Canad. Math. Bull. Vol. 26 (1), 1983

CLOSURES OF EQUIVALENCE CLASSES OF TRIVECTORS OF AN EIGHT-DIMENSIONAL COMPLEX VECTOR SPACE

BY
DRAGOMIR Ž. DJOKOVIĆ

ABSTRACT. G. B. Gurevič enumerated all the orbits of $GL_8(\mathbb{C})$ in $\Lambda^3(\mathbb{C}^8)$. There are precisely 23 orbits (including the trivial orbit). For each of these orbits, we determine its closure (for the ordinary topology).

Introduction. We shall denote by V an eight-dimensional complex vector space with a basis e_k, $1 \le k \le 8$, and by G the general linear group of V. The elements of the third exterior power $\Lambda^3 V$ will be called trivectors. The action of G in V extends canonically to $\Lambda^3 V$. Explicitly we have

$$a \cdot (x \wedge y \wedge z) = a(x) \wedge a(y) \wedge a(z)$$

for $a \in G$ and $x, y, z \in V$.

In 1935 it was shown by Gurevič [3] that there are precisely 23 orbits of G in $\Lambda^3 V$, and he has determined their representatives. We shall denote these orbits by roman numerals I–XXIII as in [4] and [1]. (In the case when the space V has dimension nine the classification problem was solved recently by Vinberg and Elašvili [6]). We shall say that two trivectors are *equivalent* if they belong to the same orbit of G.

The closure of an orbit for the ordinary topology coincides with its Zariski closure. It is also well known that a closure of an orbit is a union of this orbit and some orbits of lower dimension, see e.g. [5, p. 60]. In this note we shall determine the closures of all 23 orbits of G in $\Lambda^3 V$. We shall write $i \to j$ if the jth orbit lies in the closure of the ith orbit. The negation of $i \to j$ will be written as $i \not\to j$.

Statement of the result. In some arguments we shall need some results of our paper [1]. For that reason we shall use the same representatives for the orbits I–XXIII as in [1]. The orbit I is the trivial orbit consisting of the zero trivector only. The representatives of orbits are listed in Table I where we use the notation e_{ijk} for $e_i \wedge e_j \wedge e_k$. We have also listed in this table the dimensions of these orbits, see [1].

Received by the editors June 27, 1981.
This work was supported in part by NSERC Grant A-5285.
1980 AMS Subject Classification Numbers: Primary 15A75; Secondary 20G20.

https://doi.org/10.4153/CMB-1983-014-8 Published online by Cambridge University Press

Table I

Orbit	Representative	Dimension
I	0	0
II	e_{123}	16
III	$e_{123}+e_{145}$	25
IV	$e_{124}+e_{135}+e_{236}$	31
V	$e_{123}+e_{456}$	32
VI	$e_{123}+e_{145}+e_{167}$	28
VII	$e_{125}+e_{136}+e_{147}+e_{234}$	35
VIII	$e_{134}+e_{256}+e_{127}$	38
IX	$e_{125}+e_{346}+e_{137}+e_{247}$	41
X	$e_{123}+e_{456}+e_{147}+e_{257}+e_{367}$	42
XI	$e_{127}+e_{138}+e_{146}+e_{235}$	40
XII	$e_{128}+e_{137}+e_{146}+e_{236}+e_{245}$	43
XIII	$e_{135}+e_{246}+e_{147}+e_{238}$	44
XIV	$e_{138}+e_{147}+e_{156}+e_{235}+e_{246}$	46
XV	$e_{128}+e_{137}+e_{146}+e_{247}+e_{256}+e_{345}$	48
XVI	$e_{156}+e_{178}+e_{234}$	41
XVII	$e_{158}+e_{167}+e_{234}+e_{256}$	47
XVIII	$e_{148}+e_{157}+e_{236}+e_{245}+e_{347}$	50
XIX	$e_{134}+e_{234}+e_{156}+e_{278}$	48
XX	$e_{137}+e_{237}+e_{256}+e_{148}+e_{345}$	52
XXI	$e_{138}+e_{147}+e_{245}+e_{267}+e_{356}$	53
XXII	$e_{128}+e_{147}+e_{236}+e_{257}+e_{358}+e_{456}$	55
XXIII	$e_{124}+e_{134}+e_{256}+e_{378}+e_{157}+e_{468}$	56

THEOREM. *The closures of the orbits of G in $\Lambda^3 V$ are as indicated in the diagram on Fig.* 1. (*We have $i \to j$ if and only if there is a downward path from i to j.*)

REMARK 1. The integer attached to an edge of this diagram is the difference between the dimensions of the two orbits represented by the end-points of the edge.

REMARK 2. Given $x \in \Lambda^3 V$ there is a unique minimal subspace W of V such that $x \in \Lambda^3 W$. We say that the integer dim W is the *rank* of x. It is clear that equivalent trivectors have the same rank and hence one can speak about the rank of an orbit. The possible values for the rank are $0, 3, 5, 6, 7$ and 8. The five curves in the diagram separate the orbits of different ranks. The union of all orbits of rank $\leq k$ is closed.

Proof of the theorem: First part. First we justify each edge in our diagram in Fig. 1.

1) We have XXIII $\to$ XXII, X $\to$ IX, V $\to$ IV, III $\to$ II and II $\to$ I. Since XXIII is the open orbit of G, its closure is the whole space $\Lambda^3 V$. In particular this proves that XXIII $\to$ XXII. The reasons in the other four cases are similar. For instance X $\to$ IX is proved as follows. The intersection of the orbit X with

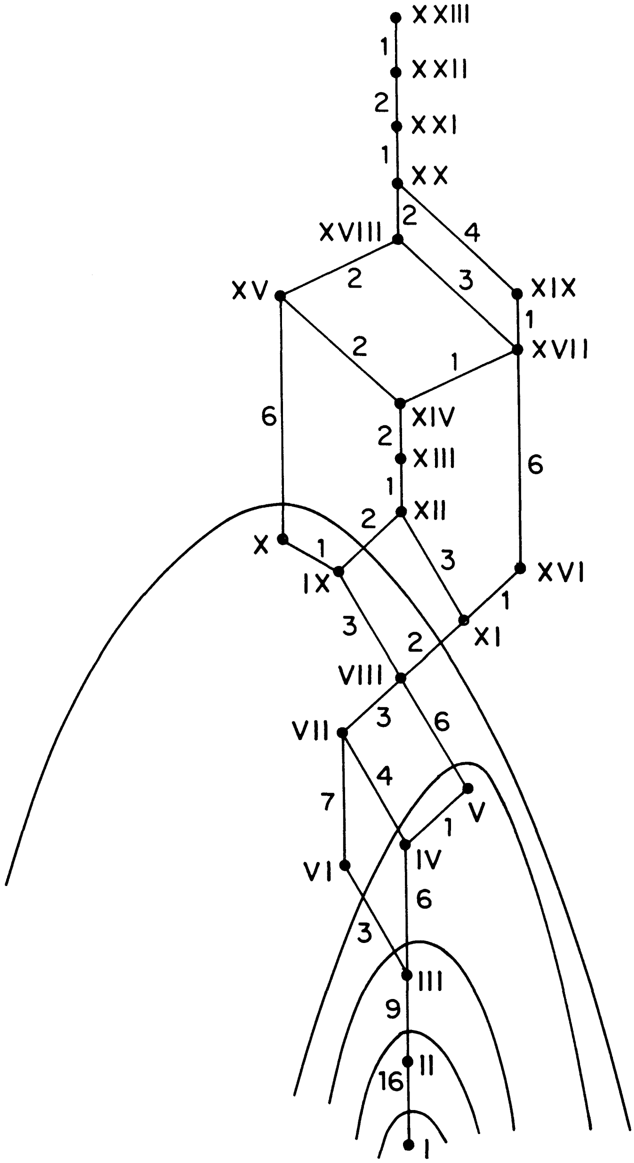

Figure 1

the subspace $\Lambda^3 W$, where $W = \langle e_1, \ldots, e_7 \rangle$, is the open orbit of $GL(W)$ in $\Lambda^3 W$. Since the representative of the orbit IX lies in $\Lambda^3 W$, and the open orbit of $GL(W)$ in $\Lambda^3 W$ is dense in $\Lambda^3 W$, we conclude that $\mathrm{X} \to \mathrm{IX}$.

2) $\mathrm{XXII} \to \mathrm{XXI}$. For $\varepsilon \neq 0$ let $a_\varepsilon \in G$ be defined by specifying the images of basic vectors as follows:

$$e_1 \to \varepsilon e_1,\ e_2 \to e_4,\ e_3 \to -e_2,\ e_4 \to \varepsilon^{-1} e_3,$$

$$e_5 \to -\varepsilon e_6,\ e_6 \to e_5,\ e_7 \to e_8,\ e_8 \to \varepsilon^{-1} e_7.$$

If x is the representative of XXII from Table I then we find that

$$a_\varepsilon \cdot x = e_{147} + e_{138} + e_{245} - \varepsilon e_{468} + e_{267} + e_{356}.$$

When $\varepsilon \to 0$ this trivector has as limit the representative of the orbit XXI, which proves our claim.

3) We have XX→XIX, XVIII→XVII, XVII→XVI, XV→XIV, XV→X, XIV→XIII, XII→XI, XII→IX, XI→VIII, IX→VIII, VIII→V, VII→VI, VII→IV, VI→III and IV→III. In each of these 15 cases the proof is the same as the one given in 2); one has only to indicate how is a_ε defined. The definition of a_ε in each case is given in Table II, where we specify the images $a_\varepsilon(e_k)$ for all basic vectors e_k except those for which $a_\varepsilon(e_k) = e_k$.

4) XX→XVIII. Let $a_\varepsilon \in G$, $\varepsilon \neq 0$, be defined by:

$$e_1 \to e_1 + \varepsilon e_2,\ e_2 \to -e_1,\ e_3 \to e_3,\ e_4 \to e_4 - \varepsilon e_7,$$
$$e_5 \to e_4,\ e_6 \to e_5,\ e_7 \to e_6,\ e_8 \to e_5 + \varepsilon e_8.$$

If x is the representative of the orbit XX from Table I then

$$\begin{aligned} a_\varepsilon \cdot x &= \varepsilon e_{236} + (e_1 + \varepsilon e_2) \wedge (e_4 - \varepsilon e_7) \wedge (e_5 + \varepsilon e_8) - e_{145} + e_3 \wedge (e_4 - \varepsilon e_7) \wedge e_4 \\ &= \varepsilon(e_{236} + e_{245} + e_{157} + e_{148} + e_{347}) + \varepsilon^2(e_{257} + e_{248} - e_{178}) - \varepsilon^3 e_{278}. \end{aligned}$$

Since $\varepsilon^{-1} a_\varepsilon \cdot x$ also belongs to the orbit XXI, and

$$\lim_{\varepsilon \to 0} (\varepsilon^{-1} a_\varepsilon \cdot x) = e_{236} + e_{245} + e_{157} + e_{148} + e_{347}$$

is the representative of the orbit XVIII, our claim is proved.

5) We have XIX→XVII, XVI→XI, XIII→XII and VIII→VII. The proofs in these four cases are similar to the proof in 4). We indicate the

Table II

	a_ε
XX→XIX	$e_1 \to \varepsilon^{-1} e_2,\ e_2 \to \varepsilon^{-1} e_1,\ e_3 \to \varepsilon e_3,\ e_4 \to \varepsilon e_7,\ e_5 \to \varepsilon e_5,\ e_7 \to e_4$
XVIII→XVII	$e_2 \to \varepsilon e_3,\ e_3 \to e_2,\ e_4 \to e_5,\ e_5 \to -e_7,\ e_6 \to -\varepsilon^{-1} e_4,\ e_7 \to e_6$
XVII→XVI	$e_2 \to \varepsilon e_2,\ e_3 \to \varepsilon^{-1} e_3,\ e_6 \to e_8,\ e_8 \to -e_6$
XV→XIV	$e_1 \to \varepsilon^{-1} e_1,\ e_2 \to e_3,\ e_3 \to \varepsilon^{-1} e_4,\ e_4 \to \varepsilon e_6,\ e_5 \to e_2,\ e_6 \to -e_5,\ e_7 \to \varepsilon^2 e_7,\ e_8 \to \varepsilon e_8$
XV→X	$e_2 \to e_5,\ e_4 \to -e_7,\ e_5 \to e_6,\ e_6 \to e_4,\ e_7 \to -e_2,\ e_8 \to \varepsilon e_8$
XIV→XIII	$e_1 \to \varepsilon e_1,\ e_2 \to \varepsilon e_2,\ e_3 \to \varepsilon^{-1} e_3,\ e_4 \to \varepsilon^{-1} e_4,\ e_5 \to e_8,\ e_8 \to e_5$
XII→XI	$e_3 \to \varepsilon e_4,\ e_4 \to e_3,\ e_6 \to e_8,\ e_7 \to \varepsilon^{-1} e_6,\ e_8 \to e_7$
XII→IX	$e_1 \to e_3,\ e_3 \to e_4,\ e_4 \to -e_1,\ e_6 \to e_7,\ e_7 \to e_6,\ e_8 \to \varepsilon e_8$
XI→VIII	$e_3 \to -e_6,\ e_6 \to -e_3,\ e_8 \to \varepsilon e_8$
IX→VIII	$e_2 \to \varepsilon e_3,\ e_3 \to e_2,\ e_4 \to e_5,\ e_5 \to \varepsilon^{-1} e_4$
VIII→V	$e_2 \to e_4,\ e_4 \to -e_2,\ e_7 \to \varepsilon e_7$
VII→VI	$e_1 \to \varepsilon^{-1} e_1,\ e_2 \to \varepsilon e_2,\ e_3 \to -\varepsilon e_7,\ e_4 \to \varepsilon e_4,\ e_5 \to e_3,\ e_7 \to e_5$
VII→IV	$e_4 \to e_6,\ e_5 \to e_4,\ e_6 \to e_5,\ e_7 \to \varepsilon e_7$
VI→III	$e_6 \to \varepsilon e_6$
IV→III	$e_3 \to e_4,\ e_4 \to e_3,\ e_6 \to \varepsilon e_6$

https://doi.org/10.4153/CMB-1983-014-8 Published online by Cambridge University Press

Table III

	a_ε
XIX → XVII	$e_1 \to e_1+\varepsilon e_2,\ e_2 \to -e_1,\ e_5 \to e_5-\varepsilon e_7,\ e_6 \to e_6+\varepsilon e_8,\ e_7 \to e_5,\ e_8 \to e_6$
XVI → XI	$e_2 \to e_1-\varepsilon e_5,\ e_3 \to e_2+\varepsilon e_8,\ e_4 \to -e_3+\varepsilon e_7,\ e_5 \to \varepsilon e_4,\ e_7 \to e_2,\ e_8 \to e_3$
XIII → XII	$e_2 \to e_2-\varepsilon e_4,\ e_3 \to e_1+\varepsilon e_3,\ e_4 \to e_2,\ e_5 \to e_7,\ e_6 \to e_5,\ e_7 \to e_6,\ e_8 \to e_6-\varepsilon e_8$
VIII → VII	$e_1 \to e_1+\varepsilon e_4,\ e_2 \to -e_1,\ e_3 \to e_2-\varepsilon e_6,\ e_4 \to e_3+\varepsilon e_5,\ e_5 \to e_2,\ e_6 \to e_3$

definition of a_ε in each case in Table III by specifying the images $a_\varepsilon(e_k)$ whenever they are different from e_k.

6) We have XXI → XX, XVIII → XV and XVII → XIV. The proofs of these claims are based on some results of [1] which we shall now summarize. There is a Z-grading of the simple complex Lie algebra g of type E_8 such that the homogeneous components g_k of g can be identified with the following spaces (V^* denotes the dual of V):

$$g_{-3}=V^*,\qquad g_{-2}=\Lambda^2 V,\qquad g_{-1}=\Lambda^3 V^*,\qquad g_0=V\otimes V^*=\mathrm{End}(V),$$
$$g_1=\Lambda^3 V,\qquad g_2=\Lambda^2 V^*,\qquad g_3=V.$$

Each of these homogeneous components is a g_0-module via the restriction of the adjoint representation of g. If $x\in\Lambda^3 V$ and $x\neq 0$ there exist $h\in g_0$ and $y\in g_{-1}$ such that

$$[x, y]=h,\qquad [h, x]=2x,\qquad [h, y]=-2y.$$

In particular $\langle x, h, y\rangle$ is a simple subalgebra of g, isomorphic to $sl_2(C)$. The eigenvalues of ad h are integers and we denote by $g(j; h)$ the eigenspace of ad h for the eigenvalue $j\in Z$. We set

$$g_k(j; h)=g_k\cap g(j; h).$$

Now let

$$l=\sum_{j\geq 0} g_0(j; h),\qquad m=\sum_{j\geq 2} g_2(j; h).$$

From the theory of $sl_2(C)$-modules it follows that $[x, l]=m$. Note that $x\in g_2(2; h)$ and so $x\in m$. If L is the connected subgroup of $G=GL(V)$ which has l as its Lie algebra then the condition $[x, l]=m$ implies that the orbit $L\cdot x$ is Zariski open in m. Hence the closure of $L\cdot x$ is the whole space m. We infer that every orbit of G in $\Lambda^3 V=g_1$ which meets m is contained in the closure of the orbit $G\cdot x$.

We shall now give the details of the proof of XVIII → XV. Let x be the representative of XVIII from Table I. Then we can choose, see [1],

$$h=\mathrm{diag}(2, 1, 1, 1, 0, 0, 0, -1),$$

where we idenitfy the elements of $g_0 = \mathrm{End}(V)$ with their matrices with respect to the basis e_k, $1 \le k \le 8$. Let us write

$$V_1 = \langle e_1 \rangle, \qquad V_2 = \langle e_2, e_3, e_4 \rangle, \qquad V_3 = \langle e_5, e_6, e_7 \rangle, \qquad V_4 = \langle e_8 \rangle.$$

With these notations we have

$$\begin{aligned} g_1(2; h) &= V_1 \otimes \Lambda^2 V_3 + \Lambda^2 V_2 \otimes V_3 + V_1 \otimes V_2 \otimes V_4, \\ g_1(3; h) &= V_1 \otimes V_2 \otimes V_3 + \Lambda^3 V_2, \\ g_1(4; h) &= V_1 \otimes \Lambda^2 V_2 \end{aligned}$$

and $g_1(j; h) = 0$ for $j > 4$. (Each of the spaces on the right hand sides of these equalities is considered as a subspace of $\Lambda^3 V$ via the obvious canonical maps.) Since

$$m = g_1(2; h) + g_1(3; h) + g_1(4; h),$$

each of the following six trivectors belongs to m:

$$e_{156}, e_{127}, e_{138}, e_{236}, e_{245}, e_{347}.$$

Thus the element

$$y = e_{127} + e_{138} + e_{156} - e_{236} + e_{245} + e_{347}$$

is in m. Let $a \in G$ be defined by:

$$e_1 \to e_1, e_2 \to e_4, e_3 \to e_2, e_4 \to e_5, e_5 \to e_3, e_6 \to e_7, e_7 \to e_6, e_8 \to e_8.$$

Then it is easy to verify that the trivector $a \cdot y$ is precisely the representative of the orbit XV in Table I. Thus the orbit XV meets m and so we have XVIII $\to$ XV.

Now let x be the representative of the orbit XVII. Then by [1] we can choose h as

$$h = \tfrac{1}{3}\,\mathrm{diag}(7, 4, 1, 1, 1, 1, -2, -2).$$

Writing

$$V_1 = \langle e_1 \rangle, \qquad V_2 = \langle e_2 \rangle, \qquad V_3 = \langle e_3, e_4, e_5, e_6 \rangle, \qquad V_4 = \langle e_7, e_8 \rangle,$$

we have

$$\begin{aligned} g_1(2; h) &= V_1 \otimes V_3 \otimes V_4 + V_2 \otimes \Lambda^2 V_3, \\ g_1(3; h) &= V_1 \otimes \Lambda^2 V_3 + V_1 \otimes V_2 \otimes V_4, \\ g_1(4; h) &= V_1 \otimes V_2 \otimes V_3. \end{aligned}$$

Thus

$$y = e_{167} + e_{138} + e_{145} + e_{234} - e_{256} \in m,$$

and let $a \in G$ be defined by:

$$e_4 \to e_5,\ e_5 \to e_6,\ e_6 \to e_4,\ e_k \to e_k \qquad (k \neq 4, 5, 6).$$

Then $a \cdot y$ is precisely the representative of the orbit XIV, and so XVII $\to$ XIV.

Finally let x be the representative of the orbit XXI. By [1] we can choose h as

$$h = \tfrac{1}{3}\operatorname{diag}(4, 4, 4, 1, 1, 1, 1, -2).$$

Writing

$$V_1 = \langle e_1, e_2, e_3 \rangle, \qquad V_2 = \langle e_4, e_5, e_6, e_7 \rangle, \qquad V_3 = \langle e_8 \rangle,$$

we find that

$$\begin{aligned} g_1(2; h) &= \Lambda^2 V_1 \otimes V_3 + V_1 \otimes \Lambda^2 V_2, \\ g_1(3; h) &= \Lambda^2 V_1 \otimes V_2, \\ g_1(4; h) &= \Lambda^3 V_1. \end{aligned}$$

Thus

$$y = e_{138} + e_{238} + e_{147} + e_{256} + e_{345} \in m,$$

and let $a \in G$ be defined by:

$$e_7 \to e_8,\ e_8 \to e_7,\ e_k \to e_k \qquad (k \neq 7, 8).$$

Then $a \cdot y$ is the representative of the orbit XX, and so we have shown that XXI $\to$ XX.

The cases 1)–6) cover all edges of our diagram in Fig. 1.

Proof of the theorem: Second part. Recall that the closure of an orbit is a union of that orbit and certain orbits of smaller dimension. To conclude the proof of the theorem it remains to show that

$$\text{XIX} \not\to \text{X}, \qquad \text{X} \not\to \text{XI}, \qquad \text{V} \not\to \text{VI},$$
$$\text{XVIII} \not\to \text{XIX}, \qquad \text{XV} \not\to \text{XVI}, \quad \text{and} \quad \text{VII} \not\to \text{V}.$$

All of these claims but the first can be proven by using arithmetical invariants $r, \rho_1, \rho_2, \sigma_1, \sigma_2, \sigma_3$ of trivectors introduced by Gurevič [2, 3]. The first of these invariants is just the rank of the trivector. The remaining five invariants are also dimensions of certain subspaces of V attached canonically to a trivector. It is immediate from his definitions of these invariants that they are upper semi-continuous. Thus if we have a convergent sequence of trivectors (x_k) and $\lim x_k = y$ then for each of the above invariants, say τ, we have $\tau(x_k) \geq \tau(y)$ for sufficiently large k. Of course, the equivalent trivectors have the same invariants and the six invariants above distinguish all 23 orbits of G in $\Lambda^3 V$, see [3].

Table IV

Orbit	Invariants
XIX	(8, 8, 8; 8, 2, 2)
XVIII	(8, 8, 8; 7, 4, 1)
XVI	(8, 8, 8; 4, 1, 1)
XV	(8, 8, 7; 5, 2, 0)
XI	(8, 6, 3; 1, 0, 0)
X	(7, 7, 7; 0, 0, 0)
VII	(7, 4, 1; 0, 0, 0)
VI	(7, 1, 1; 0, 0, 0)
V	(6, 6, 0; 0, 0, 0)

For each of the relevant orbits we list in Table IV the values of the six invariants by writing them as a sixtuple $(r, \rho_1, \rho_2; \sigma_1, \sigma_2, \sigma_3)$. This table is extracted from [3] but the reader should be warned that the designation of the 23 orbits of G in [3] is different from our notations.

The upper semi-continuity of the invariants and Table IV show that X $\not\to$ XI, V $\not\to$ VI, XVIII $\not\to$ XIX, XV $\not\to$ XVI and VII $\not\to$ V.

In order to show that XIX $\not\to$ X we shall again rely on the results of our paper [1].

For any $x \in \mathfrak{g}_1, x \neq 0$, let $h \in \mathfrak{g}_0$ and $y \in \mathfrak{g}_{-1}$ be chosen so that $[x, y] = h, [h, x] = 2x$ and $[h, y] = -2y$ hold. Then using the notation introduced in the previous section, we have

$$\dim(\operatorname{Ker}(\operatorname{ad} x) \cap \mathfrak{g}_{-2}) = \sum_{j \geq 0} [N_{-2}(j) - N_{-1}(j+2)],$$

where we write $N_k(j) = \dim \mathfrak{g}_k(j; h)$.

When $x = x_1$ is the representative of the orbit XIX we find that the above dimension is 1. On the other hand, when $x = x_2$ is the representative of the orbit X we find that $N_{-2}(j) = 0$ for all $j \geq 0$ and so the above dimension is 0. Hence the restriction $(\operatorname{ad} x_1)|_{\mathfrak{g}_{-2}}$ is singular, while $(\operatorname{ad} x_2)|_{\mathfrak{g}_{-2}}$ is non-singular. Clearly this implies that XIX $\not\to$ X.

This completes the proof of the theorem.

References

1. D. Ž. Kjoković, Classification of trivectors of an eight-dimensional real vector space to appear in *Linear and Multilinear Algebra*.

2. G. B. Gurevič, *Sur les trivecteurs dans l'éspace à sept dimensions*, Dokl. Akad. Nauk SSSR, III (1934), 567–569.

3. G. B. Gurevič, *Classifications des trivecteurs ayant le rang huit*, Dokl. Akad. Nauk SSSR, II (1935), 355–356.

4. G. B. Gurevič, *Foundations of the theory of algebraic invariants*, Noordhoff, Groningen 1964.

5. J. E. Humphries, *Linear algebraic groups*, Springer-Verlag, New York, 1975.

6. E. B. Vinberg and A. G. Elašvili, *Classification of trivectors of a nine-dimensional space*, Trudy Sem. Vekt. Tenz. Analizu, M.G.U. No. XVIII (1978), 197–233.

RUTGERS UNIVERSITY,
NEW BRUNSWICK, N.J. 08903
and
UNIVERSITY OF WATERLOO,
WATERLOO, ONTARIO, N2L 3G1