n-REFLEXIVITY FOR LINEAR SPACES OF OPERATORS

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Abstract. We discuss the relationship between the n-reflexivity of a linear sub-space $\mathcal{S}$ in $\mathcal{B}(\mathcal{H})$, property $(\mathbf{A}_{1/n})$, Class $\mathcal{C}_0$ and strictly n-separating vectors. We also show that every algebraic operator with property $(\mathbf{A}_2)$ is hyperreflexive.

1. Introduction. Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ denote the collection of all bounded linear operators on $\mathcal{H}$. For a linear subspace $\mathcal{S} \subset \mathcal{B}(\mathcal{H})$ the reflexive closure of $\mathcal{S}$ is defined as $\operatorname{Ref}\mathcal{S} = \{T \in \mathcal{B}(\mathcal{H}) : Tx \in [\mathcal{S}x] \text{ for all } x \in \mathcal{H}\}$, where $\mathcal{S}x = \{Sx : S \in \mathcal{S}\}$ and [.] denotes the closure in the norm topology. A linear subspace $\mathcal{S} \subset \mathcal{B}(\mathcal{H})$ is *reflexive* if $\mathcal{S} = \operatorname{Ref}\mathcal{S}$. It is easily proved that reflexive subspaces are weakly closed and it is also easy to verify that $\operatorname{Ref}\mathcal{A} = \operatorname{Alglat}\mathcal{A}$ if $\mathcal{A}$ is an algebra containing $I_{\mathcal{H}}$, where $\operatorname{lat}\mathcal{A}$ is the lattice of invariant subspace for $\mathcal{A}$, and $\operatorname{Alglat}\mathcal{A} = \{T \in \mathcal{B}(\mathcal{H}) : \operatorname{lat}\mathcal{A} \subset \operatorname{lat} T\}$. We recall that a subspace $\mathcal{S}$ of $\mathcal{B}(\mathcal{H})$ is *hyperreflexive* if there is a $K \geq 1$ such that, for every T in $\mathcal{B}(\mathcal{H})$, $\operatorname{dist}(T, \mathcal{S}) \leq K\sup\{\operatorname{dist}(Tx, \mathcal{S}x) : x \in \mathcal{H}, \|x\| \leq 1\}$. The smallest such $K = K(\mathcal{S})$ is the constant of hyperreflexivity of $\mathcal{S}$. Clearly, hyperreflexivity implies reflexivity, but not vice versa **[13]**. We say that an operator T is *reflexive(hyperreflexive)* if the weakly closed algebra W_T generated by $I_{\mathcal{H}}$ and T is reflexive(hyperreflexive).

For a vector $x \in \mathcal{H}$ and a linear subspace $\mathcal{S} \subset \mathcal{B}(\mathcal{H})$, we define the evaluation map $E_x : \mathcal{S} \to \mathcal{H}$ by $E_x(T) = Tx$. A vector x in $\mathcal{H}$ *separates* $\mathcal{S}$ if E_x is injective on $\mathcal{S}$ and a vector x in $\mathcal{H}$ *strictly separates* $\mathcal{S}$ if E_x is bounded below on $\mathcal{S}$. By the open mapping theorem, it is easy to see that x strictly separates $\mathcal{S}$ if and only if x separates $\mathcal{S}$ and $\mathcal{S}x$ is norm closed. We write $\mathcal{S}^{(n)} = \{S^{(n)} \in \mathcal{B}(\mathcal{H}^{(n)}) : S \in \mathcal{S}\}$ as the n-fold ampliation of $\mathcal{S}$, where $S^{(n)}$ is the direct sum of n copies of the operators acting on $\mathcal{H}^{(n)} = \mathcal{H} \oplus \ldots \oplus \mathcal{H}$.$\mathcal{S}$ is said to be *n-reflexive(n-hyperreflexive)* if and only if $\mathcal{S}^{(n)}$ is reflexive(hyperreflexive). $\mathcal{S}$ is n-reflexive(n-hyperreflexive) implies $\mathcal{S}$ is $(n+1)$-reflexive($(n+1)$-hyperreflexive), but the converse does not hold. We say that $\mathcal{S}$ has a *strictly n-separating vector* if $\mathcal{S}^{(n)}$ has a strictly separating vector. It is easy to see that if $\mathcal{S}$ has a strictly n-separating vector, then $\mathcal{S}$ has a strictly $(n+1)$-separating vector. For a linear subspace $\mathcal{S} \subset \mathcal{B}(\mathcal{H})$ and a linear subspace $\mathcal{M}$ of $\mathcal{H}$, we define $\pi : \mathcal{S} \mid \mathcal{M}$ by $\pi(S) = S \mid \mathcal{M}$. A linear subspace $\mathcal{M}$ is said to be a *strictly separating subspace* for $\mathcal{S}$ if there exists $\varepsilon > 0$ such that $\|S|\mathcal{M}\| \geq \varepsilon\|S\|$, for all $S \in \mathcal{S}$. It is easily seen that $\mathcal{M}$ is a strictly separating subspace for $\mathcal{S}$ if and only if the only member $S \in \mathcal{S}$ satisfying $S(\mathcal{M}) = \{0\}$ is $S = 0$ and $\mathcal{S} \mid \mathcal{M}$ is norm closed.

For vectors x and y in $\mathcal{H}$, we write $x \otimes y$ for the rank one operator defined by $(x \otimes y)(u) = (u, y)x$, $u \in \mathcal{H}$. Let $\mathcal{S} \subset \mathcal{B}(\mathcal{H})$ be a weak*-closed subspace and n is a positive integer. We say that $\mathcal{S}$ has property $(\mathbf{A}_{1/n})$ if every weak*-continuous functional φ on $\mathcal{S}$ can be written as

$$\varphi = \sum_{i=1}^{n}[x_i \otimes y_i] \text{ for some } x_i \text{ and } y_i \text{ in } \mathcal{H}. \tag{1}$$

Furthermore, $\mathcal{S}$ has property $(\mathbf{A}_{1/n})(r)$, $r \geq 1$ if it has property $(\mathbf{A}_{1/n})$ and if for any $C > r$ the decomposition(1) can be realized with $\sum_{i=1}^{n} \|x_i\| \|y_i\| \leq C\|\varphi\|$. Suppose m and n are cardinal numbers such that $1 \leq m, n \leq \aleph_0$. We say that $\mathcal{S} \subset \mathcal{B}(\mathcal{H})$ has property $(\mathbf{A}_{m,n})$ provided that for every family $\{\varphi_{ij} : 0 \leq i < m, 0 \leq j < n\}$ of weak*-continuous functional on $\mathcal{S}$, there exist sequences $\{x_i : 0 \leq i < m\}$ and $\{y_j : 0 \leq j < n\}$ of vectors in $\mathcal{H}$ such that

$$\varphi_{ij} = [x_i \otimes y_j]_{\substack{0 \leq i < m \\ 0 \leq j < m}}. \tag{2}$$

Furthermore, if $m, n \in \mathbf{N}$ and r is a fixed real number satisfying $r \geq 1$, then $\mathcal{S} \subset \mathcal{B}(\mathcal{H})$ is said to have property $(\mathbf{A}_{m,n})(r)$ if for every $s > r$, there exist sequences $\{x_i\}_{0 \leq i < m}$, $\{y_j\}_{0 \leq j < n}$ that satisfy (2) and also satisfy the following conditions:

$$\|x_i\| \leq \left(s \sum_{0 \leq j < n} \|\varphi_{ij}\|\right)^{1/2} \quad (0 \leq i < m), \tag{3}$$

and

$$\|y_j\| \leq \left(s \sum_{0 \leq i < m} \|\varphi_{ij}\|\right)^{1/2} \quad (0 \leq j < n). \tag{4}$$

For brevity, we shall denote $(\mathbf{A}_{n,n})$ by $(\mathbf{A}_n)$.

In [**4**], H. Bercovici, C. Foias and C. Pearcy [Proposition 9.16] proved that if a weakly closed subspace $\mathcal{S}$ of $\mathcal{B}(\mathcal{H})$ has property $(\mathbf{A}_1)$, then $\mathcal{S}$ is 3-reflexive. In Section 2 [Theorem 2.3], we prove the following generalization: a weak*-closed subspace $\mathcal{S}$ of $\mathcal{B}(\mathcal{H})$ with property $(\mathbf{A}_{1/n})$ is $(2n+1)$-reflexive. In Section 3 [Theorem 3.2], we prove that every operator of class C_0 is 2-reflexive. It was shown by L. Ding [**10**, Theorem 2.5] that if $\mathcal{S}$ is a fine dimensional linear subspace of $\mathcal{L}(\mathcal{V})$ and $\mathcal{S}$ has a k-dimensional separating subspace, then $\mathcal{S}$ is $(k+1)$-reflexive. In the last section [Theorem 4.2], we prove that a norm closed linear subspace $\mathcal{S}$ of $\mathcal{B}(\mathcal{H})$ with a strictly n-separating vector is $(n+1)$-hyperreflexive. We recover a well-known result as a special case.

2. Property $(\mathbf{A}_{1/n})$.

If $\mathcal{S}$ is a WOT-closed subspace of $\mathcal{B}(\mathcal{H})$, then $\mathcal{S}$ is weak*-closed but not conversely. B. Chevreau and J. Esterle [**6**] proved the following interesting result.

LEMMA 2.1. *Let $\mathcal{S}$ be a weak*-closed subspace of $\mathcal{B}(\mathcal{H})$ with property $(\mathbf{A}_{1/n})$ for some $n \in \mathbf{N}$. Then $\mathcal{S}$ is WOT-closed.*

The following elementary Lemma comes from [**4**, Lemma 9.15].

LEMMA 2.2. *Suppose that $n \in \mathbf{N}$, $T \in \mathcal{B}(\mathcal{H})$ and $\mathcal{S}$ is a linear subspace of $\mathcal{B}(\mathcal{H})$. Then the n-fold direct sum $T^{(n)}$ belongs to* Ref $(\mathcal{S}^{(n)})$ *if and only if whenever $\{x_1 \dots x_n\}$ and $\{y_1 \dots y_n\}$ are sequences from $\mathcal{H}$ such that $\sum_{j=1}^{n} [x_j \otimes y_j] = 0$, we have $\sum_{j=1}^{n} \langle Tx_j, y_j \rangle = 0$. Moreover, T belongs to the WOT-closure of $\mathcal{S}$ if and only if the n-fold direct sum $T^{(n)}$ belongs to* Ref $(\mathcal{S}^{(n)})$, *for every positive integer n.*

THEOREM 2.3. *Let $\mathcal{S}$ be a weak*-closed subspace of $\mathcal{B}(\mathcal{H})$ with property $(\mathbf{A}_{1/n})$. Then $\mathcal{S}$ is $(2n+1)$-reflexive.*

Proof. By Lemma 2.1, $\mathcal{S}$ is WOT-closed. Much of the proof is based on ideas of **[4]**. Suppose the $(2n+1)$-fold direct sum $T^{(2n+1)}$ belongs to Ref $(\mathcal{S}^{(2n+1)})$ for some T in $\mathcal{B}(\mathcal{H})$. We have to show that T belongs to $\mathcal{S}$. By Lemma 2.2, it suffices to show that the implication

$$\sum_{j=1}^{k}[x_j \otimes y_j] = 0 \Rightarrow \sum_{j=1}^{k}\langle Tx_j, y_j\rangle = 0 \tag{5}$$

holds for all integers k. We proceed by induction on p. By the hypothesis, we know that (5) is satisfied for $k \leq (2n+1)$. Assume that (5) has been proven for all $2n+1 \leq k < p$ and let $x_1, x_2, \ldots, x_p, y_1, y_2, \ldots, y_p$ in $\mathcal{H}$ satisfy the relation

$$\sum_{j=1}^{p}[x_j \otimes y_j] = 0. \tag{6}$$

Since $\mathcal{S}$ has property $(\mathbf{A}_{1/n})$, there exist sequences of vectors $\{u_1, \ldots, u_n\}$ and $\{v_1, \ldots, v_n\}$ in $\mathcal{H}$ such that

$$\sum_{j=1}^{n}[u_j \otimes v_j] = \sum_{j=n+2}^{p}[x_j \otimes y_j]. \tag{7}$$

Since the equality $\sum_{j=1}^{n}[-u_j \otimes v_j] + \sum_{j=n+2}^{p}[x_j \otimes y_j] = 0$ has $(p-1)$ terms, we have $\sum_{j=1}^{n}\langle -Tu_j, v_j\rangle + \sum_{j=n+2}^{p}\langle Tx_j, y_j\rangle = 0$, or, equivalently,

$$\sum_{j=1}^{n}\langle Tu_j, v_j\rangle = \sum_{j=n+2}^{p}\langle Tx_j, y_j\rangle. \tag{8}$$

Furthermore, from (6) and (7) we have $\sum_{j=1}^{n+1}[x_j \otimes y_j] + \sum_{j=1}^{n}[u_j \otimes v_j] = 0$. Thus it follows from the induction hypothesis that

$$\sum_{j=1}^{n+1}\langle Tx_j, y_j\rangle + \sum_{j=1}^{n}\langle Tu_j, v_j\rangle = 0, \tag{9}$$

since k has $(2n+1)$terms. Consequently, by Lemma 2.2, $T \in \mathcal{S}$. Hence the proof is complete.

We recover a result in **[4]** as a special case.

COROLLARY 2.4. *Suppose $\mathcal{S}$ is a weak*-closed subspace of $\mathcal{B}(\mathcal{H})$ with property $(\mathbf{A}_1)$. Then $\mathcal{S}$ is 3-reflexive.*

The following proposition improves a result of A. Loginov and V. Shulman.

PROPOSITION 2.5. *Suppose $\mathcal{S}$ is a weak*-closed subspace of $\mathcal{B}(\mathcal{H})$ and has property $(\mathbf{A}_{1/n})$ and suppose $\mathcal{S}$* is *n-reflexive. Then every weakly-closed subspace of $\mathcal{S}$ is n-reflexive, where n is a positive integer, (i.e., $\mathcal{S}$ is hereditarily n-reflexive).*

Proof. Suppose $\mathcal{M}$ be a weakly closed subspace of $\mathcal{S}$ and let the n-fold direct sum $T^{(n)}$ belong to $\operatorname{Ref}(\mathcal{M}^{(n)})$. We have to show T belongs to $\mathcal{M}$. It suffices to show the impliction $T \in \operatorname{Ref}_{E_n}(\mathcal{M}) \Rightarrow T \in \mathcal{M}$, where $E_n = E + E + \ldots + E$ (n summands). Assume that $T \in \operatorname{Ref}_{E_n}(\mathcal{M})$ and $T \notin \mathcal{M}$. Then $T \notin \mathcal{M} = (\mathcal{M}^{\perp})_{\perp}$. Thus there exists φ in $\mathcal{M}^{\perp}$ such that $\varphi(T) \neq 0$. Since $\mathcal{S}$ has property $(\mathbf{A}_{1/n})$, there exist sequences of vectors $\{x_1, \ldots, x_n\}$, $\{y_1, \ldots y_n\}$ in $\mathcal{H}$ such that

$$\varphi - \sum_{i=1}^{n}[x_i \otimes y_i] \in \mathcal{S}^{\perp}. \tag{10}$$

Thus $\varphi \in \mathcal{M}^{\perp} \cap E_n$ and $\sum_{i=1}^{n}\langle Tx_i, y_i\rangle \neq 0$. Thus we have $T \notin (\mathcal{M}^{\perp} \cap E_n)_{\perp} = \operatorname{Ref}_{E_n}(\mathcal{M})$. Consequently, $T \in \mathcal{M}$. Hence the proof is complete.

3. The Class C_0. Recall that a completely nonunitary contraction $T \in \mathcal{B}(\mathcal{H})$ (on a separable Hilbert space $\mathcal{H}$) is an operator of class C_0 if $u(T) = 0$ for some $u \in \mathbf{H}^{\infty}$, $u \neq 0$. The simplest operators of class C_0 are the Jordan Blocks $S(\Theta)$, with $\Theta \in \mathbf{H}^{\infty}$ an inner function, defined by

$$\mathrm{S}(\Theta) = \left(\mathrm{S}^* \mid (\mathbf{H}^2 \ominus \Theta\mathbf{H}^2)\right)^*, \tag{11}$$

where S is the unilateral shift. It is known [**3**] that every operator T of class C_0 is quasi-similar to a Jordan operator $\mathrm{S} = \oplus_i \mathrm{S}(\Theta_i)$, where the values of i are ordinal numbers and the inner function are subject to the conditions $\Theta_i = 1$ for some $i \geq 0$, Θ_i divides Θ_j whenever $i \geq j$ and $\Theta_i = \Theta_j$ whenever $\operatorname{card}(i) = \operatorname{card}(j)$. We start this section with the following Lemma from [**7**, Proposition 6].

LEMMA 3.1. *For an inner function Θ, the weak*-closed algebra $\mathcal{A}_{S(\Theta)}$ generated by 1 and $S(\Theta)$ has property $(\mathbf{A}_{1,2})(1)$.*

THEOREM 3.2. *Every operator of class C_0 is 2-reflexive.*

Proof. Suppose $T \in C_0$. Then T is quasi-similar to $\mathrm{S} = \oplus_i \mathrm{S}(\Theta_i)$. Since $\mathcal{A}_{\mathrm{S}(\Theta_i)}$ has property $(\mathbf{A}_{1,2})(1)$, $\mathcal{A}_{\mathrm{S}(\Theta_1)}$ is weakly closed and 2-reflexive [**4**, Proposition 9.17], for $i = 1, 2, \ldots$. This implies that $\mathcal{A}_{\mathrm{S}(\Theta_i)} \oplus \mathcal{A}_{\mathrm{S}(\Theta_2)} \oplus \ldots$ is 2-reflexive. Since $\mathcal{A}_{\mathrm{S}} = \mathcal{A}_{\mathrm{S}(\Theta_1)\oplus\mathrm{S}(\Theta_2)\oplus\ldots}$ is contained in $\mathcal{A}_{\mathrm{S}(\Theta_1)} \oplus \mathcal{A}_{\mathrm{S}(\Theta_2)} \oplus \ldots$, $\mathcal{A}_{\mathrm{S}}$ is a weakly closed and has property $(\mathbf{A}_{1,2})(1)$, $\mathcal{A}_{\mathrm{S}}$ is 2-reflexive [Propostion 2.5]. Thus $\mathrm{S} = \oplus_i \mathrm{S}(\Theta_i)$ is 2-reflexive. Since T and S are quasi-similar operators of class C_0, T is 2-reflexive [**3**, Collary 3.6]. Hence the proof is complete.

An operator $T \in \mathcal{B}(\mathcal{H})$ is algebraic if $p(T) = 0$ for some polynomial p.

COROLLARY 3.3. *Every algebraic operator $T \in \mathcal{B}(\mathcal{H})$ with $\|T\| \leq 1$ is 2-hyperreflexive. Moreover, every algebraic operator with property $(\mathbf{A}_2)$ is hyperreflexive.*

Proof. It is known that every algebraic operator with property $(\mathbf{A}_2)$ is reflexive [**5**, Corollary 6]. By [**11**, Theorem 3.14], every reflexive algebraic operator is hyperreflexive. Hence we have the corollary.

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4. Strictly n-separating vectors. In this section we study the relationship between n-hyperreflexity and strictly n-separating vectors. The following lemma comes from [**11**, Theorem 4.10].

LEMMA 4.1. *Let $\mathcal{S}$ be a norm closed linear subspace of $\mathcal{B}(\mathcal{H})$. Suppose $\mathcal{S}$ has a strictly separating vector e and a* strictly *separating closed subspace $\mathcal{M}$ such that $\overline{(\mathcal{SM})} = \overline{sp}\{Sx : S \in \mathcal{S}, x \in \mathcal{M}\}$ satisfies $\overline{(\mathcal{SM})} \cap \mathcal{S}e = \{0\}$* and *$\overline{(\mathcal{SM})} + \mathcal{S}e$ is norm closed. Then $\mathcal{S}$ is hyperreflexive.*

THEOREM 4.2. *Let $\mathcal{S}$ be a norm closed linear subspace of $\mathcal{B}(\mathcal{H})$. Suppose $\mathcal{S}$ has a strictly n-separating vector. Then $\mathcal{S}$ is $(n+1)$-hyperreflexive for some $n \in \mathbf{N}$.*

Proof. Suppose $\mathcal{S}^{(n)}$ has a strictly separating vector $e = (e_1, e_2, \ldots, e_n)$. We show that $\mathcal{S}$ has the n-dimensional strictly separating subspace $\mathcal{M}$. Suppose $= \overline{\text{span}}\{e_1, e_2, \ldots, e_n\}$. We must show that $\mathcal{S} \to \mathcal{S} \mid \mathcal{M}$ is injective and $\mathcal{S} \mid \mathcal{M}$ is norm closed. Since $\mathcal{S}^{(n)}$ has a strictly separating vector $e = (e_1, e_2, \ldots, e_n)$.

$$\|S^{(n)}e\| \geq \delta \|S^{(n)}\| \text{ where } \delta > 0. \tag{12}$$

Thus

$$\|S\| = \|S^{(n)}\| \leq \frac{1}{\delta}\|S^{(n)}e\| \tag{13}$$

$$= \frac{1}{\delta}\|S^{(n)}(e_1, e_2, \ldots, e_n)\| \tag{14}$$

$$= \frac{1}{\delta}\|(Se_1, Se_2, \ldots, Se_n)\| \tag{15}$$

$$= \frac{1}{\delta}\left(\sum_{i=1}^{n} \|Se_i\|^2\right)^{\frac{1}{2}} \tag{16}$$

$$\leq \frac{\sqrt{n}}{\delta}\text{Max}_{1 \leq i \leq n}\|Se_i\|. \tag{17}$$

Then we have

$$\|S|\mathcal{M}\| \geq \text{Max}_{1 \leq i \leq n}\|Se_i\| \geq \frac{\delta}{\sqrt{n}}\|S\|.$$

We define $\pi : \mathcal{S} \to \mathcal{S} \mid \mathcal{M}$ by $\pi(S) = S \mid \mathcal{M}$. Then we have

$$\|\pi(S)\| \geq \frac{\delta}{\sqrt{n}}\|S\| = \beta\|S\|, \text{ where } \beta = \frac{\delta}{\sqrt{n}} > 0.$$

Thus π is injective and ran π is norm closed. Let $f = e \oplus 0 \in \mathcal{H}^{(n+1)}$. Then f is a strictly separating vector for $\mathcal{S}^{(n+1)}$. Set

$$\mathcal{G} = \Big\{\underbrace{0 \oplus \ldots \oplus 0}_{(n)} \oplus m \mid m \in \mathcal{M}\Big\} \subset \mathcal{H}^{(n+1)}. \tag{18}$$

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Then $\mathcal{G}$ is a strictly separating subspace for $\mathcal{S}^{(n+1)}$. We also note that $\mathcal{S}^{(n+1)}f \cap (\mathcal{S}^{(n+1)}\mathcal{G}) = \{0\}$ and $\mathcal{S}^{(n+1)}f + (\mathcal{S}^{(n+1)}\mathcal{G})$ is norm closed. By Lemma 4.1, $\mathcal{S}^{(n+1)}$ is hyperreflexive. Hence $\mathcal{S}$ is $(n+1)$-hyperreflexive.

COROLLARY 4.3. *Suppose $\mathcal{S}$ is a norm closed linear subspace of $\mathcal{B}(\mathcal{H})$ and $\mathcal{S}$ has a finite dimensional strictly separating subspace $\mathcal{M}$. Then there is a positive integer n such that $\mathcal{S}$ is n-hyperreflexive.*

Proof. We can choose from unit sphere of $\mathcal{M}$ a finite collection of vectors $v_1, \dots, v_k$ so that $(v_1, \dots, v_k)$ is a strictly k-separating vector for $\mathcal{S}$. It follows from the above theorem that there is a positive integer $n(> k)$ such that $\mathcal{S}$ is n-hyperreflexive. Hence the proof is complete.

A subalgebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ is said to be *strictly cyclic* if $\mathcal{A}x = \mathcal{H}$, for some vector x in $\mathcal{H}$.

COROLLARY 4.4. *Suppose $\mathcal{A}^{(n)}$ is a strictly cyclic abelian algebra. Then $\mathcal{A}$* is $(n+1)$-*hyperreflexive, for some $n \in \mathbf{N}$.*

PROPOSITION 4.5. *A weak*-closed subspace $\mathcal{S}$ has a strictly n-separating vector. Such as $\mathcal{S}$ has property $(\mathbf{A}_{1/n})(nr)$, for some $r > 1$ and $n \in \mathbf{N}$.*

Proof. Suppose $\mathcal{S}$ has a strictly n-separating vector. Then $\mathcal{S}^{(n)}$ has a strictly separating vector. Thus $\mathcal{S}^{(n)}$ has property $(\mathbf{A}_1)(r)$. By Proposition 7.3(1) of [**1**], $\mathcal{S}$ has a property $(\mathbf{A}_{1/n})(nr)$. The proof is complete.

In the following example, $\mathcal{S}^{(2)} = \{S \oplus S \mid S \in \mathcal{S}\}$ has a strictly separating subspace $\mathcal{M}$ but $\mathcal{S}^{(2)}\mathcal{M}$ is not norm closed.

EXAMPLE 4.6. *Suppose $\mathcal{S}$ is a norm closed linear subspace of $\mathcal{B}(\mathcal{H})$ and $\mathcal{S}$ has a strictly separating vector e. Let $f \in \mathcal{H}$ and $\mathcal{M} = sp\{e \oplus 0, 0 \oplus f\}$. Then we have*

$$\|S^{(2)} \mid \mathcal{M}\| \geq \|S^{(2)}(e \oplus 0)\| = \|Se\| \tag{19}$$

$$\geq \varepsilon\|S\| = \varepsilon\|S^{(2)}\|. \tag{20}$$

Thus $\mathcal{M}$ is a strictly separating subspace for $\mathcal{S}^{(2)}$. But since

$$\mathcal{S}^{(2)}\mathcal{M} = \{S_1 e \oplus S_2 f \mid S_1, S_2 \in \mathcal{S}\} = \mathcal{S}e \oplus \mathcal{S}f, \tag{21}$$

$\mathcal{S}^{(2)}\mathcal{M}$ is not norm closed.

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