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Bowen Measure From Heteroclinic Points

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Abstract. We present a new construction of the entropy-maximizing, invariant probability measure on a Smale space (the Bowen measure). Our construction is based on points that are unstably equivalent to one given point, and stably equivalent to another, *i.e.*, heteroclinic points. The spirit of the construction is similar to Bowen's construction from periodic points, though the techniques are very different. We also prove results about the growth rate of certain sets of heteroclinic points, and about the stable and unstable components of the Bowen measure. The approach we take is to prove results through direct computation for the case of a Shift of Finite type, and then use resolving factor maps to extend the results to more general Smale spaces.

1 Introduction

A Smale space, as defined by David Ruelle [11], is a compact metric space, X, together with a homeomorphism, φ, which is hyperbolic. These include the basic sets of Smale's Axiom A systems [13]. Another special case of great interest are the shifts of finite type [3, 6] where the space, here usually denoted Σ, is the path space of a finite directed graph and the homeomorphism, σ, is the left shift.

The structure of (X, φ) is such that each point x in X has two local sets associated with it: $X^s(x, \epsilon)$, on which the map φ is (uniformly) contracting; and $X^u(x, \epsilon)$, on which the map φ^{-1} is contracting. We call these sets the local stable and unstable sets for x. Furthermore, x has a neighbourhood $U(x, \epsilon)$ that is isomorphic to $X^u(x, \epsilon) \times X^s(x, \epsilon)$. In other words, the sets $X^u(x, \epsilon)$ and $X^s(x, \epsilon)$ provide a coordinate system for $U(x, \epsilon)$ such that, under application of the map φ, one coordinate contracts, and the other expands.

The basic axiom for a Smale space is the existence of a map defined on pairs (x, y) in $X \times X$ that are sufficiently close. The image of (x, y) is denoted $[x, y]$ and is the unique point in $X^s(x, \epsilon) \cap X^u(y, \epsilon)$. This satisfies a number of identities and in particular defines a homeomorphism from $X^u(x, \epsilon) \times X^s(x, \epsilon) \to U(x, \epsilon)$.

There is also a notion of a global stable (unstable) set for a point x, which we denote $X^s(x)$ $(X^u(x))$. This is simply the set of all points $y \in X$ such that

$$d\big(\varphi^n(x), \varphi^n(y)\big) \longrightarrow 0$$

as $n \to +\infty$ $(-\infty)$. The collection of sets $\{X^s(y, \delta) \mid y \in X^s(x), \delta > 0\}$ forms a neighbourhood base for a topology on $X^s(x)$ that is locally compact and Hausdorff. This is the topology that we use on $X^s(x)$ (not the relative topology from X). There

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is an analogous topology on $X^u(x)$. The global stable (unstable) sets partition the Smale space X into equivalence classes. In other words, there are three equivalence relations defined on X. We say x and y are *stably equivalent* if $X^s(x) = X^s(y)$, *unstably equivalent* if $X^u(x) = X^u(y)$, and *homoclinic* if they are both stably and unstably equivalent. Finally, we say that a point z is a heteroclinic point for the pair (x, y) if z is stably equivalent to x and unstably equivalent to y (*i.e.*, $z \in X^s(x) \cap X^u(y)$).

For an irreducible Smale space, (X, φ), there is a unique φ-invariant probability measure maximizing the entropy of φ [5, 12]. This measure is known as the Bowen measure, and we denote it by μ_X, or when the space is obvious, simply by μ.

In [2], Bowen constructed the measure of maximum entropy as a limit of measures supported on periodic points. Our main goal in this paper is to present an alternative construction in which the Bowen measure is obtained as the limit of measures supported on heteroclinic points. The main result is Theorem 2.10, which is proved in Section 4. From our construction we are also able to relate the growth rate of certain sets of heteroclinic points to the topological entropy of the Smale space. A similar result concerning the growth rate of homoclinic orbits was proved by Mendoza in [7], using different techniques.

2 Main Results

It was shown in [12] that if a small subset of X is written as a product, then the Bowen measure on this set can be written as a product measure. This gives us a useful way of dealing with the Bowen measure. The following theorem makes this result precise. While this theorem is due to Ruelle and Sullivan, we will provide a new proof of the result. Along the way, we will also see how this product decomposition is preserved under resolving maps.

Theorem 2.1 *Let X be an irreducible Smale space. For each x in X, there exist measures $\mu_X^{s,x}$ and $\mu_X^{u,x}$ defined on $X^s(x)$ and $X^u(x)$, respectively. These measures are not finite, but are regular Borel measures. Moreover, these satisfy the following conditions.*

(i) *For all x in X, $\epsilon > 0$ and Borel sets $B \subset X^u(x, \epsilon)$ and $C \subset X^s(x, \epsilon)$, we have*

$$\mu([B, C]) = \mu^{u,x}(B)\mu^{s,x}(C),$$

whenever ϵ is sufficiently small so that $[B, C]$ is defined.

(ii) *For x, y in X, $\epsilon > 0$ and a Borel set $B \subset X^u(x, \epsilon)$, we have*

$$\mu^{u,y}([B, y]) = \mu^{u,x}(B),$$

whenever $d(x, y)$ and ϵ are sufficiently small so that $[B, y]$ is defined.

(iii) *For x, y in X, $\epsilon > 0$ and a Borel set $C \subset X^s(x, \epsilon)$, we have*

$$\mu^{s,y}([y, C]) = \mu^{s,x}(C),$$

whenever $d(x, y)$ and ϵ are sufficiently small so that $[y, C]$ is defined.

(iv) $\mu^{s,\varphi(x)} \circ \varphi = \lambda^{-1}\mu^{s,x}$.

(v) $\mu^{u,\varphi(x)} \circ \varphi = \lambda \mu^{u,x}$.

Here $\log(\lambda)$ *is the topological entropy of* (X, φ).

In [2] the unique entropy maximizing φ-invariant probability measure is constructed as the weak-$*$ limit of the sequence μ_n, where μ_n is defined as follows. Let $S_n = \bigcup_1^n \mathrm{Per}_k(X, \varphi)$, then

$$\mu_n = \frac{1}{\#S_n} \sum_{z \in S_n} \delta_z,$$

where δ_z is the point mass at z. In our construction we use points that are heteroclinic to a given pair of points instead of periodic points. It is worth noting that in Bowen's construction each μ_n is a φ-invariant probability measure. In our case, the measures constructed are not φ-invariant, but in the limit we recover φ-invariance.

Definition 2.2 Let (X, φ) be a mixing Smale space, x, $y \in X$, $B \subset X^u(x)$, and $C \subset X^s(y)$ open with compact closure. For each positive integer k, we define

$$h^k_{B,C} = \varphi^k(B) \cap \varphi^{-k}(C)$$

and the measure

$$\mu^k_{B,C} = \frac{1}{\#h^k_{B,C}} \sum_{z \in h^k_{B,C}} \delta_z.$$

Remark 2.3 • As $X^u(x)$ and $X^s(y)$ intersect transversally and $\varphi^k(B)$ and $\varphi^{-k}(C)$ have compact closure for each k, $\#h^k_{B,C}$ is finite for each k.

- $h^k_{B,C}$ may be empty, and hence $\mu^k_{B,C}$ may not be well defined for some positive integers k. However, for given B, C there exists a K such that for all $k > K$, $\mu^k_{B,C}$ is well defined. Since we will be interested in the (weak-$*$) limit of these measures as $k \to \infty$, we will not be concerned with the finite number of k's for which our definition is not valid.

We have the following result relating the growth of the heteroclinic sets $h^k_{B,C}$ to the topological entropy.

Theorem 2.4 *Let* (X, φ) *be a mixing Smale space,* B, C *as in Definition* 2.2*. Then we have*

$$\lim_{k \to \infty} \lambda^{-2k} \#h^k_{B,C} = \mu^{u,x}_X(B) \mu^{s,y}_X(C),$$

where $\mu^{u,x}_X$ *and* $\mu^{s,y}_X$ *are as in Theorem* 2.1, *and* $\log(\lambda) = h(X, \varphi)$ *is the topological entropy of* (X, φ)*. In consequence, we also have*

$$\lim_{k \to \infty} \frac{\log(\#h^k_{B,C})}{2k} = h(X, \varphi).$$

Theorem 2.5 *Let* (X, φ) *be a mixing Smale space, and let* $\mu^k_{B,C}$ *be as in Definition* 2.2*. For each continuous function* $f \colon X \to \mathbb{C}$ *we have*

$$\lim_{k \to \infty} \int_X f \, d\mu^k_{B,C} = \int_X f \, d\mu_X,$$

where μ_X is the Bowen measure. In other words $\mu^k_{B,C} \to \mu_X$ in the weak-$*$ topology.

Now suppose (X, φ) is an irreducible Smale space (not necessarily mixing). By Smale's spectral decomposition [13] we can find a partition of X into pairwise disjoint clopen subsets, $X_1, X_2, \ldots, X_I$ such that $\varphi(X_i) = X_{i+1}$ (with the indices interpreted modulo I) and $\varphi^I|X_i$ mixing for each i.

Definition 2.6 With the notation as above, let x, y be in the same component, X_{i_0}, of X and let $B \subset X^u(x)$ and $C \subset X^s(y)$ be open with compact closures. For each k, we define

$$h^k_{B,C} = \cup_{i=0}^{I-1}\big(\varphi^{kI+i}(B) \cap \varphi^{-kI+i}(C)\big)$$

and the measure

$$\mu^k_{B,C} = \frac{1}{\# h^k_{B,C}} \sum_{z \in h^k_{B,C}} \delta_z.$$

Remark 2.7 • The same remark applies as before concerning $h^k_{B,C}$ being empty.
- In the case that (X, φ) is mixing (and $I = 1$), this clearly reduces to the same definition as before.

With this extended definition, the analogous results as stated above for the mixing case also hold in the irreducible case.

Theorem 2.8 *Let (X, φ) be an irreducible Smale space, B, C as in Definition 2.6. Then we have*

$$\lim_{k\to\infty} \lambda^{-2kI} \# h^k_{B,C} = I \mu^{u,x}_X(B)\mu^{s,y}_X(C),$$

where $\mu^{u,x}_X$ and $\mu^{s,y}_X$ are as in Theorem 2.1, and $\log(\lambda) = h(X, \varphi)$ is the topological entropy of (X, φ). In consequence, we also have

$$\lim_{k\to\infty} \frac{\log(\# h^k_{B,C})}{2kI} = h(X, \varphi).$$

Remark 2.9 This result is essentially [7, Theorem 3.1], replacing $h^k_{B,C}$ with $\varphi^k(h^k_{B,C})$ in the case that the heteroclinic points happen to be homoclinic points.

Theorem 2.10 *Let (X, φ) be an irreducible Smale space, and let $\mu^k_{B,C}$ be as in Definition 2.6. For each continuous function $f\colon X \to \mathbb{C}$ we have*

$$\lim_{k\to\infty} \int_X f\,d\mu^k_{B,C} = \int_X f\,d\mu_X,$$

where μ_X is the Bowen measure. In other words, $\mu^k_{B,C} \to \mu_X$ in the weak-$*$ topology.

3 Resolving Factor Maps and the Bowen Measure

In the case that the Smale Space is a shift of finite type (SFT), the Bowen measure is the same as the Parry measure. We present a brief description of the Parry measure for a *mixing* SFT and prove Theorem 2.1 in this case.

Let (Σ, σ) be a mixing SFT, considered as the edge shift on a directed graph G with adjacency matrix A. See [6] for a thorough treatment of SFTs. Then (Σ, σ) is mixing precisely when A is primitive, *i.e.*, when there exists N such that, for $n \geq N$, A^n is strictly positive. This allows us to use the consequence of the Perron–Frobenius theorem ([6, Thm. 4.5.12]), which says $\lim_{n\to\infty} \lambda^{-n}A^n = u_r u_l$, where u_r, u_l are the right/left Perron–Frobenius eigenvectors of the matrix A normalized so that $u_l u_r = 1$, and λ is the Perron–Frobenius eigenvalue. This result is critical in the proof of our main result in the case of SFTs. Fix $m > N$, vertices v_i, v_j in the graph, and let ξ be a path of length $2m$, indexed from $-m+1$ to m, originating at v_i and terminating at v_j (A primitive guarantees such a ξ exists). Consider the set

$$\Sigma_{m,i,j}(\xi) = \{x \in \Sigma \mid x_k = \xi_k \text{ for } -m+1 \leq k \leq m\}.$$

The collection of such sets, as m, i, j, and ξ vary over all possible values, forms a base for the topology on Σ. The Parry measure on such a basic set is

$$\mu_\Sigma\big(\Sigma_{m,i,j}(\xi)\big) = \lambda^{-2m} u_l(i) u_r(j).$$

Fix x in Σ and suppose $t(x_m) = v_j$, $i(x_{-l+1}) = v_i$. That is to say, x_{-l+1} originates at vertex v_i, and x_m terminates at vertex v_j. Consider the sets

$$\Sigma^u(x, 2^{-m}) = \{z \in \Sigma \mid z_k = x_k \ \forall k \leq m\},$$
$$\Sigma^s(x, 2^{-l}) = \{z \in \Sigma \mid z_k = x_k \ \forall k \geq -l+1\}.$$

These sets form a base for the topology on $\Sigma^u(x)$ (respectively $\Sigma^s(x)$) in a neighbourhood of x. Suppose now that $\Sigma^u(z, 2^{-m}) \subset \Sigma^u(x, \epsilon)$ and $\Sigma^s(y, 2^{-l}) \subset \Sigma^s(x, \epsilon)$ Then the stable/unstable components of the Parry measure are

$$\mu_\Sigma^{u,x}(\Sigma^u(z, 2^{-m})) = \lambda^{-m} u_r(j), \quad \mu_\Sigma^{s,x}(\Sigma^s(y, 2^{-l})) = \lambda^{-l} u_l(i).$$

Proposition 3.1 *Theorem* 2.1 *holds for* (Σ, σ) *a mixing SFT, with* $\mu_\Sigma^{u,x}$, $\mu_\Sigma^{s,x}$ *defined as above.*

Proof We must verify the five conditions stated in Theorem 2.1.

(i) This is obvious from the formulas defining the measures on basic sets.

(ii) Consider the homeomorphism $w \mapsto [w, x']$ from $\Sigma^u(x, \epsilon)$ to $\Sigma^u(x', \epsilon')$. Under this map,

$$\Sigma^u(z, 2^{-m}) \mapsto \big\{ v \in \Sigma \mid v_k = z_k \forall 0 \leq k \leq m, v_k = x'_k \forall k \leq 0 \big\} = \Sigma^u\big([z, x'], 2^{-m}\big).$$

Now,

$$\mu_\Sigma^{u,x'}\big(\Sigma^u([z, x'], 2^{-m})\big) = \lambda^{-m} u_r(j) = \mu_\Sigma^{u,x}\big(\Sigma^u(z, 2^{-m})\big).$$

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(iii) Similarly, the map $w \mapsto [x', w]$ takes the measure $\mu_{\Sigma}^{s,x}$ to $\mu_{\Sigma}^{s,x'}$.
(iv) Now consider

$$\begin{aligned}(\mu_{\Sigma}^{u,\sigma(x)} \circ \sigma)\big(\Sigma^u(z, 2^{-m})\big) &= \mu_{\Sigma}^{u,\sigma(x)}\big(\Sigma^u(\sigma(z), 2^{-m+1})\big) \\ &= \lambda^{-m+1} u_r(j) = \lambda \mu_{\Sigma}^{u,x}\big(\Sigma^u(z, 2^{-m})\big).\end{aligned}$$

(v) Similarly,

$$\begin{aligned}(\mu_{\Sigma}^{s,\sigma(x)} \circ \sigma)\big(\Sigma^s(y, 2^{-l})\big) &= \mu_{\Sigma}^{s,\sigma(x)}\big(\Sigma^s(\sigma(y), 2^{-l-1})\big) \\ &= \lambda^{-l-1} u_l(i) = \lambda^{-1} \mu_{\Sigma}^{s,x}\big(\Sigma^s(y, 2^{-l})\big).\end{aligned}$$ ■

In the case of a SFT, the topological entropy $h(\Sigma, \sigma) = \log(\lambda)$, where λ is the Perron–Frobenius eigenvalue of the adjacency matrix associated with the SFT. Similarly, for other Smale spaces X we will write λ such that $h(X, \varphi) = \log(\lambda)$. Whenever we are talking about two or more Smale spaces, there will be an almost one-to-one factor map between them, so the entropies will be equal, hence it will be unnecessary to distinguish which space the λ comes from.

Definition 3.2 (Fried [4]) A factor map $\pi\colon (Y, \psi) \to (X, \varphi)$ is *s-resolving* (*u-resolving*) if for every $y \in Y$, $\pi|_{Y^s(y)}$ ($\pi|_{Y^u(y)}$ respectively) is injective.

We will primarily be concerned with *almost one-to-one* resolving factor maps. A factor map $\pi\colon (Y, \psi) \to (X, \varphi)$, where (Y, ψ) is irreducible, is called *almost one-to-one* if there exists $x \in X$ such that $\#\pi^{-1}(x) = 1$.

In [1], Bowen showed that for an irreducible Smale space (X, φ), there exists an irreducible SFT (Σ, σ) and an almost one-to-one factor map $\pi\colon \Sigma \to X$. Moreover, letting $E = \{x \in X \mid \#\pi^{-1}(x) = 1\}$, Bowen showed that $\mu_{\Sigma}(\pi^{-1}(E)) = 1$. In other words, π is one-to-one μ_{Σ}-a.e. It follows that for any Borel set $B \subset X$, $\mu_X(B) = \mu_{\Sigma}(\pi^{-1}(B))$ ([1, Thm. 34]).

In [9, Cor. 1.4], the second author showed that the factor map π can be realized as the composition of two resolving factor maps. In other words, given an irreducible Smale space (X, φ), we can find a Smale space (Y, ψ), a SFT (Σ, σ), and factor maps $\pi_1\colon \Sigma \to Y$, $\pi_2\colon Y \to X$ such that

(i) (Σ, σ) and (Y, ψ) are irreducible;
(ii) π_1 and π_2 are almost one-to-one;
(iii) π_1 is s-resolving and π_2 is u-resolving.

The Bowen measures on X, Y can be obtained from the Bowen measure on Σ as follows:

(i) for $E \subset Y$ the Bowen measure on (Y, ψ) is $\mu_Y(E) = \mu_{\Sigma}(\pi_1^{-1}(E))$;
(ii) for $F \subset X$ the Bowen measure on (X, φ) is

$$\mu_X(F) = \mu_Y\big(\pi_2^{-1}(F)\big) = \mu_{\Sigma}\big((\pi_2 \circ \pi_1)^{-1}(F)\big).$$

This requires only that π_1, π_2 be almost one-to-one factor maps, not that they are resolving. We now wish to define the measures on the stable and unstable equivalence classes in (Y,ψ) and (X,φ), from $\mu_\Sigma^{s,\cdot}$, $\mu_\Sigma^{u,\cdot}$, π_1, and π_2. In this case, it is not enough that the factor maps are almost one-to-one; resolving plays an important role in what follows. We begin by stating the following result which is proved by the second author in [8].

Proposition 3.3 *Let (Y,ψ) and (X,φ) be irreducible Smale spaces and let $\pi\colon Y\to X$ be an almost one-to-one u-resolving factor map. If $x\in X$ with $\pi^{-1}(x)=\{y_1,y_2,\dots,y_n\}$, then*

$$\pi^{-1}(X^u(x))=\bigcup_{i=1}^{n} Y^u(y_i),$$

and the union is disjoint. Moreover, using the topologies from the introduction, for each $1\le i\le n$

$$\pi|_{Y^u(y_i)}\colon Y^u(y_i)\longrightarrow X^u(x)$$

is a homeomorphism.

Lemma 3.4 *Let (Y,ψ) and (X,φ) be irreducible Smale spaces and let $\pi\colon Y\to X$ be an almost one-to-one u-resolving factor map. Fix $y\in Y$, the set $\{y'\in Y^s(y)\mid \pi(y')=\pi(\tilde y)$ for some $\tilde y\neq y'\}$ has $\mu_Y^{s,y}$ measure zero. In other words, $\pi|_{Y^s(y)}$ is one-to-one $\mu_Y^{s,y}$ almost everywhere.*

Proof As Y is compact, we may cover Y with a finite number of sets of the form $U_i=[Y^u(z_i,\delta_i),Y^s(z_i,\delta_i)]$. Fix U_i and $y\in U_i$, let $B_i=[Y^u(z_i,\delta_i),y]$, $C_i=[y,Y^s(z_i,\delta_i)]$, so we can write $U_i=[B_i,C_i]$.

Let $S_i=\{y'\in C_i\mid \pi(y')=\pi(\tilde y)$ for some $\tilde y\neq y'\}$. Since π is u-resolving, the set $U_i\cap\{y'\in Y\mid \pi(y')=\pi(z)$ for some $z\neq y'\}=[B_i,S_i]$. Now, we know that π is one-to-one μ_Y almost everywhere, so

$$0=\mu_Y\big([B_i,S_i]\big)=\mu_Y^{u,y}(B_i)\mu_Y^{s,y}(S_i).$$

We also know that

$$0\neq\mu_Y(U_i)=\mu_Y([B_i,C_i])=\mu_Y^{u,y}(B_i)\mu_Y^{s,y}(C_i).$$

So $\mu_Y^{u,y}(B_i)\neq 0$ and thus $\mu_Y^{s,y}(S_i)=0$. The conclusion follows. ∎

Note that the analogous result with an s-resolving map and $\mu_Y^{u,y}$ also holds.

Proposition 3.5 *Let (Y,ψ) and (X,φ) be irreducible Smale spaces and let $\pi\colon Y\to X$ be an almost one-to-one u-resolving factor map. Let $x\in X$ and $y_1,y_2\in\pi^{-1}\{x\}$. Let $B\subset X^u(x,\epsilon)$ be a Borel set, then $\mu_Y^{u,y_1}(\pi^{-1}(B))=\mu_Y^{u,y_2}(\pi^{-1}(B))$.*

Proof For each $z_1\in Y^u(y_1)$ there exists a unique $z_2\in Y^u(y_2)$ such that $\pi(z_1)=\pi(z_2)$. Consider the following set:

$$E=\{z_1\in Y^u(y_1)\mid z_2\in Y^s(y_1)\}$$

and its compliment in $Y^u(y_1)$, E^c. We will show that $\mu_Y^{u,y_1}(E^c)=0$, and that on E the map defined by $f(z_1)=z_2$ takes the measure μ_Y^{u,y_1} to μ_Y^{u,y_2}.

We begin by showing that E is non-empty. Fix $x \in X$ such that x has a unique pre-image under π, $\pi^{-1}\{x\} = \{y\}$. Now, since $Y^u(y_1)$ is dense in Y, we can find a sequence $\{z_i\} \subset Y^u(y_1)$ such that $z_i \to y$. Now consider the sequence $\{z_i'\} \subset Y^u(y_2)$, where $\pi(z_i) = \pi(z_i')$. By the compactness of Y, $\{z_i'\}$ has a convergent subsequence $\{z_{i_k}'\}$. Denote the limit of this subsequence by y'. Now by the continuity of π we have

$$\pi(y') = \pi(\lim_{k\to\infty} z_{i_k}') = \lim_{k\to\infty} \pi(z_{i_k}') = \lim_{k\to\infty} \pi(z_{i_k}) = \pi(\lim_{k\to\infty} z_{i_k}) = \pi(y) = x.$$

As x has a unique pre-image, we see that $y' = y$. It follows that for k sufficiently large, $d(z_{i_k}, z_{i_k}') < \epsilon_\pi$. Therefore, by [8, Lemma 3.3] we have $z_{i_k} \stackrel{s}{\sim} z_{i_k}'$, and hence E is non-empty.

We now show that E is open in $Y^u(y_1)$. Let $z_1 \in E$. Since $z_1 \stackrel{s}{\sim} z_2$, we can find n large enough so that $d(\psi^n(z_1), \psi^n(z_2)) < \epsilon_\pi/3$. Choose δ small enough so that $Y^u(\psi^n(z_1, \delta)) \subset Y^u(\psi^n(z_1), \epsilon_\pi/3)$ and choose $U(\psi^n(z_2)) \subset Y^u(\psi^n(z_2), \epsilon_\pi/3)$, where $\pi(U(\psi^n(z_2))) = \pi(Y^u(\psi^n(z_1), \delta))$. Let $A_1 = \psi^{-n}(Y^u(\psi^n(z_1), \delta))$, $A_2 = \psi^{-n}(U(\psi^n(z_2)))$. Now for each $z \in A_1$, the unique $z' \in A_2$ such that $\pi(z) = \pi(z')$ is such that $d(\psi^n(z), \psi^n(z')) < \epsilon_\pi$ and $\pi(\psi^n(z)) = \pi(\psi^n(z'))$. By [8, Lemma 3.3] we have that $\psi^n(z) \stackrel{s}{\sim} \psi^n(z')$ and therefore $z \stackrel{s}{\sim} z'$. So $A_1 \in E$, and hence E is open.

Now E open (and non-empty) implies that $\mu_Y^{u,y_1}(E) > 0$, and since E is ψ-invariant and $\mu_Y^{u,y_1}(\psi(E)) = \lambda\mu_Y^{u,y_1}(E)$, we must have that $\mu_Y^{u,y_1}(E) = \infty$.

We now show that on the set E the map $f(z_1) = z_2$ takes μ_Y^{u,y_1} to μ_Y^{u,y_2}. Let n, A_1, A_2 be as above. Then for $z \in A_1$ the map $f(z) = z' \in A_2$ can be written as $f(z) = \psi^{-n}([\psi^n(z), \psi^n(z_2)])$. So for a Borel set $B \subset A$ we have

$$\begin{aligned}
\mu_Y^{u,y_2} \circ f(B) &= \mu_Y^{u,y_2}\big(\psi^{-n}\big([\psi^n(B), \psi^n(z_2)]\big)\big)\\
&= \lambda^{-n}\mu_Y^{u,y_2}\big([\psi^n(B), \psi^n(z_2)]\big)\\
&= \lambda^{-n}\mu_Y^{u,y_1}\big(\psi^n(B)\big) = \mu_Y^{u,y_1}(B).
\end{aligned}$$

It remains to show that $\mu_Y^{u,y_1}(E^c) = 0$ (by the above ψ-invariance remark, the measure of E^c is either 0 or ∞). Fix $y \in Y^u(y_1)$, $\delta < \epsilon_Y/2$ and consider the sets $A_1 = [E \cap Y^u(y, \delta), Y^s(y, \delta)]$, $A_2 = [E^c \cap Y^u(y, \delta), Y^s(y, \delta)]$. We know that

$$\mu_Y(A_1) = \mu_Y^{u,y_1}\big(E \cap Y^u(y, \delta)\big)\,\mu_Y^{s,y}\big(Y^s(y, \delta)\big) > 0$$

and

$$\mu_Y(A_2) = \mu_Y^{u,y_1}\big(E^c \cap Y^u(y, \delta)\big)\,\mu_Y^{s,y}\big(Y^s(y, \delta)\big).$$

Since $\mu_Y^{s,y}(Y^s(y, \delta)) > 0$, to prove that $\mu_Y^{u,y_1}(E^c) = 0$ it suffices to show that $\mu_Y(A_2) = 0$. To this end consider $\psi(A_2)$. A typical point $z \in A_2$ can be written $z \in Y^s(z', \delta)$, where $z' = [z, y] \in E^c$. So $\psi(z) \in \psi(Y^s(z', \delta)) \subset Y^s(\psi(z'), \delta)$, where $\psi(z') \in E^c$. This shows that $A_1 \cap \psi(A_2) = \varnothing$. Similarly we can show that $A_1 \cap \psi^k(A_2) = \varnothing$ for any $k \geq 0$. However, ψ is strong mixing with respect to μ_Y, so we have

$$\mu_Y(A_1)\mu_Y(A_2) = \lim_{k\to\infty} \mu_Y\big(A_1 \cap \psi^k(A_2)\big) = \mu_Y(\varnothing) = 0.$$

Since we know $\mu_Y(A_1) > 0$, we have $\mu_Y(A_2) = 0$, and hence $\mu_Y^{u,y_1}(E^c) = 0$. ∎

We state the following result that was proved by the second author as [10, Theorem 2.5.3].

Theorem 3.6 *Let (Y,ψ) and (X,φ) be Smale spaces and let $\pi\colon Y \to X$ be an almost one-to-one u-resolving factor map. There is a constant $M \geq 1$ such that:*

(i) *for any $x \in X$ there exist $y_1, \dots, y_K$ with $K \leq M$ such that*

$$\pi^{-1}\big(X^s(x)\big) = \bigcup_{i=k}^{K} Y^s(y_k);$$

(ii) *for any $x \in X, \#\pi^{-1}\{x\} \leq M$.*

The previous two results allow us to make the following definition.

Definition 3.7 Let (Y,ψ) and (X,φ) be irreducible Smale spaces and let $\pi\colon Y \to X$ be an almost one-to-one u-resolving factor map. Let $x \in X$. Fix $y \in \pi^{-1}\{x\}$, and fix $\{y_1, \dots, y_K\}$ as in Theorem 3.6.

Define measures on $X^s(x), X^u(x)$ by

$$\mu_X^{s,x} = \sum_{k=1}^{K} \pi^* \mu_Y^{s,y_k}, \quad \mu_X^{u,x} = \pi^* \mu_Y^{u,y}.$$

Remark 3.8 We have stated Definition 3.7 in terms of an almost one-to-one u-resolving factor map. Given two Smale spaces and an almost one-to-one s-resolving factor map, we would make the analogous definition, interchanging roles of stable and unstable sets.

Proposition 3.9 *Let (Y,ψ), (X,φ) be irreducible Smale spaces and let $\pi\colon Y \to X$ be an almost one-to-one resolving factor map (s or u-resolving). Suppose Y satisfies the conclusion of Theorem* 2.1. *With the measures defined in Definition* 3.7, *X also satisfies the conclusion of Theorem* 2.1.

Proof We prove the result in the case that π is u-resolving. The s-resolving case is completely analogous. Let $x \in X$ and let $C = X^s(x_1,\delta) \subset X^s(x,\epsilon)$, $B = X^u(x_2,\delta) \subset X^u(x,\epsilon)$. Fix $y \in Y$ and $U(y) \subset Y^u(y)$ such that $\pi(y) = x_2$, $\pi(U(y)) = X^u(x_2,\delta) = B$. We need to show the following:

(i) $\mu_X([B,C]) = \mu_X^{u,x}(B)\mu_X^{s,x}(C)$;
(ii) For z close to x, $\mu_X^{u,x}(B) = \mu_X^{u,z}([B,z])$;
(iii) $\mu_X^{u,\varphi(x)}(\varphi(B)) = \lambda\mu_X^{u,x}(B)$;
(iv) For z close to x, $\mu_X^{s,x}(C) = \mu_X^{s,z}([z,C])$;
(v) $\mu_X^{s,\varphi(x)}(\varphi(C)) = \lambda^{-1}\mu_X^{s,x}(C)$.

We will prove item (ii) first, as we will use this result in the proof of item (i).

(ii) We can find $y' \in \pi^{-1}(z)$ such that y' is "close" to y. Then

$$\mu_X^{u,x}(B) = \mu_Y^{u,y}\big(U(y)\big) = \mu_Y^{u,y'}\big([U(y),y']\big) = \mu_X^{u,z}\Big(\pi\big([U(y),y']\big)\Big)$$

but $\pi([U(y),y']) = [\pi(U(y)),\pi(y')]$ and $\pi(U(y)) = B$, $\pi(y') = z$, so we have

$$\mu_X^{u,x}(B) = \mu_X^{u,z}\Big(\pi\big([U(y),y']\big)\Big) = \mu_X^{u,z}\Big([\pi\big(U(y)\big),\pi(y')]\Big) = \mu_X^{u,z}\big([B,z]\big).$$

(i) Since $C \subset X^s(x,\epsilon)$ is open with compact closure, by Lemma 3.6 we can write

$$\pi^{-1}(C) = \bigcup_{i=1}^{m} C'_i,$$

where $C'_i \subset Y^s(y_i)$ for some $y_i \in Y$. Moreover, we write each C'_i as a disjoint union of finitely many sets

$$C'_i = \bigcup_{j=1}^{k_i} C'_{ij},$$

where $C'_{ij} \subset Y^s(y_{ij}, \epsilon_Y/2)$, and $y_{ij} \in C'_{ij}$. Let $x_{ij} = \pi(y_{ij})$, and let $B_{ij} = [B, x_{ij}]$. Let $B'_{ij} \subset Y^u(y_{ij})$ be such that $\pi\colon B'_{ij} \to B_{ij}$ is a homeomorphism. We can then write

$$[B,C] = \pi\Big(\bigcup_{i,j}[B'_{ij}, C'_{ij}]\Big).$$

So

$$\mu_X\big([B,C]\big) = \mu_Y\Big(\bigcup_{i,j}[B'_{ij}, C'_{ij}]\Big) = \sum_{i,j}\mu_Y\big([B'_{ij}, C'_{ij}]\big) = \sum_{i,j}\mu_Y^{u,y_{ij}}(B'_{ij})\mu_Y^{s,y_{ij}}(C'_{ij}).$$

Now $\mu_Y^{u,y_{ij}}(B'_{ij}) = \mu_X^{u,x_{ij}}(B_{ij}) = \mu_X^{u,x}(B)$ for all i, j (by part (ii)), so we have

$$\mu_X\big([B,C]\big) = \mu_X^{u,x}(B)\sum_{i,j}\mu_Y^{s,y_{ij}}(C'_{ij}) = \mu_X^{u,x}(B)\sum_{i}\mu_Y^{s,y_i}(C'_i) = \mu_X^{u,x}(B)\mu_X^{s,x}(C)$$

(iii) $$\mu_X^{u,\varphi(x)}\big(\varphi(B)\big) = \mu_Y^{u,\psi(y)}\big(\psi(U(y))\big) = \lambda\mu_Y^{u,y}\big(U(y)\big) = \lambda\mu_X^{u,x}(B).$$

(iv) We can find $y' \in \pi^{-1}(z)$ such that $y' \in Y^u(y,\epsilon)$. Let x_{ij}, y_{ij}, C'_i, C'_{ij} be as in part (i). Let $C_{ij} = \pi(C'_{ij})$, $z_{ij} = [z, x_{ij}]$, $C(z)_{ij} = [z, C_{ij}]$, $y'_{ij} \in \pi^{-1}(z_{ij})$ s.t. $y'_{ij} \in Y^u(y_{ij},\epsilon)$ and $\tilde{C}'_{ij} = [y'_{ij}, C'_{ij}]$. Then $z_{ij} = \pi(y'_{ij})$, $C_{ij} = \pi(C'_{ij})$, $\pi(\tilde{C}'_{ij}) = [z_{ij}, C_{ij}]$, and $\cup C(z)_{ij} = [z, \cup C_{ij}] = [z, C]$, so

$$\begin{aligned}\mu_X^{s,x}(C) &= \sum_i \mu_Y^{s,y_i}(C'_i) = \sum_{i,j}\mu_Y^{s,y_{ij}}(C'_{ij})\\ &= \sum_{i,j}\mu_Y^{s,y'_{ij}}(\tilde{C}'_{ij}) = \mu_X^{s,z}(\cup_{ij}C(z)_{ij}) = \mu_X^{s,z}([z,C])\end{aligned}$$

(v) $$\mu_X^{s,\varphi(x)}(\varphi(C)) = \sum_i \mu_Y^{s,\psi(y_i)}\big(\psi(C'_i)\big) = \sum_i \lambda_{-1}\mu_Y^{s,y_i}(C'_i) = \lambda^{-1}\mu_X^{s,x}(C).$$ ■

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Proof of Theorem 2.1. As in [9, Cor. 1.4], for the irreducible Smale space (X, φ) we can find another irreducible Smale space (Y, ψ) and an irreducible SFT (Σ, σ), as well as almost one-to-one factor maps $\pi_1 \colon \Sigma \to Y$, $\pi_2 \colon Y \to X$ such that π_1 is s-resolving and π_2 is u-resolving. The conclusion then follows from Proposition 3.1 and two applications of Proposition 3.9. ∎

4 Proof of Main Result

To prove Theorem 2.5 we first establish the result for a mixing SFT and use the machinery of resolving maps to obtain the more general result.

Proposition 4.1 *Let (Σ, σ) be a mixing SFT. Fix $x, y \in \Sigma$, $n, m \in \mathbb{Z}$ and define*

$$B = \{z \in \Sigma \mid z_i = x_i \ \forall i \leq n\} = \Sigma^u_n(x) \subset \Sigma^u(x, \epsilon_\Sigma)$$
$$C = \{z \in \Sigma \mid z_i = y_i \ \forall i \geq -m+1\} = \Sigma^s_m(y) \subset \Sigma^s(y, \epsilon_\Sigma).$$

For each function $f \in C(\Sigma)$ we have

$$\lim_{k\to\infty} \int_\Sigma f \, d\mu^k_{B,C} = \int_\Sigma f \, d\mu_\Sigma.$$

In other words, $\mu^k_{B,C} \to \mu_\Sigma$ in the weak-$*$ topology.

Proof Let A be the adjacency matrix for the SFT. It suffices to prove the result for a function of the form $e_l(\xi) = \chi_{E_l(\xi)}$, where $E_l(\xi) = \Sigma_{l,i',j'}(\xi)$. Now for $k \geq \max\{n+l, m+l\}$,

$$\int_\Sigma e_l(\xi) d\mu^k_{B,C} = \mu^k_{B,C}(E_l(\xi)) = \frac{\#\left(E_l(\xi) \cap h^k_{B,C}\right)}{\# h^k_{B,C}}.$$

Consider a point $z \in E_l(\xi) \cap h^k_{B,C}$. Since $z \in E_l(\xi)$, $z_p = \xi_p$ for all $-l+1 \leq p \leq l$, and since $z \in h^k_{B,C}$, $z_p = x_p$ for all $p \leq n-k$, $z_p = y_p$ for all $p \geq -m+1+k$. Therefore the number of points in $E_l(\xi) \cap h^k_{B,C}$ is equal to the number of paths of length $-l+1-(n-k)-1 = k-(n+l)$ from $t(\sigma^k(x)_{-k+n}) = t(x_n) = v_i$ to $i(\xi_{-l+1}) = v_{i'}$, which equals $A^{k-(n+l)}_{ii'}$ times the number of paths of length $-m+1+k-l-1 = k-(m+l)$ from $t(\xi_l) = v_{j'}$ to $i(\sigma^{-k}(y)_{k-m+1}) = i(y_{-m+1}) = v_j$ or $A^{k-(m+l)}_{j'j}$. The number of points in $h^k_{B,C}$ is the number of paths from $t(\sigma^k(x)_{-k+n}) = t(x_n) = v_i$ to $i(\sigma^{-k}(y)_{k-m+1}) = i(y_{-m+1}) = v_j$, or $A^{2k-(n+m)}_{ij}$. We therefore have

$$\int_\Sigma e_l(\xi) d\mu^k_{B,C} = \frac{A^{k-(n+l)}_{ii'} A^{k-(m+l)}_{j'j}}{A^{2k-(n+m)}_{ij}}$$

and

$$
\begin{aligned}
&\lim_{k\to\infty}\int_{\Sigma} e_l(\xi)\,d\mu^k_{B,C}\\
&\quad= \lim_{k\to\infty}\frac{A^{k-(n+l)}_{ii'}A^{k-(m+l)}_{j'j}}{A^{2k-(n+m)}_{ij}} = \lim_{k\to\infty}\frac{e_iA^{k-(n+l)}e_{i'}e_{j'}A^{k-(m+l)}e_j}{e_iA^{2k-(n+m)}e_j}\\
&\quad= \lambda^{-2l}\frac{e_i\lim_k(\lambda^{-k+(n+l)}A^{k-(n+l)})e_{i'}e_{j'}\lim_k(\lambda^{-k+(m+l)}A^{k-(m+l)})e_j}{e_i\lim_k(\lambda^{-2k+(n+m)}A^{2k-(n+m)})e_j}\\
&\quad= \lambda^{-2l}\frac{e_i(u_ru_l)e_{i'}e_{j'}(u_ru_l)e_j}{e_i(u_ru_l)e_j} \quad \text{(by [6, Thm. 4.5.12])}\\
&\quad= \lambda^{-2l}\frac{u_r(i)u_l(i')u_r(j')(u_l(j)}{u_r(i)u_l(j)}\\
&\quad= \lambda^{-2l}u_l(i')u_r(j') = \mu_\Sigma(\Sigma_{l,i',j'}(\xi)) = \int_\Sigma e_l(\xi)\,d\mu_\Sigma. \qquad\blacksquare
\end{aligned}
$$

In the above, the choice of the sets B, C, is limited to certain basic sets. We now wish to extend this result to open sets with compact closure $B' \subset \Sigma^u(x), C' \subset \Sigma^s(x)$. To do this we will first need the following lemmas.

Lemma 4.2 *Let (Σ,σ) be a mixing SFT. Fix $x, y \in \Sigma$, $n, m \in \mathbb{Z}$ and define*

$$
\begin{aligned}
B &= \{z \in \Sigma \mid z_i = x_i\ \forall i \le n\} = \Sigma^u_n(x) \subset \Sigma^u(x,\epsilon_\Sigma)\\
C &= \{z \in \Sigma \mid z_i = y_i\ \forall i \ge -m+1\} = \Sigma^s_m(y) \subset \Sigma^s(y,\epsilon_\Sigma).
\end{aligned}
$$

Then

$$\lim_{k\to\infty}\lambda^{-2k}\#h^k_{B,C} = \mu^{u,x}_\Sigma(B)\mu^{s,y}_\Sigma(C),$$

where $\log(\lambda) = h(\Sigma,\sigma)$.

Proof Let $t(x_n) = v_i$ and $i(y_{-m+1}) = v_j$. We then have

$$
\begin{aligned}
\lim_{k\to\infty}\lambda^{-2k}\#h^k_{B,C} &= \lim_{k\to\infty}\lambda^{-2k}A^{2k-(n+m)}_{ij}\\
&= \lambda^{-(n+m)}\lim_{k\to\infty}\lambda^{-2k+n+m}e_iA^{2k-(n+m)}e_j = \lambda^{-(n+m)}e_iu_ru_le_j\\
&= \lambda^{-n}u_r(i)\lambda^{-m}u_l(j) = \mu^{u,x}_\Sigma(B)\mu^{s,y}_\Sigma(C). \qquad\blacksquare
\end{aligned}
$$

Lemma 4.3 *Let $B \subset \Sigma^u(x)$, $C \subset \Sigma^s(y)$ be open and compact. Then*

$$\lim_{k\to\infty}\lambda^{-2k}\#h^k_{B,C} = \mu^{u,x}_\Sigma(B)\mu^{s,y}_\Sigma(C),$$

where $\log(\lambda) = h(\Sigma,\sigma)$.

Proof If B and C are clopen, then each is a finite disjoint union of cylinder sets of the form considered in Lemma 4.2. Let

$$B = \sum_{i=1}^{n} B_i, \quad C = \sum_{i=1}^{m} C_i,$$

then for fixed k, the $h^k_{B_i,C_j}$ are pairwise disjoint and $\cup_{i,j} h^k_{B_i,C_j} = h^k_{B,C}$. Using Lemma 4.2 we can now write

$$\begin{aligned}\lim_{k\to\infty} \lambda^{-2k}\#h^k_{B,C} &= \lim_{k\to\infty} \sum_{i,j} \lambda^{-2k}\#h^k_{B_i,C_j} = \sum_{i,j} \lim_{k\to\infty} \lambda^{-2k}\#h^k_{B_i,C_j} \\ &= \sum_{i,j} \mu^{u,x}_\Sigma(B_i)\mu^{s,y}_\Sigma(C_j) = \mu^{u,x}_\Sigma(B)\mu^{s,y}_\Sigma(C).\end{aligned}$$

■

Lemma 4.4 *Let $B \subset \Sigma^u(x)$, $C \subset \Sigma^s(y)$ be open with compact closure. Then*

$$\lim_{k\to\infty} \lambda^{-2k}\#h^k_{B,C} = \mu^{u,x}_\Sigma(B)\mu^{s,y}_\Sigma(C),$$

where $\log(\lambda) = h(\Sigma, \sigma)$.

Proof Fix $\epsilon > 0$. We can find sets $B_1 \subseteq B \subseteq B_2 \subset \Sigma^u(x)$ and $C_1 \subseteq C \subseteq C_2 \subset \Sigma^s(y)$ such that B_1, B_2, C_1, and C_2 are compact and open and

$$\mu^{u,x}_\Sigma(B_2)\mu^{s,y}_\Sigma(C_2) - \epsilon < \mu^{u,x}_\Sigma(B)\mu^{s,y}_\Sigma(C) < \mu^{u,x}_\Sigma(B_1)\mu^{s,y}_\Sigma(C_1) + \epsilon.$$

Notice that $\#h^k_{B_1,C_1} \le \#h^k_{B,C} \le \#h^k_{B_2,C_2}$, so

$$\begin{aligned}\mu^{u,x}_\Sigma(B)\mu^{s,y}_\Sigma(C) - \epsilon < \mu^{u,x}_\Sigma(B_1)\mu^{s,y}_\Sigma(C_1) = \lim_{k\to\infty} \lambda^{-2k}\#h^k_{B_1,C_1} \le \liminf_{k\to\infty} \lambda^{-2k}\#h^k_{B,C}, \\ \mu^{u,x}_\Sigma(B)\mu^{s,y}_\Sigma(C) + \epsilon > \mu^{u,x}_\Sigma(B_2)\mu^{s,y}_\Sigma(C_2) = \lim_{k\to\infty} \lambda^{-2k}\#h^k_{B_2,C_2} \ge \limsup_{k\to\infty} \lambda^{-2k}\#h^k_{B,C}.\end{aligned}$$

As this hold for all $\epsilon > 0$, we have

$$\limsup_{k\to\infty} \lambda^{-2k}\#h^k_{B,C} \le \mu^{u,x}_\Sigma(B)\mu^{s,y}_\Sigma(C) \le \liminf_{k\to\infty} \lambda^{-2k}\#h^k_{B,C}$$

and hence $\lim_{k\to\infty} \lambda^{-2k}\#h^k_{B,C} = \mu^{u,x}_\Sigma(B)\mu^{s,y}_\Sigma(C)$. ■

We are now ready to prove the more general version of Propostion 4.1.

Proposition 4.5 *The result of Proposition* 4.1 *holds with $B \subset \Sigma^u(x)$, $C \subset \Sigma^s(y)$ open with compact closure.*

Proof We can write $B = \bigcup_i B_i$, $C = \bigcup_j C_i$, where each B_i, C_i is of the form considered in Proposition 4.1, and the unions are disjoint. For brevity we write

$$h^k = h^k_{B,C}, \quad \mu^k = \mu^k_{B,C},$$

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$$h_{ij}^k = h_{B_i,C_j}^k, \quad \mu_{ij}^k = \mu_{B_i,C_j}^k.$$

Notice that for fixed k, the h_{ij}^k's are pairwise disjoint and $\cup_{i,j} h_{ij}^k = h^k$. We can write

$$\lim_{k\to\infty} \int_\Sigma f d\mu^k = \lim_{k\to\infty} \sum_{i,j} \frac{\# h_{ij}^k}{\# h^k} \int_\Sigma f d\mu_{ij}^k.$$

Now let $M = \sup_{z\in\Sigma} |f(z)|$, which is finite as f is continuous and Σ is compact. For each k, $\sum_{i,j} \# h_{ij}^k / \# h^k = 1$ so for any $I \in \mathbb{N}$ we can write

$$1 = \lim_{k\to\infty} \sum_{i,j} \frac{\# h_{ij}^k}{\# h^k} = \lim_{k\to\infty} \sum_{i,j=1}^{I} \frac{\# h_{ij}^k}{\# h^k} + \lim_{k\to\infty} \sum_{I^+} \frac{\# h_{ij}^k}{\# h^k},$$

where I^+ is the set of all pairs (i, j) such that either $i > I$, or $j > I$. We also know that

$$1 = \sum_{i,j} \frac{\mu_\Sigma^{u,x}(B_i)\mu_\Sigma^{s,y}(C_j)}{\mu_\Sigma^{u,x}(B)\mu_\Sigma^{s,y}(C)},$$

and we may choose I large enough so that

$$\lim_{k\to\infty} \sum_{I^+} \frac{\# h_{ij}^k}{\# h^k} < \frac{\epsilon}{2M} \quad \text{and} \quad \Big| \sum_{i,j}^{I} \frac{\mu_\Sigma^{u,x}(B_i)\mu_\Sigma^{s,y}(C_j)}{\mu_\Sigma^{u,x}(B)\mu_\Sigma^{s,y}(C)} - 1 \Big| < \frac{\epsilon}{2M}.$$

Using Lemma 4.4 and Proposition 4.1 we now have

$$\begin{aligned}
&\Big| \lim_{k\to\infty} \int_\Sigma f d\mu^k - \int_\Sigma f d\mu_\Sigma \Big| \\
&\quad = \Big| \lim_{k\to\infty} \sum_{i,j} \frac{\# h_{ij}^k}{\# h^k} \int_\Sigma f d\mu_{ij}^k - \int_\Sigma f d\mu_\Sigma \Big| \\
&\quad = \Big| \Big(\lim_{k\to\infty} \sum_{i,j}^{I} \frac{\# h_{ij}^k}{\# h^k} + \lim_{k\to\infty} \sum_{I^+} \frac{\# h_{ij}^k}{\# h^k} \Big) \int_\Sigma f d\mu_{ij}^k - \int_\Sigma f d\mu_\Sigma \Big| \\
&\quad = \Big| \sum_{i,j}^{I} \lim_{k\to\infty} \frac{\lambda^{-2k} \# h_{ij}^k}{\lambda^{-2k} \# h^k} \int_\Sigma f d\mu_{ij}^k + \lim_{k\to\infty} \sum_{I^+} \frac{\# h_{ij}^k}{\# h^k} \int_\Sigma f d\mu_{ij}^k - \int_\Sigma f d\mu_\Sigma \Big| \\
&\quad = \Big| \sum_{i,j}^{I} \frac{\mu_\Sigma^{u,x}(B_i)\mu_\Sigma^{s,y}(C_j)}{\mu_\Sigma^{u,x}(B)\mu_\Sigma^{s,y}(C)} \int_\Sigma f d\mu_\Sigma + \lim_{k\to\infty} \sum_{I^+} \frac{\# h_{ij}^k}{\# h^k} \int_\Sigma f d\mu_{ij}^k - \int_\Sigma f d\mu_\Sigma \Big| \\
&\quad \le \Big| \int_\Sigma f d\mu_\Sigma \Big(\sum_{i,j}^{I} \frac{\mu_\Sigma^{u,x}(B_i)\mu_\Sigma^{s,y}(C_j)}{\mu_\Sigma^{u,x}(B)\mu_\Sigma^{s,y}(C)} - 1 \Big) \Big| + \Big| \lim_{k\to\infty} \sum_{I^+} \frac{\# h_{ij}^k}{\# h^k} \int_\Sigma f d\mu_{ij}^k \Big|
\end{aligned}$$

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$$\leq \Big| M\Big(\sum_{i,j}^{I} \frac{\mu_\Sigma^{u,x}(B_i)\mu_\Sigma^{s,y}(C_j)}{\mu_\Sigma^{u,x}(B)\mu_\Sigma^{s,y}(C)} - 1 \Big) \Big| + \Big| M \lim_{k\to\infty} \sum_{I^+} \frac{\#h_{ij}^k}{\#h^k} \Big| < M\frac{\epsilon}{2M} + M\frac{\epsilon}{2M}$$
$$= \epsilon.$$

This holds for all $\epsilon > 0$, so $\lim_{k\to\infty} \int_\Sigma f d\mu^k = \int_\Sigma f d\mu_\Sigma$ ■

We now wish to extend this result to the mixing Smale space case. The main tool will be resolving factor maps and the results in [9].

The following proposition allows us to extend the result of Lemma 4.4 to general mixing Smale spaces.

Proposition 4.6 *Let (X,φ) and (Y,ψ) be mixing Smale spaces, let $\pi\colon Y \to X$ be an almost one-to-one (s or u) resolving factor map, and suppose the conclusion of Lemma* 4.4 *holds for (Y,ψ). Then the conclusion of Lemma* 4.4 *holds for (X,φ).*

Proof Suppose π is u-resolving (the s-resolving case is completely analogous). Let $x_1, x_2 \in X$ and $B \subset X^u(x_1), C \subset X^s(x_2)$, and let

$$h_X^k = h_{B,C}^k, \quad \mu_X^k = \mu_{B,C}^k.$$

Now, set $C' = \pi_1^{-1}(C)$. By Theorem 3.6, $C' = \bigcup_1^m C_i'$, where the union is disjoint and $C_i' \subset Y^s(y_{2,i})$ for some $y_{2,i} \in Y$. Also, fix $y_1 \in \pi^{-1}(x_1)$, and set B' such that $\pi\colon B' \to B$ is a homeomorphism, so $B' \subset Y^u(y_1)$. Now

$$h_{B',C'}^k = \bigcup_1^m h_{B',C_i'}^k, \quad \#h_{B',C'}^k = \sum_1^m \#h_{B',C_i'}^k.$$

Notice that since $h_{B',C'}^k \subset Y^u(\varphi^{-k}(y_1))$ and π is u-resolving, π is one-to-one (and hence bijective) on $h_{B',C'}^k$. In other words $\#h_{B,C}^k = \#h_{B',C'}^k$. Also, recall from Proposition 3.9 that

$$\mu_X^{u,x_1}(B) = \mu_Y^{u,y_1}(B') \quad \text{and} \quad \mu_X^{s,x_2}(C) = \sum_1^m \mu_Y^{s,y_{2,i}}(C_i').$$

Now,

$$\lim_{k\to\infty} \lambda^{-2k}\#h_{B,C}^k = \lim_{k\to\infty} \lambda^{-2k}\#h_{B',C'}^k = \lim_{k\to\infty} \sum_1^m \lambda^{-2k}\#h_{B',C_i'}^k$$
$$= \sum_1^m \mu_Y^{u,y_1}(B')\mu_Y^{s,y_{2,i}}(C_i') = \mu_X^{u,x_1}(B)\mu_X^{s,x_2}(C). \qquad ■$$

We are now ready to prove Theorem 2.4

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Proof of Theorem 2.4 As in [9, Cor. 1.4], for the mixing Smale space (X, φ), we can find another mixing Smale space (Y, ψ) and a mixing SFT (Σ, σ), as well as almost one-to-one factor maps $\pi_1 \colon \Sigma \to Y$, $\pi_2 \colon Y \to X$ such that π_1 is s-resolving and π_2 is u-resolving. The first conclusion then follows from Lemma 4.4 and two applications of Proposition 4.6.

For the second statement notice that

$$\lim_{k\to\infty} \lambda^{-2k} \# h^k_{B,C} = \mu^{u,x}_X(B)\mu^{s,y}_X(C).$$

Hence,

$$\begin{aligned}
\lim_{k\to\infty} \log(\lambda^{-2k}\#h^k_{B,C}) &= \log(\mu^{u,x}_X(B)\mu^{s,y}_X(C))\\
\lim_{k\to\infty} \big(\log(\#h^k_{B,C}) - 2k\log(\lambda) - \log(\mu^{u,x}_X(B)\mu^{s,y}_X(C))\big) &= 0\\
\lim_{k\to\infty} \Big(\frac{\log(\#h^k_{B,C})}{2k} - h(X,\varphi) - \frac{\log(\mu^{u,x}_X(B)\mu^{s,y}_X(C))}{2k}\Big) &= 0\\
\lim_{k\to\infty} \Big(\frac{\log(\#h^k_{B,C})}{2k} - h(X,\varphi)\Big) &= 0.
\end{aligned}$$ ∎

The following proposition allows us to extend Proposition 4.5 from the mixing SFT case to the mixing Smale space case and prove Theorem 2.5.

Proposition 4.7 *Let (X, φ) and (Y, ψ) be mixing Smale spaces, let $\pi \colon Y \to X$ be an almost one-to-one (s or u) resolving factor map, and suppose the conclusion of Theorem 2.5 holds for (Y, ψ). Then the conclusion of Theorem 2.5 holds for (X, φ).*

Proof Suppose π is u-resolving (the s-resolving case is completely analogous). Let $x_1, x_2 \in X$ and $B \subset X^u(x_1)$, $C \subset X^s(x_2)$, and let

$$h^k_X = h^k_{B,C}, \quad \mu^k_X = \mu^k_{B,C}.$$

Now, set $C' = \pi^{-1}(C)$. By Theorem 3.6, $C' = \bigcup_1^m C'_i$, where the union is disjoint and $C'_i \subset Y^s(y_{2,i})$ for some $y_{2,i} \in Y$. Also, fix $y_1 \in \pi^{-1}(x_1)$, and set B' such that $\pi \colon B' \to B$ is a homeomorphism, so $B' \subset Y^u(y_1)$. Now set

$$h^k_X = h^k_{B',C'} = \bigcup_1^m h^k_{B',C'_i}, \quad \mu^k_X = \mu^k_{B',C} = \sum_1^m \frac{\#h^k_{B',C'_i}}{\#h^k_{B',C'}} \mu^k_{B',C'_i}.$$

Notice that since $h^k_Y \subset Y^u(\varphi^{-k}(y_1))$ and π is u-resolving, π is one-to-one (and hence bijective) on h^k_Y. In other words $h^k_X = \pi(h^k_Y)$, and therefore $\mu^k_Y = (\mu^k_X \circ \pi)$. Also recall from Theorem 2.4 that

$$\lim_{k\to\infty} \frac{\#h^k_{B',C'_i}}{\#h^k_{B',C'}} = \frac{\mu^{u,y_{2,i}}_Y(C'_i)}{\sum_{j=1}^k \mu^{u,y_{2,j}}_Y(C'_j)}.$$

Now, for $f \in C(X)$,

$$\begin{aligned}
\int_X f\,d\mu_X &= \int_{\pi^{-1}(X)} (f\circ\pi)\,d(\mu_X\circ\pi) = \int_Y (f\circ\pi)\,d\mu_Y \\
&= \lim_{k\to\infty}\int_Y (f\circ\pi)\,d\mu^k_{B',C'_i} \quad \text{for any } i \text{, by hypothesis} \\
&= \Big(\lim_{k\to\infty}\int_Y (f\circ\pi)\,d\mu^k_{B',C'_i}\Big)\sum_{i=1}^m \frac{\mu_Y^{u,y_{2,i}}(C'_i)}{\sum_{j=1}^k \mu_Y^{u,y_{2,j}}(C'_j)} \\
&= \Big(\lim_{k\to\infty}\sum_1^m \frac{\#h^k_{B',C'_i}}{\#h^k_{B',C'}}\int_Y (f\circ\pi)\,d\mu^k_{B',C'_i}\Big) \\
&= \lim_{k\to\infty}\int_Y (f\circ\pi)\,d\mu^k_Y = \lim_{k\to\infty}\int_Y (f\circ\pi)\,d(\mu^k_X\circ\pi) = \lim_{k\to\infty}\int_X f\,d\mu^k_X. \quad \blacksquare
\end{aligned}$$

We are now ready to prove Theorem 2.5.

Proof of Theorem 2.5 As in [9, Cor. 1.4], for the mixing Smale space (X,φ), we can find another mixing Smale space (Y,ψ) and a mixing SFT (Σ,σ), as well as almost one-to-one factor maps $\pi_1\colon \Sigma\to Y$, $\pi_2\colon Y\to X$ such that π_1 is s-resolving and π_2 is u-resolving. The conclusion then follows from Proposition 4.5 and two applications of Proposition 4.7. $\blacksquare$

Finally, we prove Theorems 2.8 and 2.10.

Proof of Theorems 2.8 and 2.10 We assume that B, C are contained in X_{i_0}. Without loss of generality, we assume $i_0 = 1$. Since for any $n \geq 0$, $\varphi^n(B), \varphi^n(C)$ are both contained in X_{1+n} (where $1+n$ is interpreted modulo I), the intersection of $h^k_{B,C}$ with X_i is $\varphi^{kI+i-1}(B)\cap\varphi^{-kI+i-1}(C)$, which we denote by h^k_i. Furthermore, we define

$$\mu^k_i = (\#h^k_i)^{-1}\sum_{z\in h^k_i}\delta_z.$$

With $1 \leq i \leq I$ fixed, consider Theorem 2.4 applied to the system $(X_i, \varphi^I \mid X_i)$ with local unstable and stable sets $\varphi^{i-1}(B)$ and $\varphi^{i-1}(C)$. Notice also that $h(X_i,\varphi^I) = Ih(X,\varphi)$, so if $\log(\lambda) = h(X,\varphi)$, $\log(\lambda^I) = h(X_i,\varphi^I)$. It now follows that

$$\begin{aligned}
\lim_k \#h^k_i(\lambda^I)^{-2k} &= \mu_X^{u,\varphi^{i-1}(x)}\big(\varphi^{i-1}(B)\big)\,\mu_X^{u,\varphi^{i-1}(y)}(\varphi^{i-1}(C)) \\
&= \lambda^{1-i}\mu_X^{u,x}(B)\lambda^{i-1}\mu_X^{s,y}(C) = \mu_X^{u,x}(B)\mu_X^{s,y}(C).
\end{aligned}$$

Noticing that $\lim_k \frac{\#h^k_{B,C}}{\#h^k_i} = I$, we have

$$\lim_k \#h^k_{B,C}\lambda^{-2kI} = I\mu_X^{u,x}(B)\mu_X^{s,y}(C).$$

It then follows as in the proof of Theorem 2.4 that

$$\lim_k \frac{h^k_{B,C}}{2kI} = h(X,\varphi).$$

We also note that Theorem 2.5 implies $\lim_k \mu^k_i = \mu_{X_i}$. Putting all of this together, we have

$$\begin{aligned}
\lim_k \mu^k_{B,C} &= \lim_k (\#h^k_{B,C})^{-1} \sum_{z\in h^k_{B,C}} \delta_z = \lim_k (\#h^k_{B,C})^{-1} \sum_{i=1}^{I} \sum_{z\in h^k_i} \delta_z \\
&= \lim_k \sum_{i=1}^{I} \frac{\#h^k_i}{\#h^k_{B,C}} (\#h^k_i)^{-1} \sum_{z\in h^k_i} \delta_z = \lim_k \sum_{i=1}^{I} \frac{\#h^k_i}{\sum_{j=1}^{I} \#h^k_j} \mu^k_i \\
&= \lim_k \sum_{i=1}^{I} \frac{\#h^k_i \lambda^{-2kI}}{\sum_{j=1}^{I} \#h^k_j \lambda^{-2kI}} \mu^k_i = \sum_{i=1}^{I} \frac{\mu^{u,x}_X(B)\mu^{s,y}_X(C)}{I\mu^{u,x}_X(B)\mu^{s,y}_X(C)} \mu_{X_i} = \sum_{i=1}^{I} \frac{1}{I}\mu_{X_i} = \mu_X. \quad \blacksquare
\end{aligned}$$

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