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A BECKMAN-QUARLES TYPE THEOREM FOR COXETER'S INVERSIVE DISTANCE

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ABSTRACT. We prove that a bijective transformation on the set of circles in the real inversive plane which preserves pairs of circles a fixed inversive distance $\rho > 0$ apart must be induced by a Möbius transformation.

1. **Introduction.** The original Beckman-Quarles theorem [2] stated that mappings from real Euclidean n-space to itself which preserve pairs of points some fixed distance ρ apart must be motions. Many generalizations and variations of this theorem exist (see the bibliography of [7]). In fact, all that is required to formulate a *Beckman-Quarles problem* is a *space* with a *distance* invariant under some suitable group of transformations; the problem is then to show that mappings on the space preserving a fixed distance belong to the group. The theorems which result can be geometric (e.g. for distances between lines in Euclidean 3-space [6]), physical (e.g. for constant light-speed in Minkowski spacetime [1]) or more abstract (e.g. for separations in Artinian planes over arbitrary fields [3,8], or many others). Here, we consider a geometric variant: the distance to be preserved is the inversive distance between pairs of circles in the Möbius plane, first introduced by Coxeter in [5].

Let $\mathcal{M}$ denote the Möbius plane (or real inversive plane) and $\mathcal{C}$, the set of circles in $\mathcal{M}$. Deleting an arbitrary *antipode* (or *point at infinity*) from $\mathcal{M}$ leaves the ordinary Euclidean plane, in which the Möbius circles become Euclidean circles or lines. If A, B in $\mathcal{C}$ become non-intersecting Euclidean circles with radii r_A, r_B and distance d between their centres, then the *inversive distance* δ_{AB} between them is given by

$$(*)\qquad \cosh \delta_{AB} := \frac{|r_A^2 + r_B^2 - d^2|}{2 r_A r_B}$$

If, instead, one of the circles (say A) becomes a line a distance h from the centre of the other then δ_{AB} is given by

$$(**)\qquad \cosh \delta_{AB} := h / r_B.$$

Inversive distances are invariant under arbitrary choices of antipode, and in fact, under all Möbius transformations of $\mathcal{M}$.

We prove the following Beckman-Quarles type theorem.

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THEOREM. *For a fixed real $\rho > 0$, let $X \longrightarrow \bar{X}$ be a bijective mapping from $\mathcal{C}$ to itself such that, for all A, B in $\mathcal{C}$,*

$$\delta_{AB} = \rho \text{ if and only if } \delta_{\bar{A}\bar{B}} = \rho.$$

Then the mapping is induced on $\mathcal{C}$ by a Möbius transformation of $\mathcal{M}$.

First, a few preliminaries. Circles in $\mathcal{C}$ are called *disjoint, tangent* or *intersecting* when they have resp. 0, 1 or 2 points in common (note that *intersecting* excludes *tangent*). We use extensively the *canonical* Euclidean representation of circle pairs: by an appropriate choice of antipode, two distinct circles can be taken to be concentric circles (if disjoint), parallel lines (if tangent) or intersecting lines (if interesecting).

Although it does give an invariant measure of *distance* between non-intersecting circles, inversive distance is not a distance in the usual sense. First, it is not even defined for all circles (just non-intersecting ones) and can be zero for distinct circles (if tangent). It *is* symmetric, but does not satisfy the triangle inequality; in fact, for *nested* circles, it reverses it. (Circle C is nested between disjoint circles A and B whenever any circle intersecting A and B must intersect C; if so, $\delta_{AC}+\delta_{CB} \leq \delta_{AB}$). The unique circle M nested between disjoint circles A and B with $\delta_{AM} = \delta_{MB} = \frac{1}{2}\delta_{AB}$ is called the *mid-circle* of A and B, and for concentric A and B, it has the same centre as A, B and radius $r_M = (r_A r_B)^{1/2}$.

2. **Lemmas.**

We first give explicitly all circles which are an inversive distance ρ from two circles in canonical form.

LEMMA 2.1. *For arbitrary circles A, B in $\mathcal{C}$ in canonical form, let C be any circle in $\mathcal{C}$ with $\delta_{AC} = \delta_{BC} = \rho$.*

i) If A and B are parallel lines a distance d apart, then C is a proper circle with radius $r_C = \frac{1}{2}d\,\mathrm{sech}\,\rho$ and centre on the line equidistant from A and B (Figure 2.1).

ii) If A and B are lines intersecting at point o, then C is a proper circle with centre on one of the four angle bisectors of A and B and radius $r_C = d\sin(\frac{1}{2}\theta)\,\mathrm{sech}\,\rho$, where θ is the angle bisected and d is the distance from o to the centre of C (Figure 2.1).

iii) If A and B are circles with common centre o, and $r_A > r_B$, then C is a proper circle inside A and outside B. If d is the distance from the centre of C to o, then (Figure 2.2) *either*

1. $r_C = \frac{1}{2}(r_A - r_B)\,\mathrm{sech}\,\rho$, $d = \{r_C^2 + r_A r_B\}^{1/2}$, and C is not nested between A and B, or

2. (possible only for $\rho \leq \frac{1}{2}\delta_{AB}$) $r_C = \frac{1}{2}(r_A + r_B)\,\mathrm{sech}\,\rho$, $d = \{r_C^2 - r_A r_B\}^{1/2}$, and C is nested between A and B.

All nested C's intersect each other and all non-nested C's. If $\rho = \frac{1}{2}\delta_{AB}$, the only nested C is the mid-circle of A and B, which is then orthogonal to all other C's.

PROOF. In all three cases, it is easily checked that C must be proper.

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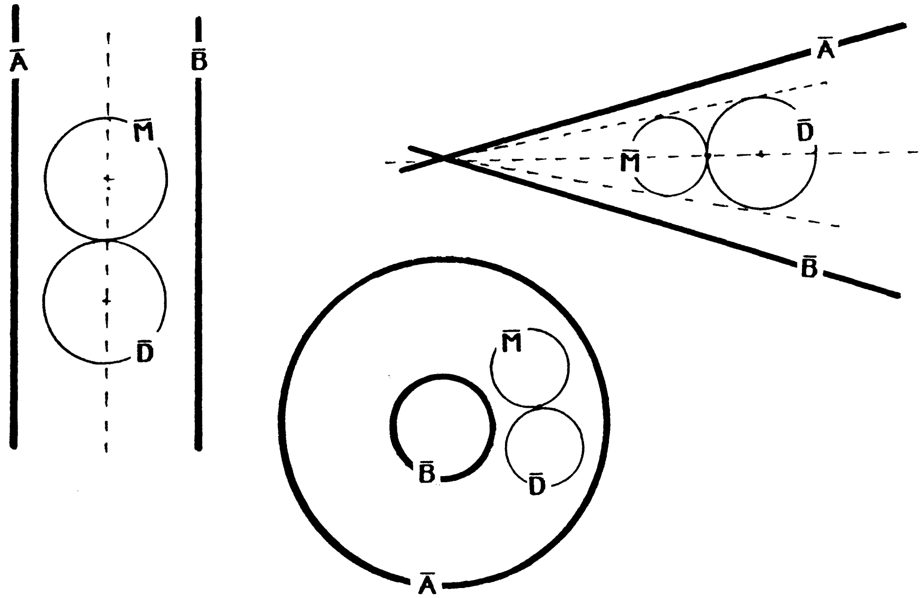

FIGURE 2.1. Possible circles C when A and B are lines.

For i) and ii), note from $(**)$ that the centre of any circle the same inversive distance from two lines must be equidistant from these lines; the rest then follows from $(**)$ (Figure 2.1).

For iii), we calculate r_C and d, and leave the remaining details as an exercise. From $(*)$, we must satisfy

$$r_B\,|r_A^2 + r_C^2 - d^2| = r_A\,|r_B^2 + r_C^2 - d^2| = 2r_Ar_Br_C\cosh\rho.$$

Suppose that $r_A^2 + r_C^2 - d^2 < 0$; then

$$-r_B(r_A^2 + r_C^2 - d^2) = \pm r_A(r_B^2 + r_C^2 - d^2) = 2r_Ar_Br_C\cosh\rho.$$

The first part implies that $r_C^2 - d^2 = \mp r_Ar_B$, so from the second part,

$$2r_Ar_Br_C\cosh\rho = \pm r_A(r_B^2 \mp r_Ar_B) = -r_Ar_B(r_A \mp r_B) < 0,$$

which is impossible. Thus $r_A^2 + r_C^2 - d^2 > 0$.

Now

$$+r_B(r_A^2 + r_C^2 - d^2) = \pm r_A(r_B^2 + r_C^2 - d^2) = 2r_Ar_Br_C\cosh\rho.$$

so $r_C^2 - d^2 = \pm r_Ar_B$ and then $2r_C\cosh\rho = r_A \pm r_B$. The lower sign gives the stated r_C and d for the non-nested C's, and works for any $\rho > 0$. The upper sign gives r_C and d for the nested C's, and works only for $r_C \geq (r_Ar_B)^{1/2}$. In this case, $\cosh\rho \leq \frac{1}{2}(r_A + r_B)/(r_Ar_B)^{1/2} = \cosh(\frac{1}{2}\delta_{AB})$, so $\rho \leq \frac{1}{2}\delta_{AB}$.

If $\rho = \frac{1}{2}\delta_{AB}$, then $d = 0$ and $r_C = (r_Ar_B)^{1/2}$, so C the mid-circle of A and B. ∎

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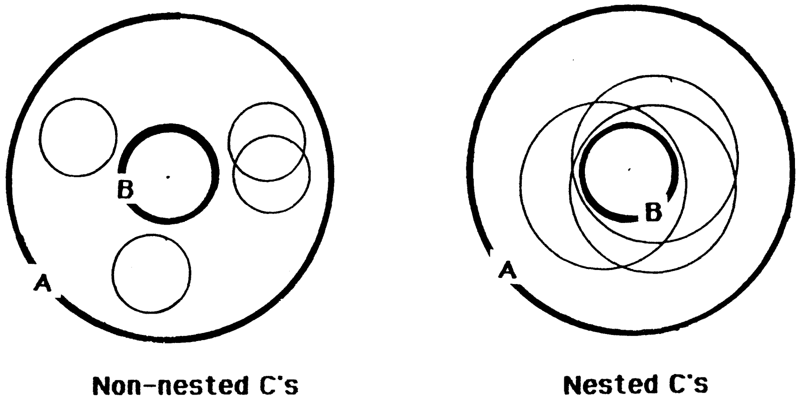

FIGURE 2.2. Possible circles C for concentric A and B.

LEMMA 2.2. *i) Let A and B be tangent circles in* $\mathcal{C}$*, then for all* $C \neq A, B$ *in* $\mathcal{C}$*, there exist at most four circles D in* $\mathcal{C}$ *with* $\delta_{AD} = \delta_{BD} = \delta_{CD} = \rho$.

ii) Let A and B be intersecting circles in $\mathcal{C}$*; then for some* $C \neq A, B$ *in* $\mathcal{C}$*, there exist eight circles D in* $\mathcal{C}$ *with* $\delta_{AD} = \delta_{BD} = \delta_{CD} = \rho$.

PROOF. i) Represent A and B by parallel lines. If C is a line, then there are no D's if C is parallel to A and B and two otherwise. If C is a proper circle, we may choose coordinates with A and B as the lines $x = \pm 1$ and C with the equation $(x-h)^2 + y^2 = r^2$ for some fixed h and $r > 0$. From Lemma 2.1, any circle D with $\delta_{AD} = \delta_{BD} = \rho$ has centre $(0, \alpha)$ on the y-axis and radius $\operatorname{sech} \rho$. The relation $\delta_{CD} = \rho$ then simplifies to

$$\alpha^2 = r^2 + \operatorname{sech}^2 \rho - h^2 \mp 2r,$$

so there are at most four α's, and thus at most four D's.

ii) Represent A and B by lines intersecting at point o, and let C be the circle with centre o and radius 1. From Lemma 2.1, any circle D with $\delta_{AD} = \delta_{BD} = \rho$ has its centre on an angle bisector of A and B a distance d from point o, and has radius $r = d \sin(\frac{1}{2}\theta) \operatorname{sech} \rho$. The relation $\delta_{CD} = \rho$ then yields the quadratic equation

$$d^2\{\sin^2(\frac{1}{2}\theta) \operatorname{sech}^2 \rho - 1\} \mp 2d \sin(\frac{1}{2}\theta) + 1 = 0$$

for d, with discriminant

$$4\{\sin^2(\frac{1}{2}\theta) \tanh^2 \rho + 1\} > 0.$$

The roots are thus distinct, so for each of the four angle bisectors, there are two D's, giving a total of eight. ∎

LEMMA 2.3. *For any orthogonal circles C and D in $\mathcal{C}$, there exist circles A and B in $\mathcal{C}$ with $\delta_{AC} = \delta_{AD} = \delta_{BC} = \delta_{BD} = \rho$ and C the mid-circle of A and B.*

PROOF. Choose Euclidean coordinates with x-axis D and y-axis C, and take A and B to be

$$(x \pm \cosh\rho)^2 + (y - \cosh\rho)^2 = 1. \qquad \blacksquare$$

3. **Proof of the theorem.** It follows immediately from Lemma 2.2 that the bijection $X \longrightarrow \bar{X}$ cannot map intersecting circles into tangent circles, or vice versa.

LEMMA 3.1. *Let A and B be disjoint circles in $\mathcal{C}$ with $\delta_{AB} = 2\rho$, and let M be the mid-circle of A and B. Then $\bar{A}$ and $\bar{B}$ are disjoint and $\bar{M}$ is the mid-circle of $\bar{A}$ and $\bar{B}$.*

PROOF. We have $\delta_{AM} = \delta_{BM} = \delta_{\bar{A}\bar{M}} = \delta_{\bar{B}\bar{M}} = \rho$. Represent the pairs A, B and $\bar{A}$, $\bar{B}$ in canonical form. Suppose that $\bar{A}$ and $\bar{B}$ are lines or concentric circles with $\bar{M}$ not nested between them. Then (Figure 3.1) there exists a circle $\bar{D}$ tangent to $\bar{M}$ with $\delta_{\bar{A}\bar{D}} = \delta_{\bar{B}\bar{D}} = \rho$. Then $\delta_{AD} = \delta_{BD} = \rho$. But D cannot intersect M, which contradicts Lemma 2.1 (M is nested between A and B). Thus $\bar{A}$ and $\bar{B}$ are disjoint with $\bar{M}$ nested between them.

Assume that $\bar{M}$ is not the mid-circle of $\bar{A}$ and $\bar{B}$; then there exists another circle $\bar{N} \neq \bar{M}$ nested between $\bar{A}$ and $\bar{B}$ with $\delta_{\bar{N}\bar{A}} = \delta_{\bar{N}\bar{B}} = \rho$. Then $\delta_{NA} = \delta_{NB} = \rho$, so N is not nested between A and B (since M is the *only* circle an inversive distance ρ from A and B and nested between them). Thus there exists a circle D with $\delta_{AD} = \delta_{BD} = \rho$ nested between A and B and tangent to N. Now $\bar{D}$ cannot intersect $\bar{N}$, which contradicts Lemma 2.1 again, since $\bar{N}$ is nested between A and B. Thus $\bar{M}$ is the mid-circle of $\bar{A}$ and $\bar{B}$. $\blacksquare$

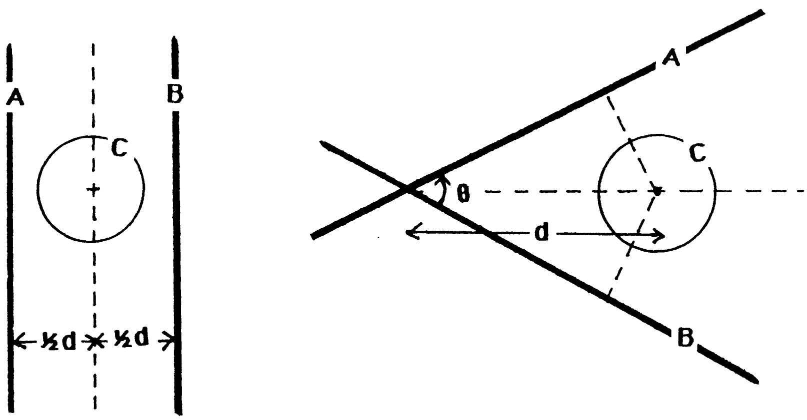

FIGURE 3.1. A circle $\bar{D}$ tangent to $\bar{M}$.

COROLLARY 3.1. *The mapping $X \longrightarrow \bar{X}$ preserves orthogonality, i.e., if C and D are orthogonal circles, then so are $\bar{C}$ and $\bar{D}$.*

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PROOF. From Lemma 2.3, there exist circles A and B in $\mathcal{C}$ with $\delta_{AC} = \delta_{AD} = \delta_{BC} = \delta_{BD} = \rho$ and C the mid-circle of A and B. Then $\bar{C}$ is the mid-circle of $\bar{A}$ and $\bar{B}$ and $\delta_{\bar{A}\bar{D}} = \delta_{\bar{B}\bar{D}} = \rho$, so by Lemma 2.1, $\bar{C}$ is orthogonal to $\bar{D}$. ∎

COROLLARY 3.2. *The mapping $X \longrightarrow \bar{X}$ preserves tangent circles.*

PROOF. The mapping preserves elliptic pencils of circles (consisting of circles orthogonal to two orthogonal circles) and hyperbolic pencils (consisting of circles orthogonal to those of an elliptic pencil). Two circles are tangent if and only if they belong to neither type pencil, so tangency is preserved. ∎

Now consider the circles through a point p. Some parabolic pencil of these circles maps into a parabolic pencil through a point $\bar{p}$ (since parabolic pencils consist of infinitely many mutually tangent circles). Any other circle in $\mathcal{C}$ passes through p if and only if no circle of the parabolic pencil is tangent to it. Since the mapping $X \longrightarrow \bar{X}$ preserves this relation, a circle passes through p if and only if its image passes through $\bar{p}$. The mapping $p \longrightarrow \bar{p}$ so defined is a transformation of the Möbius plane which preserves concyclic points, and is thus a Möbius transformation inducing the circle mapping $X \longrightarrow \bar{X}$. This concludes our proof.

A final note: for *intersecting* A and B, the righthand sides of * and ** define the cosine of their angle of intersection θ_{AB}. This angle is also a Möbius invariant, and we have the following *dual* Beckman-Quarles theorem.

THEOREM. *For $0 \leq \varphi < \pi$, let $X \longrightarrow \bar{X}$ be a bijective mapping from $\mathcal{C}$ to itself such that for all A, B in $\mathcal{C}$,*

$$\theta_{AB} = \varphi \text{ if and only if } \theta_{\bar{A}\bar{B}} = \varphi.$$

Then the mapping is induced on $\mathcal{C}$ by a Möbius transformation of $\mathcal{M}$.

PROOF. For $\varphi = 0$, or $\varphi = \frac{1}{2}\pi$, the mapping preserves tangency or orthogonality, and the proof proceeds as for inversive distances. Otherwise, any distinct circles A, B and C are not part of a parabolic pencil if and only if there exist at most eight circles D with $\theta_{AD} = \theta_{BD} = \theta_{CD} = \varphi$ [4, p. 143]. Thus tangency is preserved, and the proof continues as for inversive distances. ∎

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