **2** (2), 97–106 for a similar result not involving orthogonality.

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