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REMARKS ON SQUARE FUNCTIONS IN THE LITTLEWOOD-PALEY THEORY

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We prove that certain square function operators in the Littlewood-Paley theory defined by the kernels without any regularity are bounded on L^p_w, $1 < p < \infty$, $w \in A_p$ (the weights of Muckenhoupt). Then, we give some applications to the Carleson measures on the upper half space.

1. INTRODUCTION

In this note we shall prove weighted L^p-estimates for the Littlewood-Paley type square functions arising from kernels satisfying only size and cancellation conditions. Suppose that $\psi \in L^1(\mathbf{R}^n)$ satisfies

$$\int_{\mathbf{R}^n} \psi(x)\,dx = 0. \tag{1.1}$$

We consider a square function of Littlewood-Paley type

$$S(f)(x) = S_\psi(f)(x) = \left(\int_0^\infty \left|\psi_t * f(x)\right|^2 \frac{dt}{t}\right)^{1/2},$$

where $\psi_t(x) = t^{-n}\psi(t^{-1}x)$.

If ψ satisfies, in addition to (1.1),

$$\left|\psi(x)\right| \leqslant c\big(1+|x|\big)^{-n-\varepsilon} \text{ for some } \quad \varepsilon > 0 \tag{1.2}$$

$$\int_{\mathbf{R}^n} \left|\psi(x-y) - \psi(x)\right| dx \leqslant c\,|y|^\varepsilon \text{ for some } \quad \varepsilon > 0, \tag{1.3}$$

then it is known that the operator S is bounded on $L^p(\mathbf{R}^n)$ for all $p \in (1,\infty)$ (see Benedek, Calderón and Panzone [**1**]). Well-known examples are as follows.

EXAMPLE 1: Let $P_t(x)$ be the Poisson kernel for the upper half space $\mathbf{R}^n \times (0,\infty)$:

$$P_t(x) = c_n \frac{t}{\left(|x|^2 + t^2\right)^{(n+1)/2}}.$$

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Put

$$\psi(x) = \left(\frac{\partial}{\partial t} P_t(x)\right)_{t=1}.$$

Then $S_\psi(f)$ is the Littlewood-Paley g function.

EXAMPLE 2: Consider the Haar function ψ on $\mathbf{R}$:

$$\psi(x) = \chi_{[-1,0]}(x) - \chi_{[0,1]}(x),$$

where χ_E denotes the characteristic function of a set E. Then, $S_\psi(f)$ is the Marcinkiewicz integral

$$\mu(f)(x) = \left(\int_0^\infty \left|F(x+t) + F(x-t) - 2F(x)\right|^2 \frac{dt}{t^3}\right)^{1/2},$$

where $F(x) = \int_0^x f(y)\,dy$.

In this note, we shall prove that the L^p-boundedness of S still holds without the assumption (1.3); the conditions (1.1) and (1.2) only are sufficient. This is already known for the L^2-case (see Coifman and Meyer [**3**, p.148], and also Journé [**7**, pp. 81–82] for a proof).

To state our result more precisely, we consider the least non-increasing radial majorant of ψ

$$h_\psi(|x|) = \sup_{|y| \geqslant |x|} |\psi(y)|.$$

We also need to consider two seminorms

$$B_\varepsilon(\psi) = \int_{|x|>1} |\psi(x)|\,|x|^\varepsilon\,dx \qquad \text{for} \quad \varepsilon > 0,$$

$$D_u(\psi) = \left(\int_{|x|<1} |\psi(x)|^u\,dx\right)^{1/u} \qquad \text{for} \quad u > 1.$$

We shall prove the following result.

THEOREM 1. *Put $H_\psi(x) = h_\psi(|x|)$. If $\psi \in L^1(\mathbf{R}^n)$ satisfies (1.1) and*

(1) $B_\varepsilon(\psi) < \infty$ *for some $\varepsilon > 0$;*
(2) $D_u(\psi) < \infty$ *for some $u > 1$;*
(3) $H_\psi \in L^1(\mathbf{R}^n)$;

then the operator S_ψ is bounded on L^p_w :

$$\left\|S_\psi(f)\right\|_{L^p_w} \leqslant C_{p,w} \|f\|_{L^p_w}.$$

for all $p \in (1,\infty)$ *and* $w \in A_p$, *where* A_p *denotes the weight class of Muckenhoupt (see* [**6, 7**]*), and*

$$\|f\|_{L^p_w} = \|f\|_{L^p(w)} = \left(\int_{\mathbf{R}^n} |f(x)|^p w(x)\,dx\right)^{1/p}$$

In fact, we shall prove a more general result.

Theorem 2. *Suppose that* $\psi \in L^1(\mathbf{R}^n)$ *satisfies* (1.1) *and*

(1) $B_\varepsilon(\psi) < \infty$ *for some* $\varepsilon > 0$;

(2) $D_u(\psi) < \infty$ *for some* $u > 1$;

(3) $|\psi(x)| \leqslant h(|x|)\Omega(x')$ $\left(x' = |x|^{-1}x\right)$ *for some non-negative functions* h *and* Ω *such that*

(a) $h(r)$ *is non-increasing for* $r \in (0,\infty)$;

(b) *if* $H(x) = h(|x|)$, $H \in L^1(\mathbf{R}^n)$;

(c) $\Omega \in L^q(S^{n-1})$ *for some* q, $2 \leqslant q \leqslant \infty$.

Then, the operator S_ψ *is bounded on* L^p_w *for* $p > q'$ *and* $w \in A_{p/q'}$, *where* q' *denotes the conjugate exponent of* q.

When ψ is compactly supported, we have another formulation, which is not included in Theorem 2.

Theorem 3. *Suppose that* $\psi \in L^1(\mathbf{R}^n)$ *satisfies* (1.1) *and*

(1) ψ *is compactly supported* ;

(2) $\psi \in L^q(\mathbf{R}^n)$ *for some* $q \geqslant 2$.

Then $S_\psi : L^p_w \to L^p_w$ *for* $p > q'$ *and* $w \in A_{p/q'}$.

These results will be derived from more abstract ones. Let $\psi \in L^1(\mathbf{R}^n)$ satisfy (1.1). We also assume the following :

(1) There exists $\varepsilon \in (0,1)$ such that

$$\int_1^2 \left|\widehat{\psi}(t\xi)\right|^2 dt \leqslant c\min\left(|\xi|^\varepsilon, |\xi|^{-\varepsilon}\right) \qquad \text{for all} \quad \xi \in \mathbf{R}^n, \tag{1.4}$$

where $\widehat{\psi}$ denotes the Fourier transform

$$\widehat{\psi}(\xi) = \int \psi(x)e^{-2\pi i\langle x,\xi\rangle}\,dx, \qquad \langle x,\xi\rangle = \sum_{j=1}^n x_j\xi_j \quad (\text{the inner product in } \mathbf{R}^n).$$

(2) Let $1 \leqslant s \leqslant 2$. For all $w \in A_s$, we have

$$\sup_{k\in\mathbf{Z}} \int_{\mathbf{R}^n}\int_1^2 \left|\psi_{t2^k} * f(x)\right|^2 dt\, w(x)\,dx \leqslant C_w \|f\|^2_{L^2_w} \qquad \text{for all} \quad f \in \mathcal{S}(\mathbf{R}^n), \tag{1.5}$$

where $\mathbf{Z}$ denotes the integer group and $\mathcal{S}(\mathbf{R}^n)$ the Schwartz space.

Under these assumptions the following holds.

Proposition 1. *For $p > 2/s$ and $w \in A_{ps/2}$, the operator S_ψ is bounded on L^p_w.*

This will be used to prove the next result.

Proposition 2. *Put*

$$J_\varepsilon(\psi) = \sup_{|\xi|=1} \iint_{\mathbf{R}^n \times \mathbf{R}^n} |\psi(x)\psi(y)| \left|\langle \xi, x-y\rangle\right|^{-\varepsilon} \, dx \, dy \qquad \text{for} \quad \varepsilon \in (0,1].$$

Let $\psi \in L^1$ satisfy (1.1) and (1.5). Then if $B_\varepsilon(\psi) < \infty$ and $J_\varepsilon(\psi) < \infty$ for some $\varepsilon \in (0,1]$, the operator S_ψ is bounded on L^p_w for $p > 2/s$ and $w \in A_{ps/2}$.

In Section 2, we shall prove Proposition 1 by the method of the proof of Duoandikoetxea and Rubio de Francia [**5**, Corollary 4.2] and then Proposition 2 by using Proposition 1. Proposition 2 will be applied to prove Theorems 2 and 3 in Section 3. Finally, in Section 4, we shall give some applications of Theorem 1 to generalised Marcinkiewicz integrals and the Carleson measures on the upper half space $\mathbf{R}^n \times (0, \infty)$.

To conclude this section, we state a result for the L^2-case, from which the result of Coifman-Meyer mentioned above immediately follows, and the idea of the proof will be applied later too (see the proof of Lemma 2).

Proposition 3. *Suppose that $\psi \in L^1$ satisfies (1.1). Let*

$$L(\psi) = \sup_{|\xi|=1} \iint_{\mathbf{R}^n \times \mathbf{R}^n} |\psi(x)\psi(y)| \left|\log|\langle \xi, x-y\rangle|\right| \, dx \, dy.$$

Then, if $L(\psi) < \infty$, the operator S_ψ is bounded on L^2.

Proof: It is sufficient to show that

$$\sup_{|\xi|=1} \int_0^\infty \left|\widehat{\psi}(t\xi)\right|^2 \frac{dt}{t} < \infty.$$

We write

$$\left|\widehat{\psi}(t\xi)\right|^2 = \widehat{\psi}(t\xi)\overline{\widehat{\psi}(t\xi)} = \iint_{\mathbf{R}^n \times \mathbf{R}^n} \psi(x)\overline{\psi(y)} e^{-2\pi i t\langle \xi, x-y\rangle} \, dx \, dy,$$

and so

$$\int_0^\infty \left|\widehat{\psi}(t\xi)\right|^2 \frac{dt}{t} = \lim_{N\to\infty, \varepsilon\to 0} \iint \psi(x)\overline{\psi(y)} \left(\int_\varepsilon^N e^{-2\pi i t\langle \xi, x-y\rangle} \frac{dt}{t} \right) dx \, dy.$$

Note that

$$\int_{\varepsilon}^{N} \left(e^{-2\pi i t\langle \xi, x-y\rangle} - \cos(2\pi t)\right) \frac{dt}{t} \to -\log\left|\langle \xi, x-y\rangle\right| - i\frac{\pi}{2}\operatorname{sgn}\langle \xi, x-y\rangle$$

as $N \to \infty$ and $\varepsilon \to 0$, and the integral is bounded, uniformly in ε and N, by

$$c\Big(1 + \left|\log|\langle \xi, x-y\rangle|\right|\Big).$$

Thus, using (1.1) and the dominated convergence theorem, we get

$$\int_0^\infty \left|\widehat{\psi}(t\xi)\right|^2 \frac{dt}{t} = \iint \psi(x)\overline{\psi(y)}\left(-\log\left|\langle \xi, x-y\rangle\right| - i\frac{\pi}{2}\operatorname{sgn}\langle \xi, x-y\rangle\right) dx\, dy.$$

This immediately implies the conclusion. ∎

REMARK. In the one-dimensional case, it is easy to see that if

$$\int |\psi(x)| \log(2+|x|)\, dx < \infty \qquad \text{and} \qquad \int |\psi(x)| \log\Big(2 + |\psi(x)|\Big)\, dx < \infty,$$

then $L(\psi) < \infty$, and so $S_\psi : L^2 \to L^2$.

2. Proofs of Propositions 1 and 2

We use a Littlewood-Paley decomposition. Let $f \in \mathcal{S}(\mathbf{R}^n)$, and define

$$\widehat{\Delta_j(f)}(\xi) = \Psi(2^j\xi)\widehat{f}(\xi) \qquad \text{for} \qquad j \in \mathbf{Z},$$

where $\Psi \in C^\infty$ is supported in $\{1/2 \leqslant |\xi| \leqslant 2\}$ and satisfies

$$\sum_{j\in\mathbf{Z}} \Psi(2^j\xi) = 1 \qquad \text{for} \qquad \xi \neq 0.$$

Decompose

$$f \star \psi_t(x) = \sum_{j\in\mathbf{Z}}\sum_{k\in\mathbf{Z}} \Delta_{j+k}(f\star\psi_t)(x)\chi_{[2^k,2^{k+1})}(t) = \sum_{j\in\mathbf{Z}} F_j(x,t),$$

say, and define

$$T_j(f)(x) = \left(\int_0^\infty \left|F_j(x,t)\right|^2 \frac{dt}{t}\right)^{1/2}.$$

Then

$$S(f)(x) \leqslant \sum_{j\in\mathbf{Z}} T_j(f)(x).$$

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Put $E_j = \{2^{-1-j} \leqslant |\xi| \leqslant 2^{1-j}\}$. Then by the Plancherel theorem and (1.4) we have

$$\begin{aligned}\|T_j(f)\|_2^2 &= \sum_{k\in\mathbf{Z}} \int_{\mathbf{R}^n} \int_{2^k}^{2^{k+1}} \left|\Delta_{j+k}\left(f \star \psi_t\right)(x)\right|^2 \frac{dt}{t}\, dx \\ &\leqslant \sum_{k\in\mathbf{Z}} c \int_{E_{j+k}} \left(\int_{2^k}^{2^{k+1}} \left|\widehat{\psi}(t\xi)\right|^2 \frac{dt}{t}\right) \left|\widehat{f}(\xi)\right|^2 d\xi \\ &\leqslant \sum_{k\in\mathbf{Z}} c \int_{E_{j+k}} \min\left(\left|2^k\xi\right|^{\varepsilon}, \left|2^k\xi\right|^{-\varepsilon}\right) \left|\widehat{f}(\xi)\right|^2 d\xi \\ &\leqslant c2^{-\varepsilon|j|} \sum_{k\in\mathbf{Z}} \int_{E_{j+k}} \left|\widehat{f}(\xi)\right|^2 d\xi \\ &\leqslant c2^{-\varepsilon|j|} \|f\|_2^2,\end{aligned}$$

where the last inequality holds since the sets E_j are finitely overlapping. (We denote by $\|\cdot\|_p$ the ordinary L^p-norm.)

On the other hand, for $w \in A_s$ by (1.5) we see that

$$\begin{aligned}\|T_j(f)\|_{L^2_w}^2 &= \sum_{k\in\mathbf{Z}} \int_{\mathbf{R}^n} \int_{2^k}^{2^{k+1}} \left|\Delta_{j+k}(f) \star \psi_t(x)\right|^2 \frac{dt}{t}\, w(x)\, dx \\ &\leqslant \sum_{k\in\mathbf{Z}} c \int_{\mathbf{R}^n} \left|\Delta_{j+k}(f)(x)\right|^2 w(x)\, dx \\ &\leqslant c\|f\|_{L^2_w}^2,\end{aligned}$$

where the last inequality follows from a well-known Littlewood-Paley inequality for L^2_w since $A_s \subset A_2$.

Interpolating with change of measures between the two estimates above, we get

$$\|T_j(f)\|_{L^2(w^u)} \leqslant c2^{-\varepsilon(1-u)|j|/2} \|f\|_{L^2(w^u)}$$

for $u \in (0,1)$. If we choose u (close to 1) so that $w^{1/u} \in A_s$, then from this inequality we get

$$\|T_j(f)\|_{L^2_w} \leqslant c2^{-\varepsilon(1-u)|j|/2} \|f\|_{L^2_w},$$

and so

$$\|S(f)\|_{L^2_w} \leqslant \sum_{j\in\mathbf{Z}} \|T_j(f)\|_{L^2_w} \leqslant c\|f\|_{L^2_w}.$$

Thus the extrapolation theorem of Rubio de Francia [**8**] implies the conclusion.

To derive Proposition 2 from Proposition 1 we need the following lemmas.

LEMMA 1. *If $\psi \in L^1(\mathbf{R}^n)$ satisfies (1.1) and $B_\varepsilon(\psi) < \infty$ for $\varepsilon \in (0,1]$, then*

$$\left|\widehat{\psi}(\xi)\right| \leqslant c\,|\xi|^\varepsilon \qquad \text{for all} \quad \xi \in \mathbf{R}^n.$$

PROOF: Since $a \leqslant a^\varepsilon$ for $a, \varepsilon \in (0,1]$, we see that

$$\left|\widehat{\psi}(\xi)\right| = \left|\int \psi(x)\left(e^{-2\pi i\langle x,\xi\rangle} - 1\right) dx\right| \leqslant c\int |\psi(x)| \min\left(1, |\langle x,\xi\rangle|\right) dx$$
$$\leqslant c\,|\xi|^\varepsilon \int |\psi(x)|\,|x|^\varepsilon\, dx.$$

This completes the proof. ∎

LEMMA 2. *If $\psi \in L^1(\mathbf{R}^n)$ and $J_\varepsilon(\psi) < \infty$ for $\varepsilon \in (0,1]$, then*

$$\int_1^2 \left|\widehat{\psi}(t\xi)\right|^2 dt \leqslant c\,|\xi|^{-\varepsilon} \qquad \text{for all} \quad \xi \in \mathbf{R}^n.$$

PROOF: As in the proof of Proposition 3, we see that

$$\int_1^2 \left|\widehat{\psi}(t\xi)\right|^2 dt = \iint_{\mathbf{R}^n\times\mathbf{R}^n} \psi(x)\overline{\psi(y)}\frac{e^{-4\pi i\langle\xi,x-y\rangle} - e^{-2\pi i\langle\xi,x-y\rangle}}{-2\pi i\langle\xi, x-y\rangle}\, dx\, dy.$$

Thus

$$\int_1^2 \left|\widehat{\psi}(t\xi)\right|^2 dt \leqslant c\iint_{\mathbf{R}^n\times\mathbf{R}^n} |\psi(x)\psi(y)| \min\left(1, \left|\langle\xi, x-y\rangle\right|^{-1}\right) dx\, dy$$
$$\leqslant cJ_\varepsilon(\psi)\ |\xi|^{-\varepsilon}.$$

This completes the proof. ∎

Now, we can see that Proposition 1 implies Proposition 2, since the condition (1.4) follows from Lemmas 1 and 2.

3. PROOFS OF THEOREMS 2 AND 3

To get Theorem 2 from Proposition 2 we need Lemmas 3 and 4 below. First, we give a sufficient condition for $J_\varepsilon(\psi) < \infty$.

LEMMA 3. *Let $h(r)$, $h \geqslant 0$, be a non-increasing function for $r > 0$ satisfying $H \in L^1(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$, where $H(x) = h(|x|)$, and let $\Omega \in L^v(S^{n-1})$, $v > 1$, $\Omega \geqslant 0$. Suppose that F is a non-negative function such that*

$$F(x) \leqslant h(|x|)\Omega(x') \qquad \text{for} \quad |x| > 1$$

and $D_u(F) < \infty$ *for* $u > 1$. *Then* $J_\varepsilon(F) < \infty$ *if* $\varepsilon < \min(1/u', 1/v')$.

PROOF: For non-negative functions f, g and $\xi \in S^{n-1}$ put

$$L_\varepsilon(f, g; \xi) = \iint_{\mathbf{R}^n \times \mathbf{R}^n} f(x) g(y) \left|\langle \xi, x - y\rangle\right|^{-\varepsilon} \, dx\, dy.$$

Decompose F as $F = E + G$, where $E(x) = F(x)$ if $|x| < 1$ and $E(x) = 0$ otherwise. Then

$$L_\varepsilon(F, F; \xi) = L_\varepsilon(E, E; \xi) + 2L_\varepsilon(E, G; \xi) + L_\varepsilon(G, G; \xi).$$

We show that each of $L_\varepsilon(E, E; \xi), L_\varepsilon(E, G; \xi)$ and $L_\varepsilon(G, G; \xi)$ is bounded by a constant independent of ξ if $\varepsilon < \min(1/u', 1/v')$.

First, by Hölder's inequality and a change of variables

$$L_\varepsilon(E, E; \xi) \leqslant \|E\|_u^2 \left(\iint_{|x|<1, |y|<1} |x_1 - y_1|^{-\varepsilon u'} \, dx\, dy \right)^{1/u'},$$

where we note that $\|E\|_u = D_u(F)$.

Next, by Hölder's inequality again

$$L_\varepsilon(E, G; \xi) \leqslant \|E\|_u \left(\int_{|x|<1} \left(\int_{\mathbf{R}^n} G(y) \left|x_1 - \langle \xi, y\rangle\right|^{-\varepsilon} dy \right)^{u'} dx \right)^{1/u'}.$$

For $s > 0$, let

$$I_\varepsilon(s) = \int_{S^{n-1}} \left|x_1 - \langle \xi, s\omega\rangle\right|^{-\varepsilon} \Omega(\omega)\, d\sigma(\omega)$$

for fixed x_1 and ξ, where $d\sigma$ denotes the Lebesgue surface measure of S^{n-1} (when $n = 1$, let $\sigma(\{1\}) = \sigma(\{-1\}) = 1$). Then by Hölder's inequality

$$I_\varepsilon(s) \leqslant \left(N_{\varepsilon v'}(s)\right)^{1/v'} \|\Omega\|_v,$$

where

$$N_\varepsilon(s) = \int_{S^{n-1}} |x_1 - s\omega_1|^{-\varepsilon} \, d\sigma(\omega).$$

Thus, using Hölder's inequality,

$$\begin{aligned}
\int_{\mathbf{R}^n} G(y) \left|x_1 - \langle \xi, y\rangle\right|^{-\varepsilon} dy &\leqslant \int_0^\infty h(s) s^{n-1} I_\varepsilon(s)\, ds \\
&\leqslant \|\Omega\|_v \int_0^\infty h(s) s^{n-1} \left(N_{\varepsilon v'}(s)\right)^{1/v'} ds \\
&\leqslant c\, \|H\|_1^{1/v} \|\Omega\|_v \left(\int_0^\infty h(s) s^{n-1} N_{\varepsilon v'}(s)\, ds \right)^{1/v'} \\
&= c\, \|H\|_1^{1/v} \|\Omega\|_v \left(\int_{\mathbf{R}^n} h(|y|)\, |x_1 - y_1|^{-\varepsilon v'} \, dy \right)^{1/v'}.
\end{aligned}$$

Therefore, the desired estimate for $L_\varepsilon(E,G;\xi)$ follows if we show that

$$\sup_{x_1\in\mathbf{R}}\int_{\mathbf{R}^n} h(|y|)\,|x_1-y_1|^{-\varepsilon v'}\,dy<\infty. \tag{4.1}$$

To see this, we split the domain of the integration as follows :

$$\begin{aligned}\int_{\mathbf{R}^n} h(|y|)\,|x_1-y_1|^{-\varepsilon v'}\,dy &= \int_{|x_1-y_1|<1} h(|y|)\,|x_1-y_1|^{-\varepsilon v'}\,dy\\ &\quad+\int_{|x_1-y_1|>1} h(|y|)\,|x_1-y_1|^{-\varepsilon v'}\,dy\\ &= I_1+I_2,\quad \text{say.}\end{aligned}$$

Clearly $I_2\leqslant \|H\|_1$. To estimate I_1 we may assume that $n\geqslant 2$; the case $n=1$ can be easily disposed of since h is bounded. We need further splitting of the domain of the integration. We write $y=(y_1,y')$, $y'\in\mathbf{R}^{n-1}$. Then

$$\begin{aligned}I_1 &= \int\limits_{\substack{|x_1-y_1|<1\\ |y'|<1}} h(|y|)\,|x_1-y_1|^{-\varepsilon v'}\,dy+\int\limits_{\substack{|x_1-y_1|<1\\ |y'|>1}} h(|y|)\,|x_1-y_1|^{-\varepsilon v'}\,dy\\ &= I_3+I_4,\quad\text{say.}\end{aligned}$$

It is easy to see that

$$I_3\leqslant \|H\|_\infty\int_{|y|<2}|y_1|^{-\varepsilon v'}\,dy<\infty.$$

Next, since $h(|y|)\leqslant h(|y'|)$,

$$\begin{aligned}I_4 &\leqslant \int_{|y_1|<1}|y_1|^{-\varepsilon v'}\,dy_1\int_{|y'|>1}h(|y'|)\,dy'\\ &\leqslant c\int_{|y_1|<1}|y_1|^{-\varepsilon v'}\,dy_1\int_{|y|>1}h(|y|)\,dy<\infty.\end{aligned}$$

It remains to estimate $L_\varepsilon(G,G;\xi)$. Note that

$$L_\varepsilon(G,G;\xi)\leqslant\int_0^\infty\int_0^\infty h(r)h(s)r^{n-1}s^{n-1}I_\varepsilon(r,s)\,dr\,ds, \tag{4.2}$$

where

$$I_\varepsilon(r,s)=\iint_{S^{n-1}\times S^{n-1}}\left|\langle\xi,r\theta-s\omega\rangle\right|^{-\varepsilon}\Omega(\theta)\Omega(\omega)\,d\sigma(\theta)\,d\sigma(\omega).$$

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By Hölder's inequality

$$I_\varepsilon(r,s) \leqslant \left(N_{\varepsilon v'}(r,s)\right)^{1/v'} \|\Omega\|_v^2, \tag{4.3}$$

where

$$N_\varepsilon(r,s) = \iint_{S^{n-1}\times S^{n-1}} |r\theta_1 - s\omega_1|^{-\varepsilon}\, d\sigma(\theta)\, d\sigma(\omega).$$

Using the estimate (4.3) in (4.2) and then applying Hölder's inequality, we see that

$$\begin{aligned} L_\varepsilon(G,G;\xi) &\leqslant c\,\|H\|_1^{2/v}\,\|\Omega\|_v^2 \left(\int_0^\infty\int_0^\infty N_{\varepsilon v'}(r,s)h(r)h(s)r^{n-1}s^{n-1}\,dr\,ds\right)^{1/v'} \\ &= c\,\|H\|_1^{2/v}\,\|\Omega\|_v^2 \left(\iint_{\mathbf{R}^n\times\mathbf{R}^n} h(|x|)h(|y|)\,|x_1-y_1|^{-\varepsilon v'}\,dx\,dy\right)^{1/v'}. \end{aligned}$$

Therefore, the desired estimates follows again from (4.1). This completes the proof. □

For a non-negative function Ω on S^{n-1} we define a non-isotropic Hardy-Littlewood maximal function

$$M_\Omega(f)(x) = \sup_{r>0} r^{-n}\int_{|y|<r} |f(x-y)|\Omega\left(|y|^{-1}y\right)dy.$$

To prove Theorem 2 we also need the following (see Duoandikoetxea [**4**]).

Lemma 4. *If $\Omega \in L^q(S^{n-1})$, $q \geqslant 2$, and $w \in A_{2/q'}$, then M_Ω is bounded on L^2_w.*

Now we can prove Theorem 2. As in Stein [**10**, pp.63-64], we can show that

$$\sup_{t>0}\left|\psi_t * f(x)\right| \leqslant c\,M_\Omega(f).$$

So, by Lemma 4 we see that the condition (1.5) holds for ψ of Theorem 2 with $s = 2/q'$.

Next, applying Lemma 3, we see that $J_\varepsilon(\psi) < \infty$ for $\varepsilon < \min(1/u', 1/q')$ (note that $h(r)$ of Theorem 2 (3) is bounded for $r \geqslant 1$). Combining these facts with the assumption in Theorem 2 (1), we can apply Proposition 2 to reach the conclusion.

Finally, we give the proof of Theorem 3. Clearly $B_1(\psi) < \infty$, and $J_{1/(2q')}(\psi) < \infty$ by applying Lemma 3 suitably. Therefore, the conclusion follows from Proposition 2 if we show that the condition (1.5) holds with $s = 2/q'$. But, for $q > 2$ this is a consequence of the inequality

$$\sup_{t>0}\left|\psi_t * f(x)\right| \leqslant c\,M\left(|f|^{q'}\right)^{1/q'},$$

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where M denotes the Hardy-Littlewood maximal operator. (This inequality is easily proved from Hölder's inequality.)

To prove condition (1.5) when $q = 2$ and $w \in A_1$, we may assume that ψ is supported in $\{|x| < 1\}$. Then by Schwarz's inequality

$$\left|\psi_t \star f(x)\right|^2 \leqslant t^{-n} \|\psi\|_2^2 \int_{|y|<t} \left|f(x-y)\right|^2 dy.$$

Integrating with the measure $w(x)\,dx$ and using a property of the A_1-weight function, we get

$$\begin{aligned}\int \left|\psi_t \star f(x)\right|^2 w(x)\,dx &\leqslant \|\psi\|_2^2 \int \left|f(y)\right|^2 t^{-n} \int_{|x-y|<t} w(x)\,dx\,dy \\ &\leqslant C_w \|\psi\|_2^2 \int \left|f(y)\right|^2 w(y)\,dy\end{aligned}$$

uniformly in t. From this the desired inequality follows.

4. Applications

It is to be noted that Theorem 1 can be applied to study the L^p_w-boundedness of generalised Marcinkiewicz integrals.

Corollary 1. *For $\varepsilon > 0$, let*

$$\psi(x) = |x|^{-n+\varepsilon}\,\Omega(x')\chi_{(0,1]}(|x|),$$

where $\Omega \in L^\infty(S^{n-1})$ and $\int \Omega(x')\,d\sigma(x') = 0$. Define a Marcinkiewicz integral

$$\mu(f)(x) = \left(\int_0^\infty \left|\psi_t \star f(x)\right|^2 \frac{dt}{t}\right)^{1/2}.$$

Then, the operator μ is bounded on L^p_w for all $p \in (1,\infty)$ and $w \in A_p$:

$$\|\mu(f)\|_{L^p_w} \leqslant C_{p,w} \|f\|_{L^p_w}.$$

This result, in particular, removes the Lipschitz condition assumed for Ω in Stein [**9**, Theorem 1 (2)].

Next, we consider applications to Carleson measures on the upper half spaces.

Corollary 2. *Suppose $\psi \in L^1$ satisfies* (1.1) *and*

$$\left|\psi(x)\right| \leqslant c\left(1+|x|\right)^{-n-\varepsilon} \qquad \text{for some} \quad \varepsilon > 0.$$

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Take $b \in BMO$ *and* $w \in A_2$. *Then the measure*

$$d\nu(x,t) = \left|\psi_t \star b(x)\right|^2 \frac{dt}{t}\, w(x)\, dx$$

on the upper half space $\mathbf{R}^n \times (0,\infty)$ *is a Carleson measure with respect to the measure* $w(x)\,dx$, *that is,*

$$\nu\big(S(Q)\big) \leqslant C_w \|b\|^2_{BMO} \int_Q w(x)\, dx$$

for all cubes Q *in* $\mathbf{R}^n$, *where*

$$S(Q) = \big\{(x,t) \in \mathbf{R}^n \times (0,\infty) : x \in Q,\ 0 < t \leqslant \ell(Q)\big\},$$

with $\ell(Q)$ *denoting the sidelength of* Q.

This can be proved by using L^2_w-boundedness of the operator S_ψ (see Theorem 1) as in Journé [**7**, Chapter 6 III, pp.85–87]. In [**7**], a similar result has been proved with an additional assumption on the gradient of ψ.

Arguing as in [**7**, Chapter 6 III, p.87], by Corollary 2 we can get the following.

COROLLARY 3. *Let* ψ *and* b *be as in Corollary 2. Suppose* φ *satisfies*

$$\left|\varphi(x)\right| \leqslant c\big(1 + |x|\big)^{-n-\delta}$$

for $\delta > 0$. *Then, the sublinear operator*

$$T_b(f)(x) = \left(\int_0^\infty \left|\psi_t \star b(x)\right|^2 \left|\varphi_t \star f(x)\right|^2 \frac{dt}{t}\right)^{1/2}$$

is bounded on L^p_w *for all* $p \in (1,\infty)$ *and* $w \in A_p$:

$$\left\|T_b(f)\right\|_{L^p_w} \leqslant C_{p,w} \|b\|_{BMO} \|f\|_{L^p_w}.$$

Here again we don't need the assumption on the gradient of ψ. See Coifman and Meyer [**3**, p.149] for the L^2-case.

COROLLARY 4. *Suppose* $\eta \in L^1(\mathbf{R}^n)$ *satisfies the assumptions of Theorem 1 for* ψ. *Let* ψ, φ *and* b *be as in Corollary 3, and define a paraproduct*

$$\pi_b(f)(x) = \int_0^\infty \eta_t \star \big((\psi_t \star b)\,(\varphi_t \star f)\big)(x)\, \frac{dt}{t}.$$

Then, the operator π_b *is bounded on* L^p_w *for all* $p \in (1,\infty)$ *and* $w \in A_p$:

$$\left\|\pi_b(f)\right\|_{L^p_w} \leqslant C_{p,w} \|b\|_{BMO} \|f\|_{L^p_w}.$$

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PROOF: Let $g \in L^2(w^{-1})$, $w \in A_2$. Then, since $w^{-1} \in A_2$, by Schwarz's inequality, Theorem 1 and Corollary 3, for $0 < u < v$, we see that

$$\left|\int\int_u^v \eta_t \star \left((\psi_t \star b)(\varphi_t \star f)\right)(x) \frac{dt}{t} g(x)\,dx\right|$$
$$\leqslant \left(\int\int_u^v \left|\widetilde{\eta}_t \star g(x)\right|^2 \frac{dt}{t} w^{-1}(x)\,dx\right)^{1/2} \|T_b(f)\|_{L^2(w)}$$
$$\leqslant C_w \|b\|_{BMO} \|g\|_{L^2(w^{-1})} \|f\|_{L^2(w)},$$

where $\widetilde{\eta}(x) = \eta(-x)$. From this estimate we can see that $\pi_b(f)$ is well-defined (see Christ [**2,** III, Section 3]). Taking the supremum over g with $\|g\|_{L^2(w^{-1})} \leq 1$, we get the L^2_w-boundedness, and so the extrapolation theorem of Rubio de Francia implies the conclusion. This completes the proof. ∎

See Coifman and Meyer [**3,** p.149, Proposition 1] for a similar result in the L^2-case.

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