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THE DUALS OF THE CAMILLO-ZELMANOWITZ FORMULAS FOR GOLDIE DIMENSION

BY
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ABSTRACT. The duals of the Camillo-Zelmanowitz formulas for Goldie dimension are shown to hold for Varadarajan's notion of corank, subject to the existence of certain cocomplements. In particular, the formulas hold for modules over perfect rings. Also, if R is semiperfect, then the vector space dimension formulas hold for all modules over R for Goldie dimension iff they hold for corank iff R is semisimple.

The *Goldie dimension* of a module M, written $d(M)$, is defined to be n if there is a direct sum of n non-zero submodules of M contained in M, but no direct sum of $n+1$ submodules, or to be infinite if no such integer n exists. Camillo and Zelmanowitz have pointed out that Goldie dimension does not satisfy the familiar formulas for vector space dimension:

$$\dim(M)=\dim(M/A)+\dim(A), \tag{1}$$

and

$$\dim(A+B)=\dim(A)+\dim(B)-\dim(A\cap B) \tag{2}$$

for $A, B \subseteq M$, and have found the corrections required [3, Lemma 3 and Theorem 4]:

(1) If $\bar{A}$ is a maximal essential extension of A in M, then

$$d(M)+d(\bar{A}/A)=d(M/A)+d(A);$$

and

(2) If A and B are submodules of M, f is a maximal monic extension of the identity map $1_{A\cap B}$ considered as a homomorphism from A to B, and $D=\operatorname{dom} f$, then

$$d(A+B)=d(A)+d(B)-d(D)+d(D/(A\cap B))$$

Varadarajan has proposed the notion of corank as a dualization of Goldie dimension; the *corank* of M, written $\operatorname{cor}(M)$, is defined to be n if there is a

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direct product of non-zero factor modules of M that is an epimorphic image of M and no such product of $n+1$ factor modules, or to be infinite if no such n exists [8]. Corank is equivalent to the *codimension* of Reiter [6], there defined to be n if there is a *direct intersection* $K_1 \cap \cdots \cap K_n$ of n proper submodules of M, that is, a family $\{K_1, \ldots, K_n\}$ of proper submodules of M such that

$$K_i + \left(\bigcap_{j \neq i} K_j\right) = M, \qquad i = 1, \ldots, n,$$

and no direct intersection of $n+1$ proper submodules. (For the equivalence, see [8, Lemma 1.4].) Here, we demonstrate that formulas dual to those of Camillo and Zelmanowitz hold for corank, subject to the existence of the appropriate submodules, and we find necessary conditions for the existence of these submodules in terms of Fleury's notion of the supplement of a submodule [4]. (See also [5, 7, 8, 9].) As a corollary, it follows that the submodules required to express the dualization of the first Camillo-Zelmanowitz formula exist for modules over perfect rings, and those required for the dual of the second exist for artinian modules over perfect rings. Finally, we show that if R is semiperfect, then the vector space dimension formulas hold for all modules over R for Goldie dimension iff they hold for corank iff R is semisimple.

Let N be a submodule of M. A useful notion in the Goldie theory is that of a *complement of* N *in* M, that is, a submodule T of M with $N \cap T = 0$ and $N+T$ essential in M. A complement T can be obtained by Zorn's Lemma as being maximal with respect to $N \cap T = 0$. (For this and other results in module theory, see [1].) Zorn's Lemma cannot be used in general to find a *cocomplement of* N *in* M, defined to be a submodule T of M with $N+T=M$ and $N \cap T$ small in M (written $N \cap T \ll M$), but Reiter has shown that if $\operatorname{cor}(M) < \infty$, then a cocomplement of N in M exists [6, Theorem 4.1]. A *supplement of* N *in* M is a submodule T of M minimal with respect to $N+T=M$, and T is said to be a *supplement in* M if there is a submodule N of M such that T is a supplement of N in M.

We shall need the following propositions. The first is proven by using modularity twice.

1. Proposition (Reiter [6, Lemma 2.3]). *If* N, T, *and* L *are submodules of* M *such that* $N+T=(N \cap T)+L=M$, *then* $N+(T \cap L)=T+(N \cap L)=M$.

2. Proposition (Varadarajan [8, Lemma 2.7]). *Let* $N \subseteq M$. *If* T *is a supplement of* N *in* M, *then* $T \cap N \ll T$. *Hence a supplement of* N *in* M *is a minimal cocomplement of* N.

3. Proposition. *Let* $Y \subseteq N \subseteq M$. *If* T *is any cocomplement of* N *in* M, *then* $Y+T=M$ *iff* $N/Y \ll M/Y$. *Hence* Y *is a supplement of* T *iff* Y *is minimal with respect to* $N/Y \ll M/Y$. *Further,* N *is a supplement in* M *iff* N *has a cocomplement in* M *and* $N/Y \ll M/Y$ *implies* $N=Y$ *for all* $Y \subseteq N$.

Proof. Let T be a cocomplement of N in M and suppose $Y \subseteq N$ is such that $Y+T=M$. Assume that $N/Y+L/Y=M/Y$, so $N+L=M$. Now $Y \subseteq N \cap L$, so $(N \cap L)+T \supseteq Y+T=M=N+T$. Then by Proposition 1, also $(N \cap T)+L=M$. Since T is a cocomplement of N, $N \cap T \ll M$ and thus $L=M$, so $N/Y \ll M/Y$. Conversely, assume $N/Y \ll M/Y$. Then since $N+T=M$,

$$N/Y+(T+Y)/Y=(N+T+Y)/Y=M/Y.$$

Thus $(T+Y)/Y=M/Y$, so $T+Y=M$. The last two statements now follow, using Proposition 2.

4. Proposition (Varadarajan).
(a) [8, Remark 1.12] *If* $K \subseteq M$, *then* $\operatorname{cor}(M) \geqslant \operatorname{cor}(M/K)$.
(b) [8, Theorem 1.20] *If* $K \ll M$, *then* $\operatorname{cor}(M)=\operatorname{cor}(M/K)$.
(c) [7, Corollary 1.9] $\operatorname{cor}(A \oplus B)=\operatorname{cor}(A)+\operatorname{cor}(B)$.

We are now in a position to establish a dual of the first Camillo-Zelmanowitz formula, extending [8, Proposition 2.31] and [4, Theorem 4.2].

5. Theorem. *Assume that* $N \subseteq M$ *has a cocomplement* T *in* M *and that* $Y \subseteq N$ *is a supplement of* T *in* M. *Then*

$$\operatorname{cor}(M)+\operatorname{cor}(N/Y)=\operatorname{cor}(M/N)+\operatorname{cor}(N).$$

Proof. Let T be a cocomplement of N in M, so $T+N=M$ and $T \cap N \ll M$. Then

$$\begin{aligned}\operatorname{cor}(M)&=\operatorname{cor}(M/(T\cap N))=\operatorname{cor}(T/(T\cap N)\oplus N/(T\cap N))\\&=\operatorname{cor}(T/(T\cap N))+\operatorname{cor}(N/(T\cap N))\\&=\operatorname{cor}((T+N)/N)+\operatorname{cor}((T+N)/T)\\&=\operatorname{cor}(M/N)+\operatorname{cor}(M/T)\end{aligned}$$

Let $Y \subseteq N$ be a supplement of T in M. Then

$$N=N\cap M=N\cap(T+Y)=(N\cap T)+Y$$

by modularity. Now if $X \subseteq Y$ with $N=(N \cap T)+X$, then

$$M=T+N=T+(N\cap T)+X=T+X,$$

so $X=Y$. Thus Y is a supplement, hence a cocomplement of $N \cap T$ in N. Calculating as above,

$$\operatorname{cor}(N)=\operatorname{cor}(N/(N\cap T))+\operatorname{cor}(N/Y)=\operatorname{cor}(M/T)+\operatorname{cor}(N/Y).$$

If $\operatorname{cor}(N/Y)=\infty$ then also $\operatorname{cor}(N)=\infty$, and the formula holds. If $\operatorname{cor}(N/Y)<\infty$ then $\operatorname{cor}(N)-\operatorname{cor}(N/Y)=\operatorname{cor}(M/T)$, and the formula follows.

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6. Corollary. *If either*

(a) *M is a module over a perfect ring R, or*

(b) *M is a finitely generated module over a semiperfect ring R, or*

(c) $\operatorname{cor}(M) < \infty$ *and a submodule $Z \subseteq N$ is minimal with respect to $N/Z \ll M/Z$,*

then the hypotheses of Theorem 5 are satisfied, so there is a submodule $Y \subseteq N$ with

$$\operatorname{cor}(M) + \operatorname{cor}(N/Y) = \operatorname{cor}(M/N) + \operatorname{cor}(N).$$

Proof. Varadarajan has shown that in case either (a) or (b) holds, then T can be chosen to be a supplement of N and Y a supplement of T [9, Theorems 1.6 and 1.7]. If (c) holds, then by [6, Theorem 4.1], N has a cocomplement T in M; then Proposition 3 implies that Z is a supplement of T. Theorem 5 finishes the proof.

If A and B are submodules of M and $X \subseteq A+B$, denote by η^X the natural epimorphism $\eta^X : M/X \to M/(A+B)$. Order the set of ordered pairs (D, g) where $D \subseteq A+B$ and $g : M/B \to M/D$ by $(D, g) \leq (D', g')$ iff $D \subseteq D'$ and $g' = \delta g$, where δ is the natural epimorphism $\delta : M/D \to M/D'$. A dual version of the second Camillo-Zelmanowitz formula may be stated as follows. (The theorem is a consequence of Propositions 10 and 11 below.)

7. Theorem. *Let A and B be submodules of M. Assume that there is a submodule D with $A \subseteq D \subseteq A+B$ that is minimal with respect to the existence of an epimorphism $g : M/B \to M/D$ such that $\eta^D g = \eta^B$. Assume further that cocomplements exist for $\{(m+A, n+B) \mid m+D = g(n+B)\}$ in $M/A \oplus M/B$ and for $\{m + A \cap B \mid m+D = g(m+B)\}$ in $M/(A \cap B)$. Then*

$$\operatorname{cor}(M/(A \cap B)) + \operatorname{cor}(M/D) = \operatorname{cor}(M/A) + \operatorname{cor}(M/B) + \operatorname{cor}((A+B)/D).$$

In order to obtain the correction term in the second Camillo-Zelmanowitz formula, one must find a monomorphism f with domain $D \subseteq A$ that is maximal with respect to extending $1_{A \cap B} : A \cap B \to B$. The map f and module D can be shown always to exist by a simple Zorn's Lemma argument. For the dual version in Theorem 7, the existence of both certain cocomplements and a pair (D, g) minimal with respect to the stated property is required; here, Zorn's Lemma is ineffective. We shall find in Proposition 10 that the existence of a minimal pair (D, g) requires that $\{(m+A, n+B) \mid m+D = g(n+B)\}$ be a supplement in $M/A \oplus M/B$.

The next two lemmas give constructions we shall need both in the proof of Theorem 7 and in our examination of conditions for the existence of the correction term.

8. Lemma. *Let M be a module with submodules A and B. If $M \supseteq C \supseteq A \cap B$ with $C + A = C + B = M$, then there exists a module $D \supseteq A$ and an epimorphism $g : M/B \to M/D$ such that $C = \{m \mid m + D = g(m+B)\}$.*

Proof. Let $D = A + (C \cap B)$. Define

$$\bar{g} : M \to M/D \quad \text{via} \quad \bar{g} : m \mapsto c + D,$$

where $m = c + b$ with $c \in C$, $b \in B$. If also $m = c' + b'$ with $c' \in C$ and $b' \in B$, then $c - c' = b - b' \in C \cap B \subseteq D$, so $c + D = c' + D$ and $\bar{g}$ is well-defined. Given $m + D$, choose $c \in C$ and $a \in A$ with $m = c + a$. Then $\bar{g}(c) = c + D = m + D$, so $\bar{g}$ is onto. That $B \subseteq \ker \bar{g}$ is clear. Thus $g : M/B \to M/D$ via $g(m + B) = \bar{g}(m)$ is an epimorphism. Now if $c \in C$, then $g(c + B) = \bar{g}(c) = c + D$, so $C \subseteq \{m \mid m + D = g(m + B)\}$. Conversely, if $g(m + B) = m + D$, write $m = c + b$ with $c \in C$, $b \in B$, so that

$$c + D = g(c + B) = g(m + B) = m + D.$$

Then

$$b = m - c \in B \cap D = B \cap (A + (C \cap B)) = (B \cap A) + (C \cap B) \subseteq C.$$

Thus $m = b + c \in C$.

For Lemma 9, Propositions 10 and 11, and the proof of Theorem 7, we fix the following notation. Let A and B be submodules of a module M and set $M' = M/A \oplus M/B$, $A' = M/A \oplus 0 \subseteq M'$, $B' = 0 \oplus M/B \subseteq M'$, and $U = \{(m + A, n + B) \mid m + A + B = n + A + B\} \subseteq M'$. Let Δ be the set of ordered pairs (D, g) where $A \subseteq D \subseteq A + B$ and $g : M/B \to M/D$ is an epimorphism such that $\eta^D g = \eta^B$. Partially order Δ by $(D, g) \leqslant (D', g')$ if $D \subseteq D'$ and $g' = \delta g$, where δ is the natural epimorphism $\delta : M/D \to M/D'$. (The pair $(A + B, \eta^B)$ is the maximal element of Δ.)

Let Σ be the set of submodules S of U such that $A' + S = M' = S + B'$, partially ordered by set inclusion. (U is the maximal element of Σ.)

9. Lemma. *Let M, A, B, Δ, Σ be as above. The correspondence*

$$\Theta : (D, g) \mapsto \{(m + A, n + B) \mid m + D = g(n + B)\}$$

is an order isomorphism from Δ to Σ.

Proof. Let $S = \{(m + A, n + B) \mid m + D = g(n + B)\}$. We first show that $S \in \Sigma$. Since $\eta^D g = \eta^B$, $S \subseteq U$. Let $\pi_A(\pi_B)$ be the projection of M' onto A' along B' (onto B' along A'). Since $M/B = \operatorname{dom} g$, $\pi_B(S) = B'$ and $A' + S = M'$; since g is onto, $\pi_A(S) = A'$ and $S + B' = M'$. Thus $S \in \Sigma$. To see that Θ is injective, let $S = \Theta(D, g)$ and $S' = \Theta(D', g')$. If $D \nsubseteq D'$, let $d \in D \setminus D'$; then $(d + A, 0 + B) \in S \setminus \acute{S}'$. If $D = D'$ but $g \neq g'$, choose m and n with $m + D = g(n + B) \neq g'(n + B)$; then $(m + A, n + B) \in S \setminus S'$.

To show that Θ is surjective, let $S \in \Sigma$ be given. Since $A' + S = M' = S + B'$ and $A' \cap B' = 0 \subseteq S$, we may use Lemma 8 to obtain a module $T \supseteq B'$ and an epimorphism $h : M'/A' \to M'/T$ such that

$$S = \{(m + A, n + B) \mid (m + A, n + B) + T = h((m + A, n + B) + A')\}.$$

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Notice that since $T \supseteq B'$, $M' = A' + B' = A' + T$, so the map

$$\bar{\phi} : M \to M'/T \quad \text{via} \quad \bar{\phi} : m \mapsto (m+A, 0+B)+T$$

is an epimorphism. Let $D = \ker \bar{\phi}$ and let $\phi : M/D \to M'/T$ be the induced isomorphism. Define an epimorphism

$$g : M/B \to M/D \quad \text{via} \quad g : m+B \mapsto \phi^{-1}(h((0+A, m+B)+A')).$$

Now $S = \{(m+A, n+B) \mid m+D = g(n+B)\}$, for if $m+D = g(n+B)$, then

$$\begin{aligned}(m+A, n+B)+T &= (m+A, 0+B)+T = \phi(m+D)\\ &= \phi(g(n+B)) = h((0+A, n+B)+A'),\end{aligned}$$

so $(m+A, n+B) \in S$. A similar calculation gives the reverse containment. Now we show that $(D, g) \in \Delta$. If $a \in A$ then $\bar{\phi}(a) = 0$, so $A \subseteq \ker \bar{\phi} = D$. If $d \in D$ then $d+D = 0 = g(0+B)$, so $(d+A, 0+B) \in S \subseteq U$; thus $d+A+B = 0+A+B$ and $D \subseteq A+B$. Finally, let $n \in M$. Choose m with $m+D = g(n+B)$. Then since $(m+A, n+B) \in S \subseteq U$,

$$\eta^D g(n+B) = \eta^D(m+D) = m+A+B = n+A+B = \eta^B(n+B).$$

Hence $(D, g) \in \Delta$.

It remains to be shown that $(D, g) \leqslant (D', g')$ iff $S \subseteq S'$, where $S = \Theta(D, g)$ and $S' = \Theta(D', g')$. If $D \subseteq D'$, $\delta : M/D \to M/D'$ is the natural epimorphism, $g' = \delta g$, and $(m+A, n+B) \in S$, then $m+D = g(n+B)$, so also $m+D' = \delta(m+D) = \delta g(n+B) = g'(n+B)$ and $(m+A, n+B) \in S'$. Conversely, if $S \subseteq S'$ and $d \in D$, then $(d+A, 0+B) \in S \subseteq S'$, so also $d+D' = g'(0+B) = 0+D'$ and $d \in D'$. Now let $n \in M$; choose m with $g(n+B) = m+D$. Then $(m+A, n+B) \in S \subseteq S'$, so also $g'(n+B) = m+D' = \delta(m+D) = \delta g(n+B)$. Hence, $S \subseteq S'$ iff $(D, g) \leqslant (D', g')$.

Thus the existence of a minimal element of Δ is equivalent to the existence of a minimal element of Σ.

10. PROPOSITION.. *If the partially ordered set Δ has a minimal element (D, g) and the corresponding element $S = \{(m+A, n+B) \mid m+D = g(n+B)\}$ of Σ has a cocomplement in M', then S is a supplement in M'.*

Proof. By Lemma 9 and Proposition 3, it will be sufficient to show that if S is a minimal element of Σ and $Y \subseteq S$, then $S/Y \ll M'/Y$ implies $Y = S$. To this end, assume that $S/Y \ll M'/Y$. Then since

$$S/Y + (A'+Y)/Y = M'/Y = (B'+Y)/Y + S/Y,$$

both $A'+Y = M'$ and $B'+Y = M'$. Hence $Y \in \Sigma$. Since S is minimal, $S = Y$ and S is a supplement in M'.

11. PROPOSITION. *Let $M, A, B, \Delta, U, \Sigma, \Theta$ and M' be as above. Let $S \in \Sigma$ be a supplement in M' and assume that $T = \{m + A \cap B \mid (m+A, m+B) \in S\}$ has a*

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cocomplement in $M/(A\cap B)$. Let (D, g) correspond to S under Θ^{-1}. Then

$$\operatorname{cor}(M/(A\cap B))+\operatorname{cor}(M/D)=\operatorname{cor}(M/A)+\operatorname{cor}(M/B)+\operatorname{cor}((A+B)/D).$$

Proof. Because S is a supplement in M', Theorem 5 (or [8, Proposition 2.31]) implies that

$$\operatorname{cor}(M')=\operatorname{cor}(M'/S)+\operatorname{cor}(S).$$

Let $(D, g)\in\Delta$ be such that $S=\{(m+A, n+B)\mid m+D=g(n+B)\}$; then $T=\{m+A\cap B\mid m+D=g(m+B)\}$. We shall show that T is a supplement in $M/(A\cap B)$, so that

$$\operatorname{cor}(M/(A\cap B))=\operatorname{cor}((M/(A\cap B))/T)+\operatorname{cor}(T),$$

then use the isomorphisms $T\cong S$, $(M/(A\cap B))/T\cong(A+B)/D$ and $M'/S\cong M/D$ to complete the proof. Let $\phi: M/(A\cap B)\to M/A\oplus M/B$ be the monomorphism defined by $\phi(m+A\cap B)=(m+A, m+B)$. Clearly $\phi(T)\subseteq S$, and if $(m+A, n+B)\in S\subseteq U$, then there exist $a\in A$, $b\in B$ with $m+a=n+b$, so that

$$m+a+D=m+D=g(n+B)=g(n+b+B)=g(m+a+B),$$

so

$$m+a+A\cap B\in T \quad\text{and}\quad (m+A, n+B)=\phi(m+a+A\cap B)\in\phi(T).$$

Thus $\phi(T)=S$ and $T\cong S$. To show that T is a supplement in $M/(A\cap B)$, it suffices by Proposition 3 to show that if $Y\subseteq T$ and $T/Y\ll(M/(A\cap B))/Y$, then $T=Y$. Using ϕ we see that

$$S/\phi(Y)\ll(\operatorname{im}\phi)/\phi(Y)\subseteq M'/\phi(Y),$$

so $\phi(Y)=S$ since S is a supplement. Thus $Y=T$. Now define $\psi: M/(A\cap B)\to M/D$ via $\psi(m+A\cap B)=(m+D)-g(m+B)$. Then $\ker\psi=T$. The image of ψ is contained in $(A+B)/D$ since $\eta^B=\eta^D g$. Given $a\in A\subseteq D$, $b\in B$,

$$\psi(b+A\cap B)=(b+D)-g(b+B)=a+b+D,$$

so ψ is onto $(A+B)/D$ and $(M/(A\cap B))/T\cong(A+B)/D$. Finally, define $\theta: M'\to M/D$ via $\theta(m+A, n+B)=(m+D)-g(n+B)$. Then $\ker\theta=S$, and if $m\in M$, then $\theta(m+A, 0+B)=m+D$, so θ is onto M/D.

Theorem 7 now follows from Propositions 10 and 11.

12. Corollary. *Let A and B be submodules of a module M over a ring R. If either*

(a) *R is perfect, or*

(b) *$M/(A\cap B)$ is finitely generated and R is semiperfect, or*

(c) $\operatorname{cor}(M/(A\cap B))<\infty$,

and D is a submodule of M with $A\subseteq D\subseteq A+B$ that is minimal with respect to *admitting an epimorphism* $g: M/B \to M/D$ *such that* $\eta^D g = \eta^B$, *then*

$$\operatorname{cor}(M/(A \cap B)) + \operatorname{cor}(M/D) = \operatorname{cor}(M/A) + \operatorname{cor}(M/B) + \operatorname{cor}((A+B)/D).$$

Proof. The required cocomplements exist in cases (a) and (b) by [9, Theorems 1.6 and 1.7], and in case (c) by [6, Theorem 4.1]. (Case (b) also follows from case (c).)

13. EXAMPLES. (a) The formulas in Theorems 5 and 7 extend the corresponding formulas for composition length in the case of semisimple modules; for if X is semisimple, then $\operatorname{cor}(X) = c(X)$ where $c(X)$ is the composition length of X, and every submodule of X is a supplement in X.

(b) Let N_1 be a uniserial module of length 2 and let $\phi: N_1 \to N_2$ be an isomorphism. Let $M = \{(n_1, n_2) \mid \phi(n_1) + \operatorname{Soc} N_2 = n_2 + \operatorname{Soc} N_2\}$ be the pullback

$$\begin{tikzcd} M \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & N_2 \arrow[d, "\eta_2"] \\ N_1 \arrow[r, "\eta_2\phi"] & N_2/\operatorname{Soc} N_2 \end{tikzcd}$$

Let $A = \pi_1^{-1}(\operatorname{Soc} N_1)$, $B = \pi_2^{-1}(\operatorname{Soc} N_2)$. Then $M/A \cong N_2$, $M/B \cong N_1$, $A \cap B = 0$, $M/(A+B)$ and $(A+B)/A$ are simple, and $g: M/B \to M/A$ via $g((n_1, n_2) + B) = (n_1, \phi(n_1)) + A$ satisfies $\eta^A g = \eta^B$ since $\phi(n_1) - n_2 \in \operatorname{Soc} N_2$. Thus $D = A$. Theorem 7 applies to yield

$$\begin{aligned} \operatorname{cor}(M) &= \operatorname{cor}(M/(A \cap B)) = \operatorname{cor}(M/A) + \operatorname{cor}(M/B) + \operatorname{cor}((A+B)/A) - \operatorname{cor}(M/A) \\ &= 1 + 1 + 1 - 1 = 2. \end{aligned}$$

Thus $c(M/J(M)) = \operatorname{cor}(M/J(M)) = \operatorname{cor}(M) = 2$. Since also $c(\operatorname{Soc} M) = 2$ and $c(M) = 3$, M must decompose as the direct sum of a simple submodule and a uniserial module of length 2. In fact,

$$M = \{(0, n_2) \mid n_2 \in \operatorname{Soc} N_2\} \oplus \{(n_1, \phi(n_1)) \mid n_1 \in N_1\} \cong \operatorname{Soc} N_2 \oplus N_1.$$

(c) Now let N_1 and N_2 be uniserial of length 2 with $N_1 \not\cong N_2$ but $\psi: N_1/\operatorname{Soc} N_1 \cong N_2/\operatorname{Soc} N_2$. Let $M = \{(n_1, n_2) \mid \psi(n_1 + \operatorname{Soc} N_1) = n_2 + \operatorname{Soc} N_2\}$ be the pullback

$$\begin{tikzcd} M \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & N_2 \arrow[d, "\eta_2"] \\ N_1 \arrow[r, "\psi\eta_1"] & N_2/\operatorname{Soc} N_2. \end{tikzcd}$$

Let $A = \pi_1^{-1}(\operatorname{Soc} N_1)$, $B = \pi_2^{-1}(\operatorname{Soc} N_2)$. Then as before, $M/A \cong N_2$, $M/B \cong N_1$,

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$A \cap B = 0$, and $M/(A+B)$ is simple, but there is no epimorphism from $M/B \cong N_1$ to $M/A \cong N_2$. Since $c((A+B)/A) = 1$, necessarily $D = A+B$, so that

$$\begin{aligned}\operatorname{cor}(M) &= \operatorname{cor}(M/(A\cap B)) = \operatorname{cor}(M/A) + \operatorname{cor}(M/B) + \operatorname{cor}((A+B)/(A+B)) \\ &\quad - \operatorname{cor}(M/(A+B)) = 1+1+0-1 = 1,\end{aligned}$$

so M has simple top and is therefore indecomposable.

Example 13.b provides one of the examples required for the following proposition.

14. Proposition. *Let R be a semiperfect ring, and let $d(\ )$ denote Goldie dimension and* $\operatorname{cor}(\ )$ *denote corank. Then the following are equivalent, where* (b)–(g) *are interpreted to hold for all left* (*right*) *R-modules $A, B, N \subseteq M$*:

(a) *R is semisimple.*

(b) $d(M) = d(M/N) + d(N)$.

(c) $\operatorname{cor}(M) = \operatorname{cor}(M/N) + \operatorname{cor}(N)$.

(d) $d(A+B) + d(A\cap B) = d(A) + d(B)$.

(e) $\operatorname{cor}(A+B) + \operatorname{cor}(A\cap B) = \operatorname{cor}(A) + \operatorname{cor}(B)$.

(f) $d(M/(A\cap B)) + d(M/(A+B)) = d(M/A) + d(M/B)$.

(g) $\operatorname{cor}(M/(A\cap B)) + \operatorname{cor}(M/(A+B)) = \operatorname{cor}(M/A) + \operatorname{cor}(M/B)$.

Proof. If R is semisimple and X is an R-module, then $d(X) = \operatorname{cor}(X) =$ composition length of X. Hence (a) implies (b) through (g). If R is semiperfect but not semisimple, choose a primitive idempotent $e = e^2 \in R$ with $J(R)e \neq 0$. Re has a uniserial factor of length 2, call it N_1. Then the modules A, B and M of Example 13.b show that (f) and (g) fail. Letting $M = N_1$ and $N = \operatorname{Soc} N_1$ shows that (b) and (c) fail. Finally let $\theta : N_1 \cong N_2$, let $K = \{(n_1, -\theta(n_1)) \mid n_1 \in \operatorname{Soc} N_1\}$, let $M = (N_1 \oplus N_2)/K$ be the pushout

$$\begin{tikzcd} \operatorname{Soc} N_1 \arrow[r, "i_1"] \arrow[d, "\theta i_1"'] & N_1 \arrow[d, "\iota_1"] \\ N_2 \arrow[r, "\iota_2"'] & M \end{tikzcd}$$

and let $A = \iota_1(N_1) \cong N_1$ and $B = \iota_2(N_2) \cong N_2$. Then $A \cap B$ is simple, so by the second Camillo-Zelmanowitz formula with $D = A$, $f : A \to B$ via $f((n_1, 0) + K) = (0, -\theta(n_1)) + K$,

$$\begin{aligned} d(M) = d(A+B) &= d(A) + d(B) - d(A) + d(A/(A\cap B)) \\ &= 1+1-1+1 = 2. \end{aligned}$$

Also $\operatorname{cor}(M) = 2$, so both (d) and (e) fail.

15. Remarks. (a) Assume that $\operatorname{cor}(M/A)$ and $\operatorname{cor}(M/B)$ are finite. If C is any submodule with $A \subseteq C \subseteq A+B$ admitting a map $g : M/B \to M/C$ such

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that $\eta^B = \eta^C g$, then a necessary condition for the finiteness of $\operatorname{cor}(M/(A \cap B))$ is $\operatorname{cor}((A+B)/C) < \infty$, since the map $\psi : M/(A \cap B) \to (A+B)/C$ via $\psi(m + A \cap B) = (m + C) - g(m + B)$ is surjective (just as the map ψ in the proof of Proposition 11 is surjective). Conversely, if there is a supplement in Σ with corresponding pair (D, g) in Δ, then the finiteness of $\operatorname{cor}((A+B)/D)$ guarantees the finiteness of $\operatorname{cor}(M/(A \cap B))$.

(b) Even if $\operatorname{cor}(M/A) < \infty$ and $\operatorname{cor}(M/B) < \infty$, it is possible that $\operatorname{cor}(M/(A \cap B)) = \infty$. For example, as in [3], let R be a commutative local ring with radical J such that $J^3 = 0$, J^2 is simple, and J/J^2 has infinite composition length. Let $M = \{(r_1, r_2) \mid r_1 + J = r_2 + J\}$, $A = \{(r_1, 0) \mid r_1 \in J\}$ and $B = \{(0, r_2) \mid r_2 \in J\}$. Then $M/A \cong R \cong M/B$ and each has corank 1, but $A \cap B = 0$,

$$\begin{aligned} M &= \{(0, r_2) \mid r_2 \in J\} \oplus \{(r_1, r_1) \mid r_1 \in R\} \\ &\cong J \oplus R \end{aligned}$$

has infinite corank.

(c) In fact, if $a \leq b \leq m$ are integers, then there exists a commutative local finite dimensional algebra R over a field F and R-modules $A, B \subseteq M$ such that $\operatorname{cor}(M/A) = a$, $\operatorname{cor}(M/B) = b$, and $\operatorname{cor}(M/(A \cap B)) = m$. To see this, use Camillo's example in [2] and apply the functor $\operatorname{Hom}_F(-, F_F)$.

(d) Note that in Theorem 7, if the ring R is perfect and the module $(A+B)/A$ is artinian, in particular if M is artinian, then all the hypotheses of Theorem 7 involving cocomplements are satisfied and the module D is guaranteed to exist.

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