25

$\mathcal{H}(b)$ spaces generated by an extreme symbol b

In this chapter, we study the specific properties of $\mathcal{H}(b)$ spaces when b is an extreme point of the closed unit ball of H^∞. Thus, by Theorem 6.7, we assume

$$\int_{\mathbb{T}} \log(1 - |b(e^{i\theta})|^2)\, d\theta = -\infty.$$

In particular, this happens if $b = \Theta$ is an inner function. In this case, $\mathcal{H}(b)$ is precisely the model space K_Θ. Roughly speaking, when b is an extreme point, the space $\mathcal{H}(b)$ looks like the model space K_Θ.

In Section 25.1, we introduce a unitary operator from $L^2(\rho)$ onto $\mathcal{H}(\bar{b})$, where $\rho = 1 - |b|^2$ on $\mathbb{T}$. This unitary operator is important, in particular to compute the norm of functions $f \in \mathcal{H}(b)$. For example, we do this computation for $f = S^*b$. In Section 25.2, we prove that a nonzero element of $\mathcal{H}(\bar{b})$ cannot be analytically continued across all of $\mathbb{T}$. This fact is used to show that $b \notin \mathcal{H}(b)$. In contrast to the nonextreme case, we also show that the space $\mathcal{H}(b)$ is not invariant under the forward shift operator S. Despite the situation for $\mathcal{H}(\bar{b})$, some elements of $\mathcal{H}(b)$ can be extended across all of $\mathbb{T}$. In Section 25.3, we characterize such functions. This characterization is used to prove that $k_\lambda \in \mathcal{H}(b)$ if and only if $b(\lambda) = 0$. In Section 25.4, we give a formula for $\|X_b f\|_b$, $f \in \mathcal{H}(b)$, and show that the defect operator D_{X_b} has rank one. In Section 25.5, we show that the only function in $\mathcal{H}(\bar{b})$ that has a bounded-type meromorphic pseudocontinuation across $\mathbb{T}$ to the exterior disk $\mathbb{D}_e$ is the zero function. We also prove a similar result for $\mathcal{H}(b)$ functions. In Section 25.6, we exhibit an important orthogonal decomposition of $\mathcal{H}(b)$. We use this orthogonal decomposition to show, in Section 25.7, that the closure of $\mathcal{H}(\bar{b})$ in $\mathcal{H}(b)$ is $\mathcal{H}([b])$, where $[b]$ is the outer factor of b. This is dramatically different from the nonextreme case, where we have seen that $\mathcal{H}(\bar{b})$ is always dense in $\mathcal{H}(b)$. Finally, in Section 25.8, we give a characterization of $\mathcal{H}(b)$ spaces when b is an extreme point. The corresponding result for the nonextreme case was given in Section 23.7.

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25.1 A unitary map between $\mathcal{H}(\bar{b})$ and $L^2(\rho)$

We recall that ρ is the function

$$\rho = 1 - |b|^2 \in L^\infty(\mathbb{T})$$

and K_ρ is the application defined on $L^2(\rho)$ by the formula $K_\rho(g) = P_+(\rho g)$ into H^2 (see Section 13.4). In this situation, we can give further information about K_ρ and its range.

Theorem 25.1 *Let b be an extreme point of the closed unit ball of H^∞. Then*

$$H^2(\rho) = L^2(\rho)$$

and K_ρ is an isometry from $L^2(\rho)$ onto $\mathcal{H}(\bar{b})$. If μ is the Clark measure associated with b, then we also have $H^2(\mu) = L^2(\mu)$.

Proof That $H^2(\rho) = L^2(\rho)$ and $H^2(\mu) = L^2(\mu)$ were established in Corollary 13.34. According to Theorem 20.1, K_ρ is a partial isometry from $L^2(\rho)$ onto $\mathcal{H}(\bar{b})$ and $\ker K_\rho = (H^2(\rho))^\perp$. Since $H^2(\rho) = L^2(\rho)$, we conclude that K_ρ is in fact an isometry from $L^2(\rho)$ onto $\mathcal{H}(\bar{b})$. □

Even though in Theorem 25.1 we assumed that b is an extreme point to deduce that $H^2(\rho) = L^2(\rho)$ and $H^2(\mu) = L^2(\mu)$, we emphasize that the last two identities occur precisely when b is an extreme point of the closed unit ball of H^∞. Hence, Theorem 25.1 can be rewritten in a proper way to give a characterization of the identity $H^2(\rho) = L^2(\rho)$.

Corollary 25.2 *Let b be an extreme point of the closed unit ball of H^∞ and let f be a function in $\mathcal{H}(\bar{b})$. Then there is a unique function $g \in L^2(\rho)$ such that*

$$f = P_+(\rho g).$$

Moreover, we have $\log|\rho g| \notin L^1(\mathbb{T})$.

Proof The first part follows immediately from Theorem 25.1.

For the second part, write

$$\log|\rho g| = \log|g\rho^{1/2}| + \tfrac{1}{2}\log\rho. \tag{25.1}$$

On the one hand, since b is an extreme point of the closed unit ball of H^∞, we have

$$\int_{\mathbb{T}} \log\rho\, dm = -\infty.$$

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On the other hand, using Jensen's inequality, we see that

$$\int_{\mathbb{T}} \log |g\rho^{1/2}|\, dm \le \log \left(\int_{\mathbb{T}} |g\rho^{1/2}|\, dm \right)$$
$$\le \log \left(\int_{\mathbb{T}} |g|^2 \,\rho\, dm \right)^{1/2}.$$

Since $g \in L^2(\rho)$, we deduce that

$$\int_{\mathbb{T}} \log |g\rho^{1/2}|\, dm < +\infty.$$

Thus, the conclusion follows from (25.1). □

Corollary 25.3 *Let b be an extreme point of the closed unit ball of H^∞. Then*

$$\|S^* b\|_b^2 = 1 - |b(0)|^2.$$

Proof By Theorem 17.8, $T_{\bar{b}} S^* b \in \mathcal{H}(\bar{b})$ and

$$\|S^* b\|_b^2 = \|S^* b\|_2^2 + \|T_{\bar{b}} S^* b\|_{\bar{b}}^2. \tag{25.2}$$

To compute the norm of $T_{\bar{b}} S^* b$ in $\mathcal{H}(\bar{b})$, we use the operator Z_ρ, which was introduced in Section 8.1. Recall that Z_ρ denotes the operator on $L^2(\rho)$ of multiplication by the independent variable z, i.e.

$$(Z_\rho f)(z) = z f(z),$$

where $z \in \mathbb{T}$ and $f \in L^2(\rho)$. Hence, by Corollary 13.18, we have

$$K_\rho Z_\rho^* \chi_0 = S^* K_\rho \chi_0 = S^* P_+ \rho = S^* P_+ (1 - |b|^2)$$
$$= -S^* P_+ |b|^2 = -S^* T_{\bar{b}} b.$$

Since $S^* T_{\bar{b}} = T_{\bar{b}} S^*$, we obtain

$$T_{\bar{b}} S^* b = -K_\rho Z_\rho^* \chi_0. \tag{25.3}$$

Now, using Theorem 25.1 and the fact that Z_ρ is a unitary operator, we can write

$$\|T_{\bar{b}} S^* b\|_{\bar{b}} = \|K_\rho Z_\rho^* \chi_0\|_{\bar{b}} = \|Z_\rho^* \chi_0\|_{L^2(\rho)} = \|\chi_0\|_{L^2(\rho)}.$$

Therefore, (25.2) becomes

$$\|S^* b\|_b^2 = \|S^* b\|_2^2 + \|\chi_0\|_{L^2(\rho)}^2. \tag{25.4}$$

But it is easy to see that

$$\|\chi_0\|_{L^2(\rho)}^2 = \int_{\mathbb{T}} \rho\, dm = \int_{\mathbb{T}} (1 - |b|^2)\, dm = 1 - \|b\|_2^2$$

and, by (8.16), that

$$\|S^*b\|_b^2 = \|b\|_2^2 - |b(0)|^2.$$

Plug the last two identities into (25.4) to get the result. $\square$

25.2 Analytic continuation of $f \in \mathcal{H}(\bar{b})$

A nonzero element of $\mathcal{H}(\bar{b})$ certainly has a singularity somewhere on $\mathbb{T}$. This fact is stated in the following form.

Theorem 25.4 *Let b be an extreme point in the closed unit ball of H^∞, and let $f \in \mathcal{H}(\bar{b})$. If f can be analytically continued across all of $\mathbb{T}$, then $f \equiv 0$.*

Proof Let f be a function in $\mathcal{H}(\bar{b})$ that can be analytically continued across all of $\mathbb{T}$. Hence, Theorem 5.7 implies that there is a $c > 0$ such that $|\hat{f}(n)| = O(e^{-cn})$, as $n \longrightarrow +\infty$. We know from 25.1 that there is a unique function $g \in L^2(\rho)$ such that $f = P_+(\rho g)$ and $\log|g\rho| \notin L^1(\mathbb{T})$. In particular, the condition $f = P_+(\rho g)$ implies that

$$\hat{f}(n) = \widehat{g\rho}(n) \qquad (n \geq 0).$$

Put $h = \overline{g\rho}$. We prove that h satisfies the hypotheses of Theorem 4.31. First note that h belongs to $L^2(\mathbb{T})$, since it is the product of the $L^2(\mathbb{T})$ function $g\rho^{1/2}$ and the $L^\infty(\mathbb{T})$ function $\rho^{1/2}$. Moreover, an easy computation shows that

$$\hat{h}(-n) = \overline{\hat{\bar{h}}(n)} = \overline{\widehat{g\rho}(n)} = \overline{\hat{f}(n)} \qquad (n \geq 0).$$

Thus, $|\hat{h}(-n)| = O(e^{-cn})$, as $n \longrightarrow +\infty$. But, since $\log|h| = \log|g\rho| \notin L^1(\mathbb{T})$, Theorem 4.31 ensures that $h \equiv 0$. Therefore, $f \equiv 0$. $\square$

In Corollary 23.9, we saw that, if b is a nonextreme point of the closed unit ball of H^∞, then $b \in \mathcal{H}(b)$. But this is in fact the only case where the inclusion $b \in \mathcal{H}(b)$ is possible.

Corollary 25.5 *Let b be a point in the closed unit ball of H^∞. Then the following are equivalent.*

(i) $b \in \mathcal{H}(b)$.
(ii) *$\mathcal{H}(b)$ is invariant under the forward shift operator S.*
(iii) *$\mathcal{H}(\bar{b})$ has a nonzero element that is analytic on $\bar{\mathbb{D}}$.*
(ii) *b is a nonextreme point of the closed unit ball of H^∞.*

Proof (i) $\Longrightarrow$ (ii) According to Theorem 18.22, we have

$$X_b^* f = Sf - \langle f, S^*b\rangle_b b \qquad (f \in \mathcal{H}(b)). \tag{25.5}$$

Hence, the assumption $b \in \mathcal{H}(b)$ immediately implies that $Sf \in \mathcal{H}(b)$.

(ii) $\Longrightarrow$ (i) Take any function $f \in \mathcal{H}(b)$ such that $\langle f, S^*b\rangle_b \neq 0$. For instance, one can choose $f = S^*b \in \mathcal{H}(b)$ for which $\|S^*b\|_b^2 \neq 0$. Then we rewrite (25.5) as

$$b = \frac{Sf - X^*f}{\langle f, S^*b\rangle_b}$$

to deduce $b \in \mathcal{H}(b)$.

(i) $\Longrightarrow$ (iii) If $b \in \mathcal{H}(b)$, then, according to Theorem 17.8, we have $T_{\bar{b}}b \in \mathcal{H}(\bar{b})$. Since, by definition, the function $(I - T_{\bar{b}}T_b)\chi_0 = \chi_0 - T_{\bar{b}}b$ also belongs to $\mathcal{H}(\bar{b})$, we must have $\chi_0 \in \mathcal{H}(\bar{b})$. But the nonzero function χ_0 can obviously be analytically continued across all of $\mathbb{T}$.

(iii) $\Longrightarrow$ (iv) This follows from Theorem 25.4.

(iv) $\Longrightarrow$ (i) This was already proved in Corollary 23.9. $\square$

We just saw that, if b is an extreme point of the closed unit ball of H^∞, then $\mathcal{H}(b)$ is not S-invariant. Nevertheless, it could be possible for some functions $f \in \mathcal{H}(b)$ that Sf remains in $\mathcal{H}(b)$. The following result characterizes this class of functions.

Corollary 25.6 *Let b be an extreme point of the closed unit ball of H^∞, and let h be a function in $\mathcal{H}(b)$. The following are equivalent.*

(i) $Sh \in \mathcal{H}(b)$.
(ii) $\langle h, S^*b\rangle_b = 0$.

Proof Remember that

$$X_b^* h = Sh - \langle h, S^*b\rangle_b b.$$

According to Corollary 25.5, we know that $b \notin \mathcal{H}(b)$. That gives the equivalence. $\square$

25.3 Analytic continuation of $f \in \mathcal{H}(b)$

In contrast to the case of $\mathcal{H}(\bar{b})$, some elements of $\mathcal{H}(b)$ can be extended across all of $\mathbb{T}$. These elements are of a special form, which is described below.

Theorem 25.7 *Let b be an extreme point in the closed unit ball of H^∞, and let f be a function in H^2. Then the following are equivalent.*

(i) *$f \in \mathcal{H}(b)$ and can be analytically continued across all of $\mathbb{T}$.*
(ii) *f is rational and $T_{\bar{b}}f = 0$.*

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Proof (i) $\Longrightarrow$ (ii) Assume that $f \in \mathcal{H}(b)$ and can be analytically continued across all of $\mathbb{T}$. Since f can be analytically continued across all of $\mathbb{T}$, Theorem 5.7 ensures that there are $c_1 > 0$ and $c_2 > 0$ such that

$$|\hat{f}(n)| \leq c_2 e^{-c_1 n} \qquad (n \geq 0). \tag{25.6}$$

We first show that $T_{\bar{b}} f$ can also be analytically continued across all of $\mathbb{T}$. In fact, for each $n \geq 1$, we have

$$\begin{aligned}\widehat{T_{\bar{b}} f}(n) &= \langle T_{\bar{b}} f, \chi_n \rangle_2 = \langle P_+(\bar{b} f), \chi_n \rangle_2 = \langle \bar{b} f, \chi_n \rangle_2 \\ &= \langle \chi_{-n} f, b \rangle_2 = \langle P_+(\chi_{-n} f), b \rangle_2 = \langle S^{*n} f, b \rangle_2.\end{aligned}$$

We repeatedly used Lemma 4.8. Therefore,

$$|\widehat{T_{\bar{b}} f}(n)| \leq \|S^{*n} f\|_2 \, \|b\|_2 \leq \|S^{*n} f\|_2 \, \|b\|_\infty \leq \|S^{*n} f\|_2.$$

But, by (8.16),

$$\|S^{*n} f\|_2^2 = \sum_{k=0}^{\infty} |\hat{f}(k+n)|^2.$$

Hence, by (25.6),

$$\|S^{*n} f\|_2^2 \leq c_2 \sum_{k=0}^{\infty} e^{-2c_1(k+n)} = c_2 e^{-2c_1 n} \sum_{k=0}^{\infty} e^{-2c_1 k} = c_2 \frac{e^{-2c_1 n}}{1 - e^{-2c_1}}.$$

Thus, $\widehat{T_{\bar{b}} f}(n) = O(e^{-c_1 n})$ as $n \longrightarrow +\infty$. Another application of Theorem 5.7 implies that $T_{\bar{b}} f$ can be analytically continued across all of $\mathbb{T}$. Since $f \in \mathcal{H}(b)$, we have $T_{\bar{b}} f \in \mathcal{H}(\bar{b})$ and thus it follows from Theorem 25.4 that $T_{\bar{b}} f = 0$, which means that f belongs to the kernel of $T_{\bar{b}}$.

It remains to show that f is a rational function. By Theorem 14.10, f is a cyclic vector for S^* if and only if so is $T_{\bar{b}} f$. But, since $T_{\bar{b}} f = 0$, $T_{\bar{b}} f$ is not a cyclic vector of S^*, and thus nor is f. Theorem 8.42 now ensures that f is a rational function.

(ii) $\Longrightarrow$ (i) Assume that f is a rational function in H^2 that belongs to $\ker T_{\bar{b}}$. Then $T_{\bar{b}} f = 0 \in \mathcal{H}(\bar{b})$ and Theorem 17.8 implies that $f \in \mathcal{H}(b)$. The fact that f can be analytically continued across all of $\mathbb{T}$ follows from Theorem 5.8. $\square$

As the first application, we can characterize the Cauchy kernels that belong to $\mathcal{H}(b)$.

Corollary 25.8 *Let b be an extreme point of the closed unit ball of H^∞ and let $\lambda \in \mathbb{D}$. Then the following are equivalent:*

(i) $k_\lambda \in \mathcal{H}(b)$;
(ii) $b(\lambda) = 0$.

Proof The implication (ii) $\Longrightarrow$ (i) is rather obvious since, if $b(\lambda) = 0$, then we have $k_\lambda = k_\lambda^b \in \mathcal{H}(b)$. For the converse, first note that the function k_λ can be analytically continued across all of $\mathbb{T}$. Therefore, by Theorem 25.7, k_λ belongs to the kernel of $T_{\bar{b}}$. But, using (12.7), we have $T_{\bar{b}} k_\lambda = \overline{b(\lambda)} k_\lambda$ and thus we must have $b(\lambda) = 0$. □

If b is an extreme point in the closed unit ball of H^∞ and if f belongs to $\mathcal{H}(b)$ and can be analytically continued across all of $\mathbb{T}$, then $\|f\|_b = \|f\|_2$. In fact, by Theorem 25.7, we must have $T_{\bar{b}} f = 0$. Then, using Theorem 17.8, we get

$$\|f\|_b^2 = \|f\|_2^2 + \|T_{\bar{b}} f\|_{\bar{b}}^2 = \|f\|_2^2. \tag{25.7}$$

This fact is used to detect the monomials that are in $\mathcal{H}(b)$.

Corollary 25.9 *Let b be an extreme point of the closed unit ball of H^∞. Let m be a nonnegative integer. Then the following assertions are equivalent.*

(i) *The monomial z^m belongs to $\mathcal{H}(b)$.*
(ii) *b has a zero of order at least $m+1$ at the origin.*

Moreover, if $z^m \in \mathcal{H}(b)$, then $\|z^m\|_b = 1$.

Proof According to Theorem 25.7, the function z^m belongs to $\mathcal{H}(b)$ if and only if it belongs to the kernel of $T_{\bar{b}}$. But, by (12.4),

$$T_{\bar{b}}\, z^m = \sum_{k=0}^{m} \overline{\hat{b}(k)} z^{m-k}.$$

Thus, z^m belongs to $\mathcal{H}(b)$ if and only if $\hat{b}(k) = 0$, $0 \le k \le m$, which means that b has a zero of order at least $m+1$ at the origin. The statement concerning the norm of z^m follows directly from (25.7). □

In fact, the above result shows that, if b has a zero of order $m+1$ at the origin, then the set of polynomials at $\mathcal{H}(b)$ is a linear manifold of dimension $m+1$ spanned by $1, z, \dots, z^m$. This fact is mentioned in more detail in the following corollary and it shows that we can add one extra item to the list given in Corollary 25.5.

Corollary 25.10 *Let b be in the closed unit ball of H^∞. Then the set of analytic polynomials $\mathcal{P}$ is contained in $\mathcal{H}(b)$ if and only if b is a nonextreme point of the closed unit ball of H^∞.*

Proof In Theorem 23.13, we have already proved that, if b is a nonextreme point of the closed unit ball of H^∞, then the set of analytic polynomials $\mathcal{P}$ is contained in $\mathcal{H}(b)$ (in fact, it is even dense in $\mathcal{H}(b)$). For the converse, we argue by absurdity. Assume that the set $\mathcal{P}$ is contained in $\mathcal{H}(b)$, but b is an extreme point of the closed unit ball of H^∞. Then, by Corollary 25.9, b has a zero of order infinity at the origin, which is absurd (remember that b is not a constant). □

Exercise

Exercise 25.3.1 Let b be an extreme point of the closed unit ball of H^∞ and assume further that b is an outer function. Show that, if a function f in $\mathcal{H}(b)$ can be analytically continued across all of $\mathbb{T}$, then $f \equiv 0$.
Hint: Use Theorems 12.19 and 25.7.

25.4 A formula for $\|X_b f\|_b$

In this section, we give a formula for $\|X_b f\|_b$, and then easily generalize it for $\|X_b^n f\|_b$. The result of this section should be compared with those in Section 23.5, which were about the nonextreme case.

Theorem 25.11 *Let b be an extreme point of the closed unit ball of H^∞. Then we have*

$$X_b^* X_b = I - (k_0^b \otimes k_0^b),$$

which implies that

$$\|X_b f\|_b^2 = \|f\|_b^2 - |f(0)|^2$$

for every function $f \in \mathcal{H}(b)$.

Proof According to (18.15), with f replaced by $X_b f = S^* f$, we have

$$\begin{aligned} X_b^*(X_b f) &= S(S^* f) - \langle X_b f, S^* b\rangle_b b \\ &= f - f(0) - \langle X_b f, X_b b\rangle_b b \\ &= f - f(0) - \langle f, X_b^* X_b b\rangle_b b. \end{aligned}$$

Hence, we look for a formula for $X_b^* X_b b$. Once more, by (18.15) with $f = S^* b = X_b b$, we obtain

$$X_b^* X_b b = X_b^*(S^* b) = SS^* b - \|S^* b\|_b^2 b = b - b(0) - \|S^* b\|_b^2\, b.$$

But we know from Corollary 25.3 that $\|S^* b\|_b^2 = 1 - |b(0)|^2$, and thus

$$X_b^* X_b b = b - b(0) - (1 - |b(0)|^2) b = -b(0)(1 - \overline{b(0)} b) = -b(0) k_0^b.$$

Back to the first relation above, we can now write

$$\begin{aligned} X_b^* X_b f &= f - f(0) - \langle f, X_b^* X_b b\rangle_b b \\ &= f - f(0) + \overline{b(0)}\langle f, k_0^b\rangle_b b \\ &= f - \langle f, k_0^b\rangle_b + \overline{b(0)}\langle f, k_0^b\rangle_b b \end{aligned}$$

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$$
\begin{aligned}
&= f - \langle f, k_0^b\rangle_b (1 - \overline{b(0)} b) \\
&= f - \langle f, k_0^b\rangle_b k_0^b.
\end{aligned}
$$

The preceding identity is rewritten as

$$X_b^* X_b = I - (k_0^b \otimes k_0^b).$$

Moreover, from this identity, we get

$$
\begin{aligned}
\|X_b f\|_b^2 &= \langle X_b f, X_b f\rangle_b \\
&= \langle f, X_b^* X_b f\rangle_b \\
&= \langle f,\ f - f(0) k_0^b\rangle_b \\
&= \|f\|_b^2 - \overline{f(0)} \langle f, k_0^b\rangle_b \\
&= \|f\|_b^2 - |f(0)|^2.
\end{aligned}
$$

This completes the proof. □

The following result should be compared with Corollary 23.16, i.e. the analogous result in the nonextreme case.

Corollary 25.12 *Let b be an extreme point of the closed unit ball of H^∞. The operator $D_{X_b} = (I - X_b^* X_b)^{1/2}$ has rank one, its range is spanned by k_0^b and its nonzero eigenvalue equals $\|k_0^b\|_b$.*

Proof This follows immediately from Theorem 25.11. □

It is straightforward to generalize the preceding formula for $\|X_b f\|_b^2$ to $\|X_b^n f\|_b^2$.

Corollary 25.13 *Let b be an extreme point of the closed unit ball of H^∞. Then we have*

$$\|X_b^n f\|_b^2 = \|f\|_b^2 - \sum_{k=0}^{n-1} |\hat{f}(k)|^2$$

for every function $f \in \mathcal{H}(b)$ and every integer $n \geq 1$.

Proof The proof is by induction on the integer n. For $n = 1$ the equality is precisely the one proved in Theorem 25.11. Just note that $f(0) = \hat{f}(0)$. Assume that the equality holds for some n. Then, using once again Theorem 25.11 and the induction hypothesis, we have

$$
\begin{aligned}
\|X_b^{n+1} f\|_b^2 &= \|X_b(X_b^n f)\|_b^2 \\
&= \|X_b^n f\|_b^2 - |(X_b^n f)(0)|^2 \\
&= \|f\|_b^2 - \sum_{k=0}^{n-1} |\hat{f}(k)|^2 - |(X_b^n f)(0)|^2.
\end{aligned}
$$

But

$$(X_b^n f)(0) = \langle X_b^n f, \chi_0 \rangle_2 = \langle S^{*n} f, \chi_0 \rangle_2 = \langle f, S^n \chi_0 \rangle_2 = \langle f, \chi_n \rangle_2 = \hat{f}(n).$$

The proof is complete. □

Corollary 25.14 *Let b be an extreme point of the closed unit ball of H^∞. Then, for each function $f \in \mathcal{H}(b)$, we have*

$$\lim_{n\to\infty} \|X_b^n f\|_b^2 = \|f\|_b^2 - \|f\|_2^2.$$

Proof For each function $f \in H^2$, we have

$$\|f\|_2^2 = \sum_{k=0}^{\infty} |\hat{f}(k)|^2 = \lim_{n\to\infty} \sum_{k=0}^{n-1} |\hat{f}(k)|^2.$$

Now, let $n \longrightarrow \infty$ in the formula given in Corollary 25.13. □

Corollary 25.15 *Let b be in the closed unit ball of H^∞. Then, for every function $f \in \mathcal{H}(b)$, we have*

$$\|X_b f\|_b^2 \le \|f\|_b^2 - |f(0)|^2. \tag{25.8}$$

Moreover, the last inequality is an equality for all $f \in \mathcal{H}(b)$ if and only if b is an extreme point of the closed unit ball of H^∞.

Proof The inequality (25.8) has already been proved in Theorem 18.28. We have seen in Theorem 25.11 that (25.8) becomes an equality when b is extreme. Assume now that b is nonextreme. Then, according to (23.17) we have

$$\|X_b b\|_b^2 = \|b\|_b^2 - |b(0)|^2 - |a(0)|^2 \|b\|_b^2 < \|b\|_b^2 - |b(0)|^2.$$

In other words, when b is nonextreme, then the inequality (25.8) can be strict. □

25.5 S^*-cyclic vectors in $\mathcal{H}(b)$ and $\mathcal{H}(\bar{b})$

The result of Douglas, Shapiro and Shields (Theorem 8.42) completely characterizes the cyclic vectors of S^* as an operator on H^2. We saw that $\mathcal{H}(b)$ and $\mathcal{H}(\bar{b})$ are invariant under S^*. In fact, the restriction of S^* on $\mathcal{H}(b)$ was denoted by X_b. In this section, we characterize the S^*-cyclic vectors that are in $\mathcal{H}(b)$ and $\mathcal{H}(\bar{b})$. The first result says that, except of course for the zero function, all other elements of $\mathcal{H}(\bar{b})$ are cyclic vectors for S^*.

Theorem 25.16 *Let b be an extreme point of the closed unit ball of H^∞. Then each $f \in \mathcal{H}(\bar{b})$, $f \not\equiv 0$, is a cyclic vector for S^*. Hence, the only function in $\mathcal{H}(\bar{b})$ that has a bounded-type meromorphic pseudocontinuation across $\mathbb{T}$ to $\mathbb{D}_e$ is the zero function.*

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Proof Fix $f \in \mathcal{H}(\bar{b})$, $f \not\equiv 0$. It follows from Corollary 25.2 that there is a unique function $g \in L^2(\rho)$ such that

$$f = P_+(g\rho) \quad \text{and} \quad \log|g\rho| \notin L^1(\mathbb{T}).$$

Denote by M the closed invariant subspace of S^* generated by f. In other words, let

$$M = \operatorname{Span}_{H^2}(S^{*n} f : n \geq 0).$$

We need to show that $M = H^2$. We recall that Z is the bilateral forward shift operator on $L^2(\mathbb{T})$, i.e.

$$(Zg)(z) = zg(z) \qquad (z \in \mathbb{T},\ g \in L^2(\mathbb{T})).$$

The closed subspace $M \oplus \overline{H_0^2}$ of $L^2(\mathbb{T})$ is invariant under Z^*. In fact, if $f_1 \in M$ and $f_2 \in \overline{H_0^2}$, then we have

$$\begin{aligned} Z^*(f_1 + f_2) &= \bar{z} f_1 + \bar{z} f_2 \\ &= P_+(\bar{z} f_1) + P_-(\bar{z} h_1) + \bar{z} h_2 \\ &= S^* f_1 + P_-(\bar{z} f_1) + \bar{z} f_2, \end{aligned}$$

and $S^* f_1 \in M$ (remember, M is invariant under S^*) and $P_-(\bar{z} f_1) + \bar{z} f_2 \in \overline{H_0^2}$. Therefore, by Theorems 8.29 and 8.30, either $M \oplus \overline{H_0^2} = \Theta \overline{H^2}$ with Θ a unimodular function in $L^\infty(\mathbb{T})$, or $M \oplus \overline{H_0^2} = \chi_E L^2(\mathbb{T})$ with E a Borel subset of $\mathbb{T}$.

Let us show that the first case cannot occur. To do so, suppose that there is a unimodular function Θ such that $M \oplus \overline{H_0^2} = \Theta \overline{H^2}$. We find a contradiction. Since $f \in M$, the function $g\rho = f + P_-(g\rho)$ belongs to $M \oplus \overline{H_0^2}$, and thus there is an $h \in H^2$ such that $g\rho = \Theta \bar{h}$. Hence, $\log|g\rho| = \log|\Theta \bar{h}| = \log|h|$, which shows that $\log|h| \notin L^1(\mathbb{T})$. But then Lemma 4.30 implies that $h \equiv 0$, which in turn yields $g \equiv 0$ and $f \equiv 0$. This is absurd.

Therefore, for a proper Borel set E, we have $M \oplus \overline{H_0^2} = \chi_E L^2(\mathbb{T})$. Since $f \in M \subset \chi_E L^2(\mathbb{T})$, we deduce that $f \equiv 0$ almost everywhere on $\mathbb{T} \setminus E$. Then Lemma 4.30 implies that $m(\mathbb{T} \setminus E) = 0$. Hence, $\chi_E L^2(\mathbb{T}) = L^2(\mathbb{T})$, or equivalently $M \oplus \overline{H_0^2} = L^2(\mathbb{T})$. Finally, since $M \subset H^2$, we must have $M = H^2$. The second assertion follows immediately from Theorem 8.42. □

While the preceding result says that $\mathcal{H}(\bar{b})$ is filled with cyclic vectors and the only exception is the zero function, the space $\mathcal{H}(b)$ might have more noncyclic elements. More explicitly, the noncyclic vectors are precisely the elements of K_Θ, where Θ is the inner part of b.

Theorem 25.17 *Let b be an extreme point of the closed unit ball of H^∞, and let f be a function in H^2. Then the following are equivalent.*

(i) $f \in \mathcal{H}(b)$ *and* f *is not a cyclic vector for* S^*.
(ii) $f \in \mathcal{H}(b)$ *and* f *has a bounded-type meromorphic pseudocontinuation across* $\mathbb{T}$ *to* $\mathbb{D}_e$.
(iii) $T_{\bar{b}} f = 0$.
(iv) $f \in K_\Theta$, *where* Θ *is the inner part of* b.

Proof (i) $\Longleftrightarrow$ (ii) This follows immediately from Theorem 8.42.

(i) $\Longrightarrow$ (iii) Theorem 14.10 implies that $T_{\bar{b}} f$ is not a cyclic vector for S^*. But, by Theorem 17.8, $T_{\bar{b}} f \in \mathcal{H}(\bar{b})$. Hence, Theorem 25.16 implies that $T_{\bar{b}} f = 0$.

(iii) $\Longrightarrow$ (i) According to Theorem 17.8, we have $f \in \mathcal{H}(b)$. Since $T_{\bar{b}} f$ is obviously not a cyclic vector for S^*, it follows once more from Theorem 14.10 that f is not a cyclic vector for S^*.

(iii) $\Longleftrightarrow$ (iv) This follows from Theorem 12.19. □

The above result yields a statement similar to Theorem 25.16 for the S^*-cyclic elements of $\mathcal{H}(b)$, in the case where b is outer.

Corollary 25.18 *Let* b *be outer and an extreme point of the closed unit ball of* H^∞. *Then each* $f \in \mathcal{H}(b)$, $f \not\equiv 0$, *is a cyclic vector for* S^*. *Hence, the only function in* $\mathcal{H}(b)$ *that is pseudocontinuable across* $\mathbb{T}$ *is the zero function.*

If we combine Theorem 25.17 and Corollary 18.15, then we get the following necessary condition for cyclic vectors for X_b.

Corollary 25.19 *Let* b *be an extreme point of the closed unit ball of* H^∞, *let* $b = \Theta b_o$ *be the factorization of* b *into its inner part* Θ *and its outer part* b_o *and assume that* Θ *and* b_o *are not constant. Let* $f \in \mathcal{H}(b)$. *If* f *is a cyclic vector for* X_b, *then* $f \notin K_\Theta$.

25.6 Orthogonal decompositions of $\mathcal{H}(b)$

Remember that, if $b_1 = \Theta$ is an inner function and b_2 is a function in H^∞ and $b = b_1 b_2$, then, according to Corollary 18.9, the space $\mathcal{H}(b)$ can be decomposed as

$$\mathcal{H}(b) = \mathcal{H}(\Theta) \oplus \Theta \mathcal{H}(b_2),$$

and the sum is orthogonal. When b is extreme, we can give another orthogonal decomposition for $\mathcal{H}(b)$. In a sense, we can say that the roles of Θ and b_2 can be exchanged.

Theorem 25.20 *Let* $b = b_1 b_2$, $b_\ell \in H^\infty$, $\|b_\ell\|_\infty \le 1$ *and* b_ℓ *is nonconstant. Then the following assertions hold.*

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(i) *The space $\mathcal{H}(b)$ decomposes as*

$$\mathcal{H}(b) = \mathcal{H}(b_1) + b_1\mathcal{H}(b_2). \tag{25.9}$$

(ii) *If b_1 is extreme and b_2 is inner, then the sum in* (25.9) *is orthogonal, the inclusion map of $\mathcal{H}(b_1)$ into $\mathcal{H}(b)$ is an isometry and the operator T_{b_1} acts as an isometry from $\mathcal{H}(b_2)$ into $\mathcal{H}(b)$.*

(iii) *If the sum in* (25.9) *is orthogonal, then necessarily b_1 is extreme.*

Proof Part (i) has already been proved in Theorem 18.7.

Let us now prove (ii). According to Theorem 18.8, it is sufficient to check that

$$\mathcal{H}(b_2) \cap \mathcal{H}(\bar{b}_1) = \{0\}.$$

Let $f \in \mathcal{H}(b_2) \cap \mathcal{H}(\bar{b}_1)$. On the one hand, since b_2 is inner, $\mathcal{H}(b_2)$ is a closed S^*-invariant subspace of H^2, and thus f cannot be a cyclic vector for S^*. On the other hand, since $|b| = |b_1|$ (once again because b_2 is inner), then b_2 is also an extreme point of the closed unit ball of H^∞. It now follows from Theorem 25.16 that $f \equiv 0$ and we conclude that $\mathcal{H}(b_2) \cap \mathcal{H}(\bar{b}_1) = \{0\}$. It remains to show that, if b_1 is nonextreme, then the sum in (25.9) is not orthogonal. Since b_1 is nonextreme, then we have

$$\mathcal{P} \subset \mathcal{H}(b_1) \subset \mathcal{H}(b).$$

Since b is nonextreme (note that $\log(1 - |b_1|) \le \log(1 - |b|)$), Theorem 23.13 implies that $\mathcal{P}$ is dense in $\mathcal{H}(b)$. Therefore, we get that $\mathcal{H}(b_1)$ is also dense in $\mathcal{H}(b)$. In particular, its orthogonal complement is reduced to $\{0\}$ and thus the decomposition (25.9) cannot be orthogonal. □

25.7 The closure of $\mathcal{H}(\bar{b})$ in $\mathcal{H}(b)$

In Theorem 17.9, we saw that the space $\mathcal{H}(\bar{b})$, for any b, is contractively contained in $\mathcal{H}(b)$. Then, in Corollary 23.10, we showed that $\mathcal{H}(\bar{b})$ is a dense submanifold of $\mathcal{H}(b)$ whenever b is a nonextreme point of the closed unit ball of H^∞. The situation in the extreme case is different and is discussed below.

Theorem 25.21 *Let b be an extreme point in the closed unit ball of H^∞ and let $b = \Theta[b]$ be its canonical factorization, with Θ the inner part and $[b]$ the outer part of b. Then the closure of $\mathcal{H}(\bar{b})$ in $\mathcal{H}(b)$ is $\mathcal{H}([b])$. In particular, $\mathcal{H}(\bar{b})$ is dense in $\mathcal{H}(b)$ if and only if b is an outer function.*

Proof First note that, by Lemma Theorem 17.11, we have $\mathcal{H}(\bar{b}) = \mathcal{H}(\overline{[b]})$, and Theorem 17.9 implies that

$$\mathcal{H}(\bar{b}) = \mathcal{H}(\overline{[b]}) \subset \mathcal{H}([b]).$$

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But, as a consequence of the orthogonal decomposition given in Theorem 25.20, we know that $\mathcal{H}([b])$ is a closed subspace of $\mathcal{H}(b)$, whence we conclude that the closure of $\mathcal{H}(\bar{b})$ in $\mathcal{H}(b)$ is contained in $\mathcal{H}([b])$. Using once more Theorem 25.20, it only remains to show that every function in $\mathcal{H}(b)$ that is orthogonal to $\mathcal{H}(\bar{b})$ belongs to $[b]\mathcal{H}(\Theta)$.

To continue the argument, let f be a function in $\mathcal{H}(b)$ and assume that f is orthogonal to $\mathcal{H}(\bar{b})$. Since $f \in \mathcal{H}(b)$, by Corollary 20.2, there exists a function $g \in L^2(\rho)$ such that

$$T_{\bar{b}}f = K_\rho(g) = P_+(\rho g).$$

Now, take any function $h \in H^2(\rho)$ and let $k = K_\rho(h)$. Then $k \in \mathcal{H}(\bar{b})$ and it follows from Theorem 13.21 that

$$T_{\bar{b}}k = T_{\bar{b}}K_\rho h = K_\rho(\bar{b}h).$$

Using the fact that f is orthogonal to $\mathcal{H}(\bar{b})$ and applying Theorems 17.8 and 25.1, we obtain

$$\begin{aligned} 0 &= \langle f, k\rangle_b \\ &= \langle f, k\rangle_2 + \langle T_{\bar{b}}f, T_{\bar{b}}k\rangle_{\bar{b}} \\ &= \langle f, P_+(\rho h)\rangle_2 + \langle K_\rho g, K_\rho(\bar{b}h)\rangle_{\bar{b}} \\ &= \langle f, \rho h\rangle_2 + \langle g, \bar{b}h\rangle_{L^2(\rho)} \\ &= \langle f, h\rangle_{L^2(\rho)} + \langle gb, h\rangle_{L^2(\rho)} \\ &= \langle f + gb, h\rangle_{L^2(\rho)}. \end{aligned}$$

But, since this relation holds for all functions $h \in H^2(\rho)$, and since $H^2(\rho) = L^2(\rho)$ (Theorem 25.1), we deduce that $f + gb = 0$ in $L^2(\rho)$. Therefore, we have

$$f(1 - |b|^2) + bg(1 - |b|^2) = 0$$

almost everywhere on $\mathbb{T}$, which implies that

$$\frac{f}{b} = \bar{b}f - (1 - |b|^2)g = \bar{b}f - \rho g,$$

because $b \neq 0$ almost everywhere on $\mathbb{T}$. The last equality implies that the function f/b belongs to $L^2(\mathbb{T})$ and, by the definition of g, we have

$$P_+\left(\frac{f}{b}\right) = P_+(\bar{b}f) - P_+(\rho g) = T_{\bar{b}}f - K_\rho g = 0. \tag{25.10}$$

Since

$$\frac{f}{[b]} = \frac{f}{b}\,\Theta,$$

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the function $f/[b]$ also belongs to $L^2(\mathbb{T})$, and since $[b]$ is outer, Corollary 4.28 implies that $f/[b]$ is in H^2. Then, using (25.10), we get

$$T_{\bar{\Theta}}\left(\frac{f}{[b]}\right) = P_+\left(\bar{\Theta}\frac{f}{[b]}\right) = P_+\left(\frac{f}{b}\right) = 0,$$

which means that $f/[b]$ belongs to the kernel of $T_{\bar{\Theta}}$. It remains to apply Theorem 12.19 to deduce that $f/[b] \in K_\Theta$, which means that $f \in [b]K_\Theta$. That concludes the proof of the first assertion.

For the second assertion, note that $\mathcal{H}(\bar{b})$ is dense in $\mathcal{H}(b)$ if and only if $\mathcal{H}([b]) = \mathcal{H}(b)$, which is equivalent by Theorem 25.20 to $K_\Theta = \{0\}$. This last identity precisely means that Θ is a constant of modulus one, that is, b is outer. □

25.8 A characterization of $\mathcal{H}(b)$

In this section, we study an analog of Theorem 17.24, which characterizes the spaces $\mathcal{H}(b)$ when b is extreme.

Theorem 25.22 *Let $\mathcal{H}$ be a Hilbert space contained contractively in H^2. Then the following assertions are equivalent.*

(i) *$\mathcal{H}$ is S^*-invariant (and T denotes the restriction of S^* to $\mathcal{H}$), the operator $I - TT^*$ is an operator of rank one and we have*

$$\|Tf\|_{\mathcal{H}}^2 = \|f\|_{\mathcal{H}}^2 - |f(0)|^2 \qquad (f \in \mathcal{H}). \tag{25.11}$$

(ii) *There is an extreme point b in the closed unit ball of H^∞, unique up to a unimodular constant, such that $\mathcal{H} \doteq \mathcal{H}(b)$.*

Proof (i) $\Longrightarrow$ (ii) According to Theorem 16.29, we know that $\mathcal{H}$ is contained contractively in H^2 and, if $\mathcal{M}$ denotes its complementary space, then S acts as a contraction on $\mathcal{M}$. Now the strategy of the proof is quite simple and quite similar to the strategy of the proof of Theorem 23.22. We show that S acts as an isometry on $\mathcal{M}$. Then we apply Theorem 17.24 to deduce that there exists a function b in the closed unit ball of H^∞ such that $\mathcal{M} \doteq \mathcal{M}(b)$, and Corollary 16.27 enables us to conclude that $\mathcal{H} \doteq \mathcal{H}(b)$. To show that S acts as an isometry, we decompose the proof into several steps, 10 in total.

Step 1: Let $k_0^{\mathcal{H}}$ be the unique vector in $\mathcal{H}$ such that

$$f(0) = \langle f, k_0^{\mathcal{H}}\rangle_{\mathcal{H}} \qquad (f \in \mathcal{H}).$$

Then $I - T^*T = k_0^{\mathcal{H}} \otimes k_0^{\mathcal{H}}$.

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Let $f \in \mathcal{H}$. Then, according to Lemma 2.16 and (25.11), we have

$$\begin{aligned} f \in \ker(I - T^*T) \quad &\Longleftrightarrow \quad \|Tf\|_{\mathcal{H}} = \|f\|_{\mathcal{H}} \\ &\Longleftrightarrow \quad f(0) = 0 \\ &\Longleftrightarrow \quad f \perp k_0^{\mathcal{H}}, \end{aligned}$$

whence $\ker(I - T^*T) = (\mathbb{C}k_0^{\mathcal{H}})^{\perp}$. Thus, we get $\mathcal{R}(I - T^*T) = \mathbb{C}k_0^{\mathcal{H}}$ and $I - T^*T$ is a rank-one operator whose range is spanned by $k_0^{\mathcal{H}}$. Since this operator is positive and self-adjoint, we get

$$I - T^*T = ck_0^{\mathcal{H}} \otimes k_0^{\mathcal{H}},$$

for some positive real constant c. It remains to show that $c = 1$. On the one hand, we have

$$\begin{aligned} \|I - T^*T\| &= \sup_{f \in \mathcal{H},\, \|f\|_{\mathcal{H}} \le 1} |\langle (I - T^*T)f,\ f\rangle_{\mathcal{H}}| \\ &= \sup_{f \in \mathcal{H},\, \|f\|_{\mathcal{H}} \le 1} (\|f\|_{\mathcal{H}}^2 - \|Tf\|_{\mathcal{H}}^2) \\ &= \sup_{f \in \mathcal{H},\, \|f\|_{\mathcal{H}} \le 1} |\langle f, k_0^{\mathcal{H}}\rangle_{\mathcal{H}}|^2 \\ &= \|k_0^{\mathcal{H}}\|_{\mathcal{H}}^2, \end{aligned}$$

and, on the other, we have $\|I - T^*T\| = c\|k_0^{\mathcal{H}}\|^2$, whence $c = 1$, which ends the proof of Step 1.

Step 2: Let f_0 be the unique vector in $\mathcal{H}$ such that $I - TT^* = f_0 \otimes f_0$. Then

$$T^*f_0 = \frac{\langle f_0, Tk_0^{\mathcal{H}}\rangle_{\mathcal{H}}}{\|f_0\|_{\mathcal{H}}^2} k_0^{\mathcal{H}}.$$

Using $T^*(I - TT^*) = (I - T^*T)T^*$, we have

$$T^*f_0 \otimes f_0 = (k_0^{\mathcal{H}} \otimes k_0^{\mathcal{H}})T^* = k_0^{\mathcal{H}} \otimes Tk_0^{\mathcal{H}}.$$

Thus, for every $f \in \mathcal{H}$, we have

$$\langle f, f_0\rangle_{\mathcal{H}} T^*f_0 = \langle f, Tk_0^{\mathcal{H}}\rangle_{\mathcal{H}} k_0^{\mathcal{H}}. \tag{25.12}$$

In particular, this equality with $f = f_0$ gives

$$\|f_0\|_{\mathcal{H}}^2 T^*f_0 = \langle f_0, Tk_0^{\mathcal{H}}\rangle_{\mathcal{H}} k_0^{\mathcal{H}},$$

which concludes the proof of Step 2.

Step 3: If $1 \notin \mathcal{H}$, then there exist nonzero constants $c_1, c_2 \in \mathbb{C}$ such that $Sf_0 = c_1 k_0^{\mathcal{H}} + c_2$.

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Put $\alpha = \langle f_0, Tk_0^{\mathcal{H}} \rangle_{\mathcal{H}} k_0^{\mathcal{H}} / \|f_0\|_{\mathcal{H}}^2$. Then, according to Step 2, we have $T^* f_0 = \alpha k_0^{\mathcal{H}}$. Note that $\alpha \neq 0$. Indeed, applying (25.12) to $f = Tk_0^{\mathcal{H}}$ gives

$$\langle Tk_0^{\mathcal{H}}, f_0 \rangle_{\mathcal{H}} T^* f_0 = \|Tk_0^{\mathcal{H}}\|_{\mathcal{H}}^2 k_0^{\mathcal{H}},$$

whence

$$|\langle Tk_0^{\mathcal{H}}, f_0 \rangle_{\mathcal{H}}|^2 = \langle Tk_0^{\mathcal{H}}, f_0 \rangle_{\mathcal{H}} \langle T^* f_0, k_0^{\mathcal{H}} \rangle_{\mathcal{H}} = \|Tk_0^{\mathcal{H}}\|_{\mathcal{H}}^2.$$

But, since $1 \notin \mathcal{H}$, we have $Tk_0^{\mathcal{H}} \neq 0$, which implies that $\langle Tk_0^{\mathcal{H}}, f_0 \rangle_{\mathcal{H}} \neq 0$. Thus, $\alpha \neq 0$.

Now, using $(I - TT^*) f_0 = \|f_0\|_{\mathcal{H}}^2 f_0$, we get

$$(1 - \|f_0\|_{\mathcal{H}}^2) f_0 = TT^* f_0 = \alpha T k_0^{\mathcal{H}}.$$

Since $\alpha T k_0^{\mathcal{H}} \neq 0$, we necessarily have $\|f_0\|_{\mathcal{H}} \neq 1$. Hence,

$$f_0 = \frac{\alpha}{1 - \|f_0\|_{\mathcal{H}}^2} Tk_0^{\mathcal{H}} = c_1 Tk_0^{\mathcal{H}},$$

where $c_1 = \alpha/(1 - \|f_0\|_{\mathcal{H}}^2) \neq 0$. Thus,

$$\begin{aligned} Sf_0 &= S(c_1 Tk_0^{\mathcal{H}}) \\ &= SS^*(c_1 Tk_0^{\mathcal{H}}) \\ &= c_1 k_0^{\mathcal{H}} - c_1 k_0^{\mathcal{H}}(0) \\ &= c_1 k_0^{\mathcal{H}} + c_2, \end{aligned}$$

where $c_1 \neq 0$ and $c_2 = -c_1 k_0^{\mathcal{H}}(0) = -c_1 \|k_0^{\mathcal{H}}\|_{\mathcal{H}}^2 \neq 0$.

Step 4: S acts as an isometry on $\mathcal{M}$ (case $1 \notin \mathcal{H}$).

Let $f \in \mathcal{H}$ and $g \in \mathcal{M}$. Write

$$f = (I - TT^*) f + TT^* f = \langle f, f_0 \rangle_{\mathcal{H}} f_0 + TT^* f = \lambda f_0 + TT^* f,$$

where $\lambda = \langle f, f_0 \rangle_{\mathcal{H}}$. Then

$$\begin{aligned} \|g + f\|_2^2 &= \|g + \lambda f_0 + TT^* f\|_2^2 \\ &= \|g\|_2^2 + \|\lambda f_0 + TT^* f\|_2^2 + 2\,\Re \langle g, \lambda f_0 + TT^* f \rangle_2. \end{aligned}$$

But

$$\begin{aligned} \langle g, \lambda f_0 + TT^* f \rangle_2 &= \langle g, S^* T^* f \rangle_2 + \langle g, \lambda f_0 \rangle_2 \\ &= \langle Sg, T^* f \rangle_2 + \langle g, \lambda f_0 \rangle_2 \end{aligned}$$

and, using Step 3, we also have

$$\begin{aligned} \langle g, \lambda f_0 \rangle_2 &= \langle zg, \lambda z f_0 \rangle_2 = \langle zg, \lambda c_1 k_0^{\mathcal{H}} + \lambda c_2 \rangle_2 \\ &= \langle zg, \lambda c_1 k_0^{\mathcal{H}} \rangle_2 = \langle zg, \lambda h_0 \rangle_2, \end{aligned}$$

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with $h_0 = c_1 k_0^{\mathcal{H}} \in \mathcal{H}$. Thus,

$$\langle g, \lambda f_0 + TT^* f\rangle_2 = \langle zg,\ T^* f + \lambda h_0\rangle_2$$

and we obtain

$$\|g + f\|_2^2 = \|zg + T^* f + \lambda h_0\|_2^2 + \|\lambda f_0 + TT^* f\|_2^2 - \|T^* f + \lambda f_0\|_2^2.$$

This gives

$$\begin{aligned}&\|g + f\|_2^2 - \|f\|_{\mathcal{H}}^2 \\ &\quad = \|zg + T^* f + \lambda h_0\|_2^2 + \|\lambda f_0 + TT^* f\|_2^2 - \|T^* f + \lambda h_0\|_2^2 - \|f\|_{\mathcal{H}}^2.\end{aligned}$$

Now, we prove that

$$\|\lambda f_0 + TT^* f\|_2^2 - \|T^* f + \lambda h_0\|_2^2 = \|f\|_{\mathcal{H}}^2 - \|T^* f + \lambda h_0\|_{\mathcal{H}}^2. \tag{25.13}$$

Using Step 3, we have

$$f_0 = S^* S f_0 = S^*(c_1 k_0^{\mathcal{H}} + c_2) = c_1 S^* k_0^{\mathcal{H}} = T h_0.$$

Thus, on the one hand, we have

$$\begin{aligned}\|\lambda f_0 + TT^* f\|_2^2 - \|T^* f + \lambda h_0\|_2^2 &= \|T(\lambda h_0 + T^* f)\|_2^2 - \|\lambda f_0 + T^*\|_2^2 \\ &= |(\lambda h_0 + T^* f)(0)|^2,\end{aligned}$$

and, on the other, we also have

$$\begin{aligned}\|f\|_{\mathcal{H}}^2 - \|T^* f + \lambda h_0\|_{\mathcal{H}}^2 &= \|\lambda f_0 + TT^* f\|_{\mathcal{H}}^2 - \|T^* f + \lambda h_0\|_{\mathcal{H}}^2 \\ &= \|T(\lambda h_0 + T^* f)\|_{\mathcal{H}}^2 - \|\lambda h_0 + T^* f\|_{\mathcal{H}}^2 \\ &= |(\lambda h_0 + T^* f)(0)|^2.\end{aligned}$$

The last equality follows from (25.11). This concludes the proof of (25.13). Hence,

$$\begin{aligned}\|g + f\|_2^2 - \|f\|_{\mathcal{H}}^2 &= \|zg + T^* f + \lambda h_0\|_2^2 - \|T^* f + \lambda h_0\|_{\mathcal{H}}^2 \\ &\le \sup_{h \in \mathcal{H}} (\|zg + h\|_2^2 - \|h\|_{\mathcal{H}}^2) \\ &= \|zg\|_{\mathcal{M}}^2 \qquad (f \in \mathcal{H}).\end{aligned}$$

This gives $\|g\|_{\mathcal{M}}^2 \le \|zg\|_{\mathcal{M}}^2$, which with Theorem 16.29 implies that S is an isometry on $\mathcal{M}$.

From now on, we assume that $1 \in \mathcal{H}$ and $n \ge 1$ is such that $f_0 = z^{n-1} \tilde{f}_0$, with $\tilde{f}_0 \in H^2$ and $\tilde{f}_0(0) \ne 0$.

Step 5: $\|1\|_{\mathcal{H}} = 1$ *and* $k_0^{\mathcal{H}} = 1$.

Since $1 \in \mathcal{H}$, using (25.11), we have

$$\|T1\|_{\mathcal{H}}^2 = \|1\|_{\mathcal{H}}^2 - 1.$$

Hence, $\|1\|_{\mathcal{H}} = 1$, because $T1 = 0$. But, by Corollary 16.28,

$$i_{\mathcal{H}}^*(1) = 1 \quad \text{and} \quad i_{\mathcal{M}}^*(1) = 0,$$

where $i_{\mathcal{H}}$ (respectively $i_{\mathcal{M}}$) denotes the canonical injection of $\mathcal{H}$ (respectively $\mathcal{M}$) into H^2. Thus, for each $f \in \mathcal{H}$, we get

$$\begin{aligned}\langle f, 1\rangle_{\mathcal{H}} &= \langle i_{\mathcal{H}}(f), 1\rangle_{\mathcal{H}} \\ &= \langle f, i_{\mathcal{H}}^*(1)\rangle_2 \\ &= \langle f, 1\rangle_2 = f(0) \\ &= \langle f, k_0^{\mathcal{H}}\rangle_{\mathcal{H}}.\end{aligned}$$

By the uniqueness of reproducing kernels, we deduce that $k_0^{\mathcal{H}} = 1$.

Step 6: We have

$$T^*h = S(h - \langle h, f_0\rangle_{\mathcal{H}} f_0) \qquad (h \in \mathcal{H}).$$

Since $I - f_0 \otimes f_0 = TT^*$, we get

$$\begin{aligned}S(I - f_0 \otimes f_0)h = STT^*h &= SS^*T^*Th \\ &= T^*h - (T^*h)(0) \qquad (h \in \mathcal{H}).\end{aligned}$$

But, according to Step 5,

$$(T^*h)(0) = \langle T^*h, 1\rangle_{\mathcal{H}} = \langle h, T1\rangle_{\mathcal{H}} = 0.$$

Thus,

$$S(I - f_0 \otimes f_0)h = T^*h.$$

Step 7: The function $\tilde{f}_0$ belongs to $\mathcal{H}$ and

$$T^{*n}1 = z^n(1 - \overline{\tilde{f}_0(0)}\tilde{f}_0). \tag{25.14}$$

Moreover, if $n \geq 2$, we have

$$T^{*k}1 = z^k \qquad (1 \leq k \leq n-1). \tag{25.15}$$

Since $\tilde{f}_0 = S^{*n-1} f_0 = T^{n-1} f_0$, we surely have $\tilde{f}_0 \in \mathcal{H}$. Now, using Step 6, we have

$$T^*1 = S(1 - \langle 1, f_0\rangle_{\mathcal{H}} f_0)$$

and

$$\langle 1, f_0\rangle_{\mathcal{H}} = \overline{f_0(0)},$$

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because $1 = k_0^{\mathcal{H}}$. Hence,

$$T^*1 = z(1 - \overline{f_0(0)}f_0). \tag{25.16}$$

If $n = 1$, we have $f_0 = \tilde{f}_0$. This proves (25.14) in the case $n = 1$.

Now, assume that $n > 1$. We first prove (25.15) by induction. Since $n > 1$, we have $f_0(0) = 0$, and thus, by (25.16), we get $T^*1 = z$. Assume that $T^{*k}1 = z^k$, for some $k < n - 1$. Thus, using Step 6, we obtain

$$\begin{aligned} T^{*(k+1)}1 = T^* z^k &= z(z^k - \langle z^k, f_0\rangle_{\mathcal{H}} f_0) \\ &= z^{k+1} - \langle z^k, f_0\rangle_{\mathcal{H}} z f_0. \end{aligned}$$

But

$$\langle z^k, f_0\rangle_{\mathcal{H}} = \langle T^{*k}1, f_0\rangle_{\mathcal{H}} = \langle 1, T^k f_0\rangle_{\mathcal{H}} = \overline{(T^k f_0)(0)}$$

and

$$T^k f_0 = S^{*k} f_0 = P_+(\bar{z}^k z^{n-1} f_0) = P_+(z^{n-k-1} f_0) = z^{n-k-1} f_0.$$

Hence, $(T^k f_0)(0) = 0$, because $n - k - 1 > 0$, and then

$$T^{*(k+1)}1 = z^{k+1}.$$

Therefore,

$$T^{*k}1 = z^k \qquad (1 \le k \le n-1).$$

Using Step 6 once more, we have

$$\begin{aligned} T^{*n}1 = T^* z^{n-1} &= z^n - \langle z^{n-1}, f_0\rangle_{\mathcal{H}} z f_0 \\ &= z^n - \langle z^{n-1}, f_0\rangle_{\mathcal{H}} z^n \tilde{f}_0 \\ &= z^n(1 - \langle z^{n-1}, f_0\rangle_{\mathcal{H}} \tilde{f}_0). \end{aligned}$$

It just remains to note that

$$\langle z^{n-1}, f_0\rangle_{\mathcal{H}} = \langle T^{*n-1}1, f_0\rangle_{\mathcal{H}} = \langle 1, T^{n-1} f_0\rangle_{\mathcal{H}} = \langle 1, \tilde{f}_0\rangle_{\mathcal{H}} = \overline{\tilde{f}_0(0)}.$$

Step 8: $z^{n-1} \in \mathcal{H}$ and $\langle g, z^{n-1}\rangle_2 = 0$, for every $g \in \mathcal{M}$.

According to (25.15), we have $z^k \in \mathcal{H}$, for every $0 \le k \le n-1$. Moreover, $\|z^k\|_{\mathcal{H}} = 1$. Indeed, this is true for $k = 0$ by Step 5. Assume that, for some $0 \le k \le n-1$, $\|z^k\|_{\mathcal{H}} = 1$. Hence, considering (25.11), we get

$$\|z^{k+1}\|_{\mathcal{H}}^2 = \|T z^{k+1}\|_{\mathcal{H}}^2 = \|z^k\|_{\mathcal{H}}^2 = 1.$$

By induction, we then have $\|z^k\|_{\mathcal{H}} = 1$, for every $0 \le k \le n-1$. It follows from Corollary 16.28 that $i_{\mathcal{H}}^*(z^{n-1}) = z^{n-1}$ and $i_{\mathcal{M}}^*(z^{n-1}) = 0$. Hence, if $g \in \mathcal{M}$, we have

$$\langle g, z^{n-1}\rangle_2 = \langle i_{\mathcal{M}}(g), z^{n-1}\rangle_2 = \langle g, i_{\mathcal{M}}^*(z^{n-1})\rangle_{\mathcal{M}} = 0.$$

Step 9: S acts as an isometry on $\mathcal{M}$ (case $1 \in \mathcal{H}$).

Let $f \in \mathcal{H}$ and $g \in \mathcal{M}$. Argue as in Step 4, and write $f = \lambda f_0 + TT^* f$, with $\lambda = \langle f, f_0 \rangle_{\mathcal{H}}$. Then

$$\begin{aligned}\|g+f\|_2^2 &= \|g + \lambda f_0 + TT^* f\|_2^2 \\ &= \|g\|_2^2 + \|\lambda f_0 + TT^* f\|_2^2 + 2\,\Re\langle g, \lambda f_0 + TT^* f\rangle_2.\end{aligned}$$

But

$$\langle g, \lambda f_0 + TT^* f\rangle_2 = \langle g, \lambda f_0\rangle_2 + \langle Sg, T^* f\rangle_2.$$

Write $h = -\overline{\tilde{f}_0(0)}^{(-1)} T^{*n} 1$. According to Step 6, we have

$$h = -\overline{\tilde{f}_0(0)}^{(-1)} z^n + z^n \tilde{f}_0 = -\overline{\tilde{f}_0(0)}^{(-1)} z^n + z f_0.$$

Hence,

$$\langle zg, \lambda h\rangle_2 = -\tilde{f}_0(0)^{-1} \bar{\lambda} \langle zg, z^n\rangle_2 + \langle zg, \lambda z f_0\rangle_2 = \langle g, \lambda f_0\rangle_2,$$

because $\langle zg, z^n\rangle_2 = \langle g, z^{n-1}\rangle_2 = 0$, according to Step 8. Therefore,

$$\langle g, \lambda f_0 + TT^* f\rangle_2 = \langle zg, \lambda h\rangle_2 + \langle zg, T^* f\rangle_2 = \langle zg, \lambda h + T^* f\rangle_2.$$

We thus get

$$\|g+f\|_2^2 = \|zg + \lambda h + T^* f\|_2^2 + \|\lambda f_0 + TT^* f\|_2^2 - \|\lambda h + T^* f\|_2^2,$$

and then

$$\begin{aligned}&\|g+f\|_2^2 - \|f\|_{\mathcal{H}}^2 \\ &\quad = \|zg + \lambda h + T^* f\|_2^2 + \|\lambda f_0 + TT^* f\|_2^2 - \|\lambda h + T^* f\|_2^2 - \|f\|_{\mathcal{H}}^2.\end{aligned}$$

Now, we prove that

$$\|\lambda f_0 + TT^* f\|_2^2 - \|T^* f + \lambda h\|_2^2 = \|\lambda f_0 + TT^* f\|_{\mathcal{H}}^2 - \|T^* f + \lambda h\|_{\mathcal{H}}^2.$$

Denote the right-hand side by A and the left-hand side by B. Then, using Step 6, we obtain

$$\begin{aligned}A &= \|f\|_2^2 - \|z(f - \lambda f_0 - \lambda \overline{\tilde{f}_0(0)}^{(-1)} z^n + \lambda f_0)\|_2 \\ &= \|f\|_2^2 - \|f - \lambda \overline{\tilde{f}_0(0)}^{(-1)} z^n)\|_2 \\ &= \|f\|_2^2 - (\|f\|_2^2 + |\lambda|^2 |\tilde{f}_0(0)|^{-2} - 2\,\Re(\bar{\lambda} \tilde{f}_0(0)^{-1} \langle f, z^{n-1}\rangle_2)) \\ &= -|\lambda|^2 |\tilde{f}_0(0)|^{-2} + 2\,\Re(\bar{\lambda} \tilde{f}_0(0)^{-1} \langle f, z^{n-1}\rangle_2).\end{aligned}$$

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Moreover, using (25.11), we have

$$\begin{aligned} B &= \|f\|_{\mathcal{H}}^2 - \|T^*f + \lambda h\|_{\mathcal{H}}^2 \\ &= \|f\|_{\mathcal{H}}^2 - \|TT^*f + \lambda Th\|_{\mathcal{H}}^2 - |(T^*f)(0) + \lambda h(0)|. \end{aligned}$$

But $(T^*f)(0) = h(0) = 0$ and $Th = S^*h = -\overline{\tilde{f}_0(0)}^{(-1)} z^{n-1} + f_0$. Hence,

$$TT^*f + \lambda Th = TT^*f + \lambda f_0 - \lambda \overline{\tilde{f}_0(0)}^{(-1)} z^{n-1},$$

which gives

$$\begin{aligned} B &= \|f\|_{\mathcal{H}}^2 - \|f - \lambda \overline{\tilde{f}_0(0)}^{(-1)} z^{n-1}\|_{\mathcal{H}}^2 \\ &= \|f\|_{\mathcal{H}}^2 - [\, \|f\|_{\mathcal{H}}^2 + |\lambda|^2 |\tilde{f}_0(0)|^{-2} \|z^{n-1}\|_{\mathcal{H}}^2 \\ &\quad -2\,\Re(\bar{\lambda} \overline{\tilde{f}_0(0)}^{-1} \langle f, z^{n-1}\rangle_{\mathcal{H}})\,] \\ &= -|\lambda|^2 |\tilde{f}_0(0)|^{-2} + 2\,\Re(\bar{\lambda} \overline{\tilde{f}_0(0)}^{-1} \langle f, z^{n-1}\rangle_{\mathcal{H}}). \end{aligned}$$

Considering

$$\begin{aligned} \langle f, z^{n-1}\rangle_{\mathcal{H}} &= \langle i_{\mathcal{H}}(f), z^{n-1}\rangle_{\mathcal{H}} \\ &= \langle f, i_{\mathcal{H}}^*(z^{n-1})\rangle_2 \\ &= \langle f, z^{n-1}\rangle_2, \end{aligned}$$

we deduce that $A = B$. This reveals that

$$\begin{aligned} \|g + f\|_2^2 - \|f\|_{\mathcal{H}}^2 &= \|zg + \lambda h + T^*f\|_2 - \|\lambda h + T^*f\|_{\mathcal{H}}^2 \\ &\le \sup_{u \in \mathcal{H}} (\|zg + u\|_2^2 - \|u\|_{\mathcal{H}}^2) \\ &= \|zg\|_{\mathcal{M}}^2 \qquad (f \in \mathcal{H}). \end{aligned}$$

This gives $\|g\|_{\mathcal{M}}^2 \le \|zg\|_{\mathcal{M}}^2$, which, in the light of Theorem 16.29, ensures that S is an isometry on $\mathcal{M}$.

Step 10: There is an extreme point b in the closed unit ball of H^∞, unique up to a unimodular constant, such that $\mathcal{H} \doteq \mathcal{H}(b)$.

According to Steps 4 and 9, S acts as an isometry on $\mathcal{M}$. Therefore, Theorem 17.24 implies that there exists a function b in the closed unit ball of H^∞ such that $\mathcal{M} \doteq \mathcal{M}(b)$. Now, Corollary 16.27 implies that $\mathcal{H} \doteq \mathcal{H}(b)$. Finally, b is an extreme point of the closed unit ball of H^∞, since $I - T^*T$ is an operator of rank one. Remember that, according to Theorem 23.14, if b is a nonextreme point of the closed unit ball of H^∞, then the operator $I - T^*T$ is of rank two.

That finishes the proof of the implication (i) $\Longrightarrow$ (ii).

(ii) $\Longrightarrow$ (i) This follows from Corollary 18.23 and Theorem 25.11.

This completes the proof of Theorem 25.22. □

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Notes on Chapter 25

Section 25.1

The formula for the norm $\|S^*b\|_b$ that appears in Corollary 25.3 is due to Sarason [160]. However, the proof presented here comes from [166].

Section 25.2

Theorem 25.4 is due to Lotto and Sarason [123, theorem 5.1]. The equivalence between (i) and (iv) in Corollary 25.5 is due to Sarason [160] and the equivalence of (i) and (ii) is due to de Branges and Rovnyak [65]. Corollary 25.6 is due to Lotto and Sarason [123, lemma 8.1].

Section 25.3

Theorem 25.7 comes from Sarason's book [166], but the immediate corollary that is presented in Exercise 25.3.1 is due to Lotto and Sarason [123, corollary 5.2]. Corollaries 25.9 and 25.10 appear in a paper of Sarason [160], but the proofs come from his book.

Section 25.4

The determination of the defect operator of the contraction X_b made in Theorem 25.11 and Corollary 25.12 follows Sarason [160]. In that paper, he identifies the characteristic function (in the language of Sz.-Nagy and Foiaş) of X_b; see also [139]. Corollaries 25.13 and 25.14 are due to Lotto and Sarason [123].

Corollary 25.15 is due to de Branges and Rovnyak [65, theorem 16]. More precisely, they proved that (25.8) is an equality if and only if $b \notin \mathcal{H}(b)$, and we know this condition is equivalent to b being an extreme point of the closed unit ball of H^∞. See also Nikolskii and Vasyunin [139, corollary 8.8] for a generalization of this result in the vector-valued situation.

As already mentioned, de Branges called (25.8) the inequality for difference quotients.

Section 25.5

Theorem 25.16 has been proved by Lotto and Sarason [123, theorem 5.3], who applied this to the problem of multipliers of $\mathcal{H}(b)$. See Section 26.2 for results in this direction. It should be noted that Suárez [181] described the invariant subspaces of X_b but, as already mentioned, the problem of determining the cyclic vector of X_b (in the extreme case) is an open problem. Corollary 25.18 is also due to Lotto and Sarason [123, theorem 5.4].

Section 25.6

Theorem 25.20 is a slight generalization of a result of Lotto and Sarason [123, theorem 6.1].

Section 25.7

Theorem 25.21 has been proved by Lotto and Sarason [123, theorem 6.2].

Section 25.8

The characterization of the $\mathcal{H}(b)$ spaces in the extreme case obtained in Theorem 25.22 is due to de Branges and Rovnyak [64, appdx, theorem 6] and [65, theorem 15]. The proof of de Branges and Rovnyak is based on the construction of an auxiliary Hilbert space of analytic functions. Our proof here is different and is inspired by the analogous result in the nonextreme case due to Guyker [96]; see Theorem 23.22.

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