GRAPHS WITH NEAR v- AND e-NEIGHBOURHOODS

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1. Introduction. All the graphs considered in this paper are connected finite undirected graphs without loops and multiple edges.

By the *vertex-neighbourhood* (v-*neighbourhood*) of any vertex x in the graph G we mean the subgraph $N_G^v(x)$ induced by the set of all vertices adjacent to x. Analogously by the *edge-neighbourhood* (e-*neighbourhood*) of any edge f with end vertices x, y we mean the subgraph $N_G^e(f)$ (or $N_G^e(xy)$) induced by the set of all vertices which are adjacent to at least one vertex of the pair x, y and which are different from x, y.

Zykov **[5]** proposed the following problem: does there exist a graph G with the property that $N_G^v(x)$ is isomorphic to a given graph H for each vertex x of G? Then Zelinka **[4]** proposed the edge version of this problem: does there exist a graph G with the property that $N_G^e(f)$ is isomorphic to a given graph H for each edge f of G? If the answer to the first (second) question is positive, then H is said to be v-*realizable* (e-*realizable*), and G is called a v-*realization* (e-*realization*) *of* H.

There exist some classes of graphs which are both v-realizable and e-realizable—even cycles, for example. Further, Nedela **[3]** has constructed a class $\mathcal{F}$ of graphs in which the v-neighbourhoods and the e-neighbourhoods are "similar" graphs. For each even positive integer n there exists a graph $F_n \in \mathcal{F}$ which is the v-realization of the cycle C_n and the e-realization of C_{2n-4}.

By a simple observation we can see that (for each $n \geqslant 2$ and $r \geqslant 3$) the complete multipartite graph $K_{n,n,\ldots,n} = K_{rn} - rK_n$ is both the v-realization of $K_{(r-1)n} - (r-1)K_n$ and the e-realization of $K_{n-1,n-1,n,\ldots,n} = K_{rn-2} - 2K_{n-1} - (r-2)K_n$; these two graph are, in a certain sense, "similar". Further, these are the only v-realizable or e-realizable complete multipartite graphs ([**1**], [**2**]).

In view of all this, it is natural to ask the following question. Does there exist a pair of graphs G, H such that

$$N_G^v(x) \simeq N_G^e(f) \simeq H \tag{1.1}$$

for each vertex x and each edge f of G?

In this paper we prove that the answer is negative if H is connected, but that there exists a family of graphs in which the v-neighbourhoods and the e-neighbourhoods differ by just one edge.

2. The main results. Suppose that there exists a pair of graphs G, H such that (1.1) holds for each vertex x and each edge f of G. Then G is regular of degree r where we can supose that $r \geqslant 2$. We denote the vertex set of a graph F by $V(F)$; so if f is any edge of G and y_1 and y_2 are its end vertices, then

$$V(N_G^e(y_1y_2)) = V(N_G^v(y_1)) \cup V(N_G^v(y_2)) - \{y_1, y_2\}. \tag{2.1}$$

Let $V(N_G^e(y_1y_2)) = \{x_1, x_2, \ldots, x_r\}$. Since it follows from (1.1) that

$$|V(N_G^e(y_1y_2))| = |V(N_G^v(y_1))| = |V(N_G^v(y_2))| = r,$$

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it follows from (2.1) that

$$|V(N_G^v(y_1)) \cap V(N_G^v(y_2))| = r - 2.$$

Thus, if we assume that $r > 2$, $N_G^e(y_1 y_2)$ contains exactly $r-2$ vertices which are adjacent to both y_1 and y_2. Without loss of generality we can suppose that these are the vertices $x_1, \ldots, x_{r-2}$, and that the vertex x_{r-1} is adjacent to y_1 (but not to y_2), while x_r is adjacent to y_2 (but not to y_1). A vertex x which is adjacent to just one end vertex of an edge f will be called a *solo-vertex* of $N_G^e(f)$: thus x_{r-1} and x_r are solo-vertices of $N_G^e(y_1 y_2)$, and, by (1.1), every $N_G^e(f)$ with f an edge of G will have two solo-vertices.

Suppose that the graph $N_G^e(y_1 x_{r-1})$ is induced by the vertices $x_1, x_2, \ldots, x_{r-2}, y_2, z$. Since y_2 is not adjacent to x_{r-1} and z is not adjacent to y_1, the solo-vertices of $N_G^e(y_1 x_{r-1})$ are y_2 and z; thus all of $x_1, \ldots, x_{r-2}$ are adjacent to both y_1 and x_{r-1}. By a similar argument applied to $N_G^e(y_2 x_r)$, we see that each of $x_1, \ldots, x_{r-2}$ is adjacent to x_r. Thus $N_G^e(y_1 y_2)$ contains the subgraph $F = K_{2,r-2}$ which is a bipartite graph with partite sets $P_1 = \{x_{r-1}, x_r\}$ and $P_2 = \{x_1, \ldots, x_{r-2}\}$.

We now turn our attention to the e-neighbourhood of an edge $y_1 x_i$ where $1 \leqslant i \leqslant r-2$. Since $V(N_G^e(y_1 x_i)) = \{y_2, x_1, x_2, \ldots, x_r\} - \{x_i\}$ and neither y_2 nor x_{r-1} is a solo-vertex, it follows that the two solo-vertices are x_r and x_j for some $x_j \in \{x_1, \ldots, x_{r-2}\} - \{x_i\}$. Thus for each $i \in \{1, 2, \ldots, r-2\}$, x_i is adjacent to exactly $r-4$ vertices of the set $\{x_1, \ldots, x_{r-2}\}$; therefore the graph F_1 induced by the set P_2 is isomorphic to $K_{r-2} - \dfrac{r-2}{2} K_2$, and r is even, $r \geqslant 4$. Furthermore, we see that the solo-vertices x_r and x_j of $N_G^e(y_1 x_i)$ are adjacent to each other, and so the solo-vertices x_{r-1} and x_r of $N_G^e(y_1 y_2)$ must also be adjacent to each other. Since $N_G^e(y_1 y_2)$ contains the subgraph $F \simeq K_{2,r-2}$ and the subgraph F_1, it follows from the adjacency of x_{r-1} and x_r that $N_G^e(y_1 y_2) \simeq K_r - \dfrac{r-2}{2} K_2$. Hence $N_G^e(y_1 y_2)$ contains the vertex x_r of degree $r-1$ in $N_G^e(y_1 y_2) \simeq H$. Since each $N_G^v(y)$ is also isomorphic to H, $N_G^v(y)$ must also contain a vertex x of degree $r-1$. But then $N_G^e(xy)$ contains $r-1$ vertices adjacent to both x and y, giving a contradiction.

If $r = 2$, then G is a disjoint union of cycles. If $y_1 y_2$ is an edge of a cycle C_n with $n \geqslant 5$ then $N_G^e(y_1 y_2)$ is not connected; if each cycle is a C_4 then each $N_G^v(x)$ is not connected. So we have proved the following result.

Theorem 2.1. *Let H_1 and H_2 be connected graphs. Let the* v-*neighbourhood of each vertex of a graph G be isomorphic to H_1 and the* e-*neighbourhood of each edge of G be isomorphic to H_2. Then H_1 is not isomorphic to H_2.*

Remark. There *do* exist graphs with *disconnected* v- and e-neighbourhoods which are isomorphic to each other. For example, if $G \simeq C_n$, $n \geqslant 5$, then $N_G^v(x) \simeq N_G^e(f) \simeq 2K_1$.

Let H_1 be a graph obtained from H_2 by deletion of an edge e. Then the graphs H_1 and H_2 will be called *near graphs*.

In view of Theorem 2.1, we now vary the conditions slightly and ask the following question. Do there exist connected graphs G, H_1, H_2 such that $|V(H_1)| = |V(H_2)|$ and such that G is a v-realization of H_1 and an e-realization of H_2? We show that the answer to this question is positive and that, in addition, H_1 and H_2 must be near graphs. Indeed, it follows from the construction of the graph H in the previous proof that

$$H_2 = H \simeq K_r - \frac{r-2}{2} K_2,$$

so that $H_1 \simeq K_r - \frac{r}{2} K_2$ and $G \simeq K_{r+2} - \frac{r+2}{2} K_2$; and it is clear that these are the only graphs which satisfy the required conditions. So the following theorem is now proved.

THEOREM 2.2. *Let G, H_1, H_2 be connected graphs such that*
(i) $N_G^v(x) \simeq H_1$ *for each vertex* x *of* G,
(ii) $N_G^e(f) \simeq H_2$ *for each edge* f *of* G,
(iii) $|V(H_1)| = |V(H_2)| = r$.
Then $r = 2k$ *for some* $k \geq 2$, H_1 *and* H_2 *are near graphs, and*

$$G \simeq K_{2k+2} - (k+1)K_2; \qquad H_1 \simeq K_{2k} - kK_2; \qquad H_2 \simeq K_{2k} - (k-1)K_2.$$

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