J. Austral. Math. Soc. (Series A) 44 (1988), 138–142

PAIR-COUNTABLE, CLOSURE-PRESERVING COVERS OF COMPACT SETS

HENRY POTOCZNY

(Received 29 November 1983; revised 26 February 1986)

Communicated by J. H. Rubinstein

Abstract

In this paper, we prove the following results: (1) if a topological space X has a pair-countable, closure-preserving cover of compact sets, then X is locally paracompact at each point of X and X has a dense open subspace which is locally σ-compact. In addition, if X is also collectionwise-T_2, then X is paracompact. Locally paracompact is taken to mean that each point X has an open set with paracompact closure.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): primary 54 D 18; secondary 54 D 45.

Introduction

In [8], Tamano asked whether or not a space which is the closure-preserving union of compact sets must be paracompact. It was shown in [4] that the answer is negative, an example being given there of a topological space which is the union of a closure-preserving family of finite sets, is completely regular and T_2, even has a basis consisting of open and closed sets, yet fails to be even normal, hence certainly not paracompact.

In a positive direction, it was shown in [5] that if a topological space X has a point-countable closure-preserving cover of compact sets, then X is paracompact, in fact, X is the pairwise disjoing union of a family of open and closed σ-compact subsets.

https://doi.org/10.1017/S1446788700031426 Published online by Cambridge University Press

As regards historical perspective, the following results are of interest.

(1) If a regular space X is strongly collectionwise-normal with respect to compact sets, and has a closure-preserving cover by compact sets, then X is paracompact.

(2) If a T_1 space has a closure-preserving cover by closed compact sets, it is metacompact.

The first is essentially shown in [6]: The phrase, "with respect to compact sets", is omitted, but that is all that is used in the proof. The second result was shown in [7], and independently, in [2].

The first result gives a much-needed perspective on Theorem 3, since it makes the following question natural: If a regular space X is strongly collectionwise-T_2, and has a closure-preserving cover by compact sets, is X paracompact?

In this paper, we prove the following related results: (1) If a topological space X has a pair-countable, closure-preserving cover of compact sets, then X is locally paracompact at each point of X and X has a dense open subspace which is locally σ-compact. In addition, if X is also collectionwise-T_2, then X is paracompact. Locally paracompact is taken to mean that each point X has an open neighborhood with paracompact closure.

Definitions. A family $\mathscr{F}$, of sets, is said to be *pair-countable* provided that for each two distinct points x and y, at most countably many members of $\mathscr{F}$ contain both x and y. A family $\mathscr{F}$ of sets is of *countable order* at x if x is an element of at most countably many members of $\mathscr{F}$. Otherwise, $\mathscr{F}$ is said to be of *uncountable order* at x.

In what follows, regular is understood to include T_2.

Lemma 1. *Let x be a topological space and let $\mathscr{F} = \{F(\alpha) \mid \alpha \in A\}$ be a closure-preserving closed cover of A by compact sets. For each $x \in X$, let $K(x) = \{y \in X \mid$ if $y \in F(\alpha)$, then $x \in F(\alpha)\}$. Then*

(i) $K(x) = X - \bigcup\{F(\alpha) \mid x \notin F(\alpha)\}$,

(ii) *$K(x)$ is open, and if $y \in K(x)$, then $K(y) \subset K(x)$,*

(iii) *if $\mathscr{F}$ is of countable order at $x \in X$, then $K(x)$ has σ-compact closure.*

Lemma 2. *Let X be a regular space with a point x such that the complement of every open neighborhood of x is paracompact. Then x is paracompact.*

Proof. The proof is straightforward and is left as an exercise.

Lemma 3. *Let X be a regular space with a closure-preserving cover $\mathscr{F} = \{F(\alpha) \mid \alpha \in A\}$, of compact sets. If $\mathscr{F}$ is of countable order at all but at most one point, then X is paracompact.*

PROOF. Let W be an open set about X. The family $\{F(\alpha) \cap (X - W) \mid \alpha \in A\}$ is a closure-preserving, point-countable cover of $X - W$ by compact sets. By Corollary 2 of [2], $X - W$ is the disjoint union of open and closed σ-compact subspaces, hence is certainly paracompact. Paracompactness of X follows from Lemma 2.

LEMMA 4. *Let X be a regular topological space and let $\mathcal{F} = \{F(\alpha) \mid \alpha \in A\}$ be a pair-countable, closure-preserving cover of X by compact sets. Let M be the set of all points of uncountable order. Then M is closed and every subset of M is closed.*

PROOF. Let N be a subset of M, and let x be a point of $X - N$. Consider the open set $K(x)$. If $K(x) \cap N$ is not empty, let $p \in K(x) \cap N$. Then p is certainly of uncountable order. Moreover, by Lemma 1, if p appears in a member of $\mathcal{F}$, so does x. But then p and x appear together in an uncountable number of members of $\mathcal{F}$, and this violates the pair-countability of $\mathcal{F}$. Hence $K(x) \cap N = \varnothing$, and N is seen to be closed.

THEOREM 1. *Let X be a regular topological space and let $\mathcal{F} = \{F(\alpha) \mid \alpha \in A\}$ be a pair-countable, closure-preserving cover of X by compact sets. Then X is locally paracompact, that is, each point x, of X, has an open set with paracompact closure.*

PROOF. If x is a point of countable order, then by Lemma 1, x has an open neighborhood with σ-compact closure which, in the presence of regularity, implies the existence of the Lindelöf property, hence implies paracompactness. If x is a point of uncountable order, then consider the open set $K(x)$. This is an open set whose closure may contain some points of M, the set of points of uncountable character. By the previous lemma, however, $M - \{x\}$ is closed, and since X is regular, there is an open set V such that $x \in V \cap \overline{V}$ and $\overline{V} \cap (M - \{x\}) = \varnothing$. Now the family $\{\overline{V} \cap F(\alpha) \mid \alpha \in A\}$ is easily seen to be a closure-preserving pair-countable cover of $\overline{V}$ by compact sets. Moreover, $\overline{V}$ clearly contains only one point of uncountable order, x itself, so that Lemma 3 applies, and $\overline{V}$ is seen to be paracompact.

THEOREM 2. *Let X be a regular topological space. Let $\mathcal{F} = \{F(\alpha) \mid \alpha \in A\}$ be a closure-preserving pair-countable cover of X by compact sets. Then X has a dense open subspace which is locally σ-compact.*

PROOF. Let M be the set of points of uncountable order. Let $Y = (X - M) \cup \{x \mid x \in M, \{x\} \text{ is open}\}$. Then Y is a dense open subspace of X, and is locally σ-compact. To see that Y is open, note that if p is of countable order so is every

https://doi.org/10.1017/S1446788700031426 Published online by Cambridge University Press

point of $K(p)$, whence $p \in K(p) \subset Y$; and certainly, $\{x \mid x \in M, \{x\} \text{ is open}\}$, is an open set.

To see that Y is locally σ-compact, it clearly suffices to show that for each $p \in X - M$, p has an open set with σ-compact closure in Y. By Lemma 1, each point p has the open neighborhood $K(p)$ whose closure in the entire space X, is σ-compact. But since X is regular, there is an open set V such that $p \in V \subset \overline{V} \subset Y$. Then $K(p) \cap V$ contains the point p and has σ-compact closure in Y.

To see that Y is a dense subspace of X, notice that only points of M are candidates to be limit points of Y. Let $x \in M$, and suppose X has an open set V which does not meet Y. By Lemma 4, $M - \{x\}$ is closed so that there is an open set W such that $x \in W$, and $W \cap (M - \{x\}) = \varnothing$. But then $v \cap W$ is evidently $\{x\}$, so that $x \in Y$ after all.

DEFINITION. A topological space X is said to be *collectionwise-T_2* provided that for each discrete family of singleton sets, there is a pairwise disjoint family of open sets, each containing one of the singletons.

DEFINITION. A topological space X is said to be *strongly collectionwise-T_2* provided that if $\{\{x(\alpha)\} \mid \alpha \in A\}$ is a discrete family of singletons, then there is a discrete family $\{V(\alpha) \mid \alpha \in A\}$, of open subsets of X such that for each $\alpha \in A$, $x(\alpha) \in V(\alpha)$.

THEOREM 3. *Let X be a regular topological space. Let $\mathscr{F} = \{F(\alpha) \mid \alpha \in A\}$ be a closure-preserving pair-countable cover of X by compact sets. If X is also strongly collectionwise-T_2, then X is paracompact.*

PROOF. Let M be the set of points of uncountable order. Then, by Lemma 4, the set of singletons $\{\{x\} \mid x \in M\}$ is a discrete family.

Now let $\mathscr{B} = \{B(x) \mid x \in M\}$ be a discrete family of open subsets of X such that for each $x \in M$, $x \in B(x)$. To see that X is paracompact, let $\mathscr{V}$ be an open cover of x. For each $x \in M$, let $V(x)$ be a member of such that $x \in V(x)$. For each $x \in M$, let $W(x) = B(x) \cap V(x)$. Then the family $\{W(x) \mid x \in M\}$ is a discrete family of open subsets of X, each set containing the point on which it is indexed. Since it is a discrete family, it is certainly closure-preserving so that if we set $W = \bigcup\{W(x) \mid x \in M\}$, then $\overline{W} = \overline{\cup\{W(x) \mid x \in M\}} = \bigcup\{\overline{W(x)} \mid x \in M\}$ which is a subset of $V = \bigcup\{V(x) \mid x \in M\}$. We now have M, the set of points of uncountable order inside the open set W, and $\overline{W} \subset V$. We may now observe that $X - W$ is a closed subset of X, so that $\{F(\alpha) \cap (X - W) \mid \alpha \in A\}$ is pair-countable, closure-preserving cover of $X - W$ by compact sets. Moreover, each point is of countable order, so that $X - W$ is the disjoint union of open and closed σ-compact subspaces, hence is paracompact. The proof may now proceed along lines similar to those of Lemma 2.

Corollary. *Let X be a topological space and let* $\mathscr{F} = \{F(\alpha) \mid \alpha A\}$ *be a closure-preserving, pair-countable family of compact sets which covers X. The following are equivalent*

(i) *X is paracompact,*
(ii) *X is normal and collectionwise-*T_2.
(iii) *X is regular and strongly collectionwise-*T_2.

The implications (i) → (ii) → (iii) *are well known for topological spaces in general, while* (iii) → (i) *is the content of Theorem* 3.

Examples. An example to illustrate the main results is the Pixley-Roy space $\mathscr{F}(R)$, on R, and its subspaces, (see [3]). Elements are finite non-empty subsets of R. Basic open sets are of the form $[B, U] = \{C \in \mathscr{F}(R) \mid B \subset C \subset U\}$, where U is an open subset of R such that $B \subset U$. The space has a closure-preserving cover by the finite sets $\{\mathscr{P}(B) \mid B \in \mathscr{F}(R)\} - \{\varnothing\}$, but does not, however, have a dense paracompact subspace, hence "pair-countable" cannot be omitted from Theorem 3. On the other hand, it has a subspace $\{B \mid B \subset R, 0 \leqslant |B| < 2\}$, which does have a pair-countable closure-preserving cover by finite sets. The subspace, however, has no point-countable closure-preserving cover by compact sets. In fact, this subspace is simply "Heath's Tangent V Space" (see [1]).

This latter space also serves as an example to show that "strongly collectionwise-T_2" cannot be omitted from Theorem 3.

References

[1] R. Heath, 'Screenability, pointwise paracompactness, and metrization of Moore spaces', *Canad. J. Math.* **16** (1964), 763–770.
[2] Y. Katuta, 'On spaces which admit closure-preserving covers by compact sets', *Proc. Japan Acad.* **50**, No. 10 (1974).
[3] D. Lutzer, 'Pixley-Roy topology', *Topology Proc.* **3** (1978), 139–158.
[4] H. Potoczny, 'A nonparacompact space which admits a closure-preserving cover of compact sets', *Proc. Amer. Math. Soc.* **32** (1972), 309–311.
[5] H. Potoczny, 'On a problem of Tamano', *Fund. Math.* **75** (1972), 29–31.
[6] H. Potoczny, Closure-preserving families of compact sets, *General Topology Appl.* **3** (1973), 243–248.
[7] H. Potoczny and H. Junnila, 'Closure-preserving families and metacompactness', *Proc. Amer. Math. Soc.* **53** (1975), 523–529.
[8] H. Tamano, 'A characterization of paracompactness', *Fund. Math.* **72** (1971), 189–201.

Department of Mathematics and Computer Science
U.S. Air Force Institute of Technology
Wright-Patterson, Ohio 45433