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COVERING THE INTEGERS WITH ARITHMETIC PROGRESSIONS

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A *regular covering system* is a collection of arithmetic progressions such that every integer belongs to at least one arithmetic progression in the collection, and no proper subcollection has this property.

An *exact covering system* is a regular covering system with the property that every integer belongs to exactly one of the arithmetic progressions.

The thesis contains three principal results.

1. Let P be the lowest common multiple of the common differences of the arithmetic progressions in a regular covering system and suppose P has prime factorisation

$$P = \prod_{i=1}^{t} p_i^{\alpha_i} .$$

Then the number of arithmetic progressions in the collection is at least

$$\sum_{i=1}^{t} \alpha_i (p_i - 1) + 1 .$$

A similar result has been proved by Korec [4] applied to exact covering systems. In both cases the results are the best possible.

2. An exact covering system in which each common difference occurs at most M times is called an ECS(M). I prove the following result. If

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$p_1 < p_2 \cdots < p_t$ are the distinct prime divisors of the lowest common multiple of the common differences of the arithmetic progressions in an ECS(M) then

$$M \prod_{i=1}^{t-1} p_i/(p_i-1) \geq p_t$$

Burshtein [1] showed that a similar inequality applied in the case of a special type of exact covering system called a naturally exact covering system. Our result has several consequences. For instance it follows that in any ECS(M) we have $p_1 \leq M$ and that there exists a number $B(M)$ such that any ECS(M) contains an arithmetic progression with common difference less than $B(M)$.

3. The last part of this thesis concerns the following conjecture due to Crittenden and Vanden Eynden [3].

Let S be the union of n arithmetic progressions, each with common difference not less than k where $k \leq n$. It is conjectured that if S contains the closed interval $[1, k\,2^{n-k+1}]$ then S contains all integers.

Crittenden and Vanden Eynden [2] proved the conjecture in the (equivalent) cases corresponding to $k = 1$ and $k = 2$. I prove the conjecture in the case $k = 3$ and show that if a counterexample exists for a given k then a counterexample exists for that k with the following properties:

(a) Each common difference in the counterexample is either a prime $\geq k$ or a product of primes $< k$.

(b) If p is a prime, $p \geq k$, then the number of arithmetic progressions with common difference p is less than $\log p/\log 2$.

(c) The cardinality of the collection is less than an explicit function of k, that function being asymptotically equal to $3k(1 + 1/\log 2)$ as $k \to \infty$.

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