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THE RELATIONSHIP BETWEEN POLAR AND AC OPERATORS

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Abstract. This paper gives examples to show that a polar operator is not necessarily AC and an AC operator is not necessarily polar.

1. Introduction. Well-bounded operators are the building blocks of polar and AC operators. First introduced by Smart in 1960, well-bounded operators were originally studied by Smart and Ringrose [**6**], [**7**], [**8**]. These operators admit a spectral decomposition which is, in some sense, analogous to that for self-adjoint operators on Hilbert space. The spectral decomposition is simplified when we consider well-bounded operators of type (B).

By their definition, well-bounded operators have real spectra. In order to extend the concept of well-boundedness to operators with complex spectra, we consider trigonometrically well-bounded, polar and AC operators. The relevant facts about well-bounded, polar and AC operators are outlined in the next section. For a detailed account of the theory well-bounded operators see [**4**].

2. Background and notation. Throughout the following X will denote a complex Banach space with dual space X^* and $\mathcal{B}(X)$ will denote the algebra of all bounded linear operators mapping X into itself. Given a compact interval $J = [a, b]$ of the real line, let $BV(J)$ denote the Banach algebra of complex-valued functions of bounded variation on J with norm

$$\|f\|_{BV(J)} = |f(b)| + \operatorname*{var}_J f$$

where $\operatorname*{var}_J f$ represents the total variation of f on J. Similarly, using $\mathbf{T}$ to represent the unit circle, let $BV(\mathbf{T})$ denote the Banach algebra of complex-valued functions of bounded variation on $\mathbf{T}$ with norm

$$\|f\|_{BV(\mathbf{T})} = |f(1)| + \operatorname*{var}_{\mathbf{T}} f,$$

where $\operatorname*{var}_{\mathbf{T}} f$ is the total variation of f on $\mathbf{T}$. Furthermore, the notation $AC(J)$ (respectively $AC(\mathbf{T})$) will denote the closed subalgebra of $BV(J)$ (respectively $BV(\mathbf{T})$) consisting of the absolutely continuous functions on J (respectively $\mathbf{T}$).

Definition 2.1. An operator $T \in \mathcal{B}(X)$ is said to be *well-bounded* if there exists a constant K and a compact interval $J \subseteq \mathbf{R}$ such that

$$\|p(T)\| \leq K\|p\|_{BV(J)},$$

for all polynomials p.

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Note that, in this case, the spectrum of T must be a subset of J.

Definition 2.2. Let J be a compact interval of the real line. An *$AC(J)$-functional calculus* (respectively an *$AC(\mathbf{T})$-functional calculus*) for an operator $T \in \mathcal{B}(X)$ is a norm-continuous algebra-homomorphism γ of $AC(J)$ into $\mathcal{B}(X)$ (resp. $AC(\mathbf{T})$ into $\mathcal{B}(X)$) which sends the identity map $v(t) = t$ to T and the function identically 1 to I, the identity operator of $\mathcal{B}(X)$. In addition, γ is said to be weakly compact if, for each $x \in X$, $\gamma(\cdot)x$ is a weakly compact linear mapping of the domain of γ into X.

Since the polynomials are dense in the set of absolutely continuous functions [**7**, Lemma 10], we can say that an operator T is well-bounded if there exists a compact interval J for which T has an $AC(J)$-functional calculus.

Definition 2.3. An operator T is said to be *well-bounded of type (B)* if, for some compact interval J, T has a weakly compact $AC(J)$-functional calculus. (Note that if X is a reflexive space then every well-bounded operator on X is automatically of type (B). See [**5**, p. 68].)

Definition 2.4. A *spectral family in X* is a projection-valued function $E(\cdot) : \mathbf{R} \to \mathcal{B}(X)$ satisfying the following conditions:

(i) $\sup\{\|E(\lambda)\| : \lambda \in \mathbf{R}\} < \infty$;
(ii) $E(\lambda)E(\mu) = E(\mu)E(\lambda) = E(\min\{\lambda, \mu\})(\lambda, \mu, \in \mathbf{R})$;
(iii) $E(\cdot)$ is strongly right continuous;
(iv) $E(\cdot)$ has a strong left-hand limit at each point of $\mathbf{R}$;
(v) $E(\lambda) \to 0$ (respectively $E(\lambda) \to I$) in the strong operator topology of $\mathcal{B}(X)$ as $\lambda \to -\infty$ (respectively $\lambda \to +\infty$).

Note. If $E(\lambda) = 0$ for all $\lambda < a$, and $E(\lambda) = I$ for all $\lambda \geq b$, then $E(\cdot)$ is said to be *concentrated on* $[a, b]$.

If $E(\cdot)$ is a spectral family in X concentrated on $J = [a, b]$ and $f \in BV(J)$, then

$$\int_J^{\oplus} f(\lambda) dE(\lambda) \equiv f(a)E(a) + \int_a^b f(\lambda) dE(\lambda)$$

exists as the strong limit of the Riemann-Stieltjes sums

$$\mathcal{S}(f, u) = f(a)E(a) + \sum_{j=1}^{n} f(\lambda_j)\{E(\lambda_j) - E(\lambda_{j-1})\},$$

where $u = (\lambda_0, \lambda_1, \ldots, \lambda_n)$ is a partition of J. Rearranging the above in the style of integration by parts gives

$$\mathcal{S}(f, u) = f(b)E(b) - \sum_{j=1}^{n} \{f(\lambda_j) - f(\lambda_{j-1})\} E(\lambda_{j-1}).$$

The following results concerning well-bounded operators maybe found in [**4**, Part V].

PROPOSITION 2.5. *The mapping*

$$f \to \int_J^{\oplus} f\,dE$$

is an identity-preserving algebra homomorphism of $BV(J)$ into $\mathcal{B}(X)$ satisfying

$$\left\| \int_J^{\oplus} f\,dE \right\| \leq \|f\|_{BV(J)} \sup\left\{ \|E(\lambda)\| : \lambda \in \mathbf{R} \right\}$$

for every $f \in BV(J)$.

PROPOSITION 2.6. *Let $T \in \mathcal{B}(X)$. Then T is well-bounded of type (B) if and only if there exists a spectral family $E(\cdot)$ in X such that*

(i) *$E(\cdot)$ is concentrated on a compact interval $[a, b]$, and*
(ii) *$T = \int_{[a,b]}^{\oplus} \lambda dE(\lambda)$.*

In this case $E(\cdot)$ is uniquely determined and is called the spectral family of T.

PROPOSITION 2.7. *Let $T \in \mathcal{B}(X)$ be well-bounded of type (B) and let $E(\cdot)$ be its spectral family. Then an operator S commutes with T if and only if S commutes with $E(\lambda)$, for all $\lambda \in \mathbf{R}$.*

PROPOSITION 2.8. *Let $T \in \mathcal{B}(X)$ be well-bounded of type (B) and let $E(\cdot)$ be its spectral family. Then for each $\lambda \in \mathbf{R}$, $\left\{E(\lambda) - E(\lambda^-)\right\}$ is a projection operator and*

$$\left\{E(\lambda) - E(\lambda^-)\right\}X = \{x \in X : Tx = \lambda x\},$$

where $E(\lambda^-)$ denotes the strong limit of $E(s)$ as $s \to \lambda^-$.

Trigonometrically well-bounded, polar and AC operators all arise from well-bounded operators. Their definitions are given below.

DEFINITION 2.9. An operator $T \in \mathcal{B}(X)$ is said to be *trigonometrically well-bounded* if there exists a well-bounded operator A of type (B) on X such that $T = e^{iA}$.

PROPOSITION 2.10. *If T is a trigonometrically well-bounded operator on the Banach space X, then there is a unique well-bounded operator A of type (B) on X such that $T = e^{iA}$, $\sigma(A) \subseteq [0, 2\pi]$, and such that $\sigma_p(A)$, the point spectrum of A, does not contain 2π.*

DEFINITION 2.11. The unique operator A in Proposition 2.10 is called the *argument* of T and is denoted by $\arg T$. For more on trigonometrically well-bounded operators see [**3**].

DEFINITION 2.12. An operator $T \in \mathcal{B}(X)$ is said to be a *polar operator* if there exist commuting type (B) well-bounded operators R and A on X such that $T = Re^{iA}$. The following results about polar operators will be required in Section 3.

THEOREM 2.13. See [**1**, Theorem 1]. *Let $T \in \mathcal{B}(X)$ be polar. Then T has a decomposition $T = Re^{iA}$ such that*

(i) *R and A are commuting well-bounded operators of type (B);*
(ii) $\sigma(R) \geq 0$;
(iii) $F(0)e^{iA} = F(0)$, *where $F(\cdot)$ is the spectral family of R;*
(iv) $\sigma(A) \subseteq [0, 2\pi]$, $2\pi \notin \sigma_p(A)$.

This decomposition is unique and is called the *canonical decomposition of T.*

THEOREM 2.14. See [**1**, Theorem 3.18(i)]. *Let T be a polar operator with canonical decomposition $T = Re^{iA}$. Then the commutants of T, R and A satisfy the equality $\{T\}' = \{R\}' \cap \{A\}'$.*

Polar operators are discussed further in [**1**] and [**9**]. The final definition required is that of an AC operator.

DEFINITION 2.15. An operator $T \in \mathcal{B}(X)$ is said to be an *AC operator* if there exist commuting well-bounded operators C and D such that $T = C + iD$. AC operators are studied in [**2**], from which the following result is taken.

THEOREM 2.16. See [**2**, Lemma 4]. *Let C and D be commuting well-bounded operators of type (B) on X and let $S \in \mathcal{B}(X)$ commute with $C + iD$. Then S commutes with C and D.*

It has been shown [**3**, Theorem 3.4] that an operator T is trigonometrically well-bounded if and only if there exist commuting well-bounded operators A and B of type (B) such that

$$T = A + iB \tag{1}$$

and

$$A^2 + B^2 = I. \tag{2}$$

With this in mind, it seems natural to pose the following questions.

(1) If T is polar, do there exist commuting well-bounded operators A and B (of type (B)) such that (1) holds?
(2) If $T = A + iB$ with A and B commuting well-bounded operators of type (B), does it follow that T is polar?

We shall now give examples to show that the answer to both these questions is negative. We shall use the following definitions and results from [**4**].

3. Examples. Let $a \in l^2$ and, for $n \in N$, let $P_n : l^2 \to l^2$ be defined by

$$P_n a = \langle a, y_n \rangle x_n,$$

where

$$x_{2n-1} = e_{2n-1} + \sum_{i=n}^{\infty} \alpha_{i-n+1} e_{2i},$$

$$x_{2n} = e_{2n},$$

$$y_{2n-1} = e_{2n-1},$$

$$y_{2n} = \sum_{i=1}^{n} (-\alpha_{n-i+1}) e_{2i-1} + e_{2n} \ (n \in \mathbf{N}),$$

$$\alpha_1 = 0, \alpha_n = \frac{1}{n \log n}, (n = 2, 3, \ldots),$$

and ε_n is the element of l^2 with 1 in its nth position and 0 elsewhere. Then each P_n is a projection, $P_n P_m = 0$ whenever $n \neq m$, and $I = \sum_{n=1}^{\infty} P_n$, the series converging in the strong operator topology of $\mathcal{B}(l^2)$.

Proposition 3.1. See ([**4**, 18.5]). *Let $\{\lambda_n\}$ be a monotonic bounded sequence in* $\mathbf{R}$ *and, for each $n \in \mathbf{N}$, let P_n be as above. Then the series $\sum_{n=1}^{\infty} \lambda_n P_n$ converges strongly in $\mathcal{B}(l^2)$ and its sum is a well-bounded operator.*

Note that in the proof of 18.4 in [**4**] it is shown that

$$\left\| \sum_{j=1}^{n} P_{2j} \right\| \to \infty \text{ as } n \to \infty. \tag{3}$$

We shall use this result in the example below.

Example. Let $X = l^2$, P_n be defined as above, and define sequences $\{\lambda_n\}$ and $\{\mu_n\}$ by

$$\lambda_n = \frac{n+1}{n}$$

and

$$\mu_{2n-1} = \mu_{2n} = \cos^{-1}\left(\frac{4n^2 - 1/2}{4n^2 + 2n}\right),$$

for all $n \in \mathbf{N}$. In addition, define

$$C = \sum_{n=1}^{\infty} \lambda_n P_n \text{ and } D = \sum_{n=1}^{\infty} \mu_n P_n.$$

By Proposition 3.1, each series converges strongly in $\mathcal{B}(l^2)$ and C and D are well-bounded operators (of type (B)). Furthermore, since C and D commute, it follows that Ce^{iD} is polar.

Now suppose that $C\cos D$ is well-bounded with spectral family $E(\cdot)$. Then

$$(C\cos D)P_n = (\lambda_n \cos \mu_n)P_n$$
$$= \begin{cases} \left(\dfrac{n^2 - 1/2}{n^2}\right)P_n & \text{when } n \text{ is even,} \\ \left(\dfrac{n^2 + 2n + 1/2}{n^2 + 2n}\right)P_n & \text{when } n \text{ is odd.} \end{cases}$$

Fix $x \in l^2$ and suppose that n is even. Observe that

$$P_n x \in \left\{x \in l^2 : (C\cos D)x = \left(\frac{n^2 - 1/2}{n^2}\right)x\right\}. \tag{4}$$

Furthermore, if

$$(C\cos D)x = \frac{n^2 - 1/2}{n^2}x,$$

then

$$P_m x = \frac{n^2}{n^2 - 1/2}(C\cos D)P_m x$$

and it follows that $P_m x = 0$ if $m \neq n$. Thus

$$\left\{x \in l^2 : (C\cos D)x = \left(\frac{n^2 - 1/2}{n^2}\right)x\right\} \subseteq P_n X \tag{5}$$

Combining statements (4) and (5), we see that

$$P_n X = \left\{x \in l^2 : (C\cos D)x = \left(\frac{n^2 - 1/2}{n^2}\right)x\right\}. \tag{6}$$

Also, by Proposition 2.8,

$$P_n X = \left\{E\left(\frac{n^2 - 1/2}{n^2}\right) - E\left(\left(\frac{n^2 - 1/2}{n^2}\right)^-\right)\right\}X.$$

Since

$$\frac{n^2 - 1/2}{n^2} < 1$$

for all $n \in \mathbf{N}$, it follows that $E(1)P_n = P_n$ whenever n is even.

When n is odd, an argument similar to that above shows that

$$P_n X = \left\{x \in l^2 : C\cos Dx = \left(\frac{n^2 + 2n + 1/2}{n^2 + 2n}\right)x\right\}. \tag{7}$$

As

$$\frac{n^2 + 2n + 1/2}{n^2 + 2n} > 1,$$

for all $n \in \mathbf{N}$, it follows that $E(1)P_n = 0$ for n odd.

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Combining even and odd cases, we see that

$$E(1) = \sum_{n=1}^{\infty} E(1)P_n = \sum_{n=1}^{\infty} P_{2n},$$

with the series converging in the strong operator topology of $\mathcal{B}(l^2)$. Since $E(1)$ is bounded, the partial sums of the series $\sum_{n=1}^{\infty} P_{2n}$ must be bounded in norm, giving a contradiction to (3). Hence $C \cos D$ is not well-bounded.

Now suppose that $T = Ce^{iD} = A + iB$, where A and B are commuting well-bounded operators. Since C and D commute with T it follows that $C \cos D$ commutes with T and, by Theorem 2.16, $C \cos D$ commutes with A and B.

Next fix $n \in \mathbf{N}$ and suppose that $y \in P_n X = \{x \in X : C \cos Dx = \lambda_n \cos \mu_n x\}$. Then

$$Ay = (\lambda_n \cos \mu_n)^{-1} A(C \cos D)y = (\lambda_n \cos \mu_n)^{-1}(C \cos D)Ay$$

and hence $Ay \in P_n X$. A similar argument shows that $P_n X$ is invariant under B. It readily follows that A and B commute with P_n.

Now, since each $P_n X$ is one-dimensional (the $P_n X$ are the eigenspaces of $C \cos D$ corresponding to the distinct eigenvalues $\lambda_n \cos \mu_n$), there exist α_n and $\beta_n \in \mathbf{R}$ such that

$$A|P_n X = \alpha_n \text{ and } B|P_n X = \beta_n.$$

Thus, on $P_n X$, we have

$$T = Ce^{iD} = \lambda_n \cos \mu_n + i\lambda_n \sin \mu_n$$

and also

$$T = A + iB = \alpha_n + i\beta_n.$$

Equating real and imaginary parts gives $\alpha_n = \lambda_n \cos \mu_n$ and $\beta_n = \lambda_n \sin \mu_n$. Thus $A = C \cos D$ on each $P_n X$ and it follows that $A = C \cos D$. This contradicts the fact that A is well-bounded. Hence T cannot be AC.

An example of an AC operator which is not polar now follows.

Example. Let $X = l^2$ and, for $n \in \mathbf{N}$, let P_n be as in the example above. Now define

$$C = \sum_{n=1}^{\infty} \lambda_n P_n \text{ and } D = \sum_{n=1}^{\infty} \mu_n P_n,$$

where, for $n \in \mathbf{N}$,

$$\lambda_n = \sqrt{\frac{n+1}{n}}$$

and

$$\mu_{2n-1} = \mu_{2n} = \sqrt{\frac{(2n-1) - (4n-2)^{-1}}{2n}}.$$

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By Proposition 3.1, C and D are well-bounded operators (of type (B)). Moreover, as C and D commute, $T = C + iD$ is AC.

Now suppose that T is polar with canonical form Re^{iA}. By Theorems 2.14 and 2.16, R and A commute with C and D. It is readily checked that, for each $n \in \mathbf{N}$, $P_nX = \{x \in X : Cx = \lambda_n x\}$, P_nX is invariant under both R and A, and P_n commutes with R and A.

Now, given $n \in \mathbf{N}$, there exist r_n and $\theta_n \in \mathbf{R}$ such that

$$R|P_nX = r_n \text{ and } A|P_nX = \theta_n.$$

Furthermore, on P_nX,

$$T = Re^{iA} = r_n e^{i\theta_n}$$

and

$$T = C + iD = \lambda_n + i\mu_n.$$

Hence $r_n e^{i\theta_n} = \lambda_n + i\mu_n$ and $r_n = (\lambda_n^2 + \mu_n^2)^{1/2}$, for all $n \in \mathbf{N}$. Observe that

$$r_n = \begin{cases} \left(2 - \dfrac{1}{2n(n-1)}\right)^{1/2} & \text{for } n \text{ even,} \\ \left(\dfrac{2n^2 + 2n + 1/2}{n^2 + n}\right)^{1/2} & \text{for } n \text{ odd,} \end{cases}$$

so that $r_n < \sqrt{2}$ if n is even and $r_n > \sqrt{2}$ if n is odd. Notice also that

$$P_nX = \{x \in X : Rx = r_n x\}.$$

If $E(\cdot)$ is the spectral family of R then $E(\sqrt{2})$ is a bounded projection on X and, by Proposition 2.8 and 2.4(ii),

$$\begin{aligned} E(\sqrt{2})P_nX &= E(\sqrt{2})\{E(r_n) - E(r_n^-)\}X \\ &= \begin{cases} \{E(r_n) - E(r_n^-)\}X & \text{for } n \text{ even,} \\ 0 & \text{for } n \text{ odd.} \end{cases} \end{aligned}$$

It follows that

$$E(\sqrt{2})P_n = \begin{cases} P_n & \text{for } n \text{ even,} \\ 0 & \text{for } n \text{ odd.} \end{cases}$$

This gives

$$E(\sqrt{2}) = \sum_{n=1}^{\infty} E(\sqrt{2})P_n = \sum_{n=1}^{\infty} P_{2n},$$

where the series converges in the strong operator topology of $\mathcal{B}(l^2)$. As $E(\sqrt{2})$ is bounded, the partial sums of the series $\sum_{n=1}^{\infty} P_{2n}$, must be bounded in norm. Again we have a contradiction to (3) so that R cannot be well-bounded and T is not polar.

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