ON THE L^1-CONVERGENCE OF FOURIER TRANSFORMS

DǍNG VŨ GIANG and FERENC MÓRICZ

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Abstract

We study cosine and sine Fourier transforms defined by $F(t) := (2/\pi) \int_0^\infty f(x) \cos tx \, dx$ and $\tilde{F}(t) := (2/\pi) \int_0^\infty f(x) \sin tx \, dx$, where f is L^1-integrable over $[0, \infty)$. We also assume that F and $\tilde{F}$ are locally absolutely continuous over $[0, \infty)$. In particular, this is the case if both $f(x)$ and $xf(x)$ are L^1-integrable over $[0, \infty)$. Motivated by the inversion formulas, we consider the partial integrals $s_\nu(f, x) := \int_0^\nu F(t) \cos xt \, dt$ and $\tilde{s}_\nu(f, x) := \int_0^\nu \tilde{F}(t) \sin xt \, dt$, the modified partial integrals $u_\nu(f, x) := s_\nu(f, x) - F(\nu)(\sin \nu x)/x$ and $\tilde{u}_\nu(f, x) := \tilde{s}_\nu(f, x) + \tilde{F}(\nu)(\cos \nu x)/x$, where $\nu > 0$. We give necessary and sufficient conditions for the $L^1[0, \infty)$-convergence of $u_\nu(f)$ and $\tilde{u}_\nu(f)$ as well as for the $L^1[0, X]$-convergence of $s_\nu(f)$ and $\tilde{s}_\nu(f)$ to f as $\nu \to \infty$, where $0 < X < \infty$ is fixed. On the other hand, in certain cases we conclude that $s_\nu(f)$ and $\tilde{s}_\nu(f)$ cannot belong to $L^1[0, \infty)$. Consequently, it makes no sense to speak of their $L^1[0, \infty)$-convergence as $\nu \to \infty$.

As an intermediate tool, we use the Cesàro means of Fourier transforms. Then we prove Tauberian type results and apply Sidon type inequalities in order to obtain Tauberian conditions of Hardy-Karamata kind.

We extend these results to the complex Fourier transform defined by $G(t) := 1/(2\pi) \int_{-\infty}^\infty g(t) e^{-ixt} \, dt$, where g is L^1-integrable over $(-\infty, \infty)$.

1. Notation and basic assumptions

We consider a complex-valued function f that is Lebesgue integrable over $[0, \infty)$, in symbols : $f \in L^1[0, \infty)$. The cosine and sine Fourier transforms of f are defined

respectively by

(1.1)
$$F(t) := \frac{2}{\pi} \int_0^\infty f(x) \cos tx \, dx,$$

(1.2)
$$\tilde{F}(t) := \frac{2}{\pi} \int_0^\infty f(x) \sin tx \, dx \quad (t \geq 0).$$

As is well-known (see for example [5, pp. 13 and 16–17]), the inversion formulas

(1.3)
$$f(x) = \lim_{\nu \to \infty} \int_0^\nu F(t) \cos xt \, dt$$
$$= \lim_{\nu \to \infty} \int_0^\nu \tilde{F}(t) \sin xt \, dt \quad (x > 0)$$

hold true under various conditions. The aim of this paper is to recover f from F or $\tilde{F}$ by means of L^1-convergence.

To this effect, we consider the partial integrals of (1.3) defined by

$$s_\nu(f, x) := \int_0^\nu F(t) \cos xt \, dt,$$
$$\tilde{s}_\nu(f, x) := \int_0^\nu \tilde{F}(t) \sin xt \, dt;$$

the Cesàro means, or $(C, 1)$ means of (1.3) defined by

$$\sigma_\nu(f, x) := \int_0^\nu \left(1 - \frac{t}{\nu}\right) F(t) \cos xt \, dt,$$
$$\tilde{\sigma}_\nu(f, x) := \int_0^\nu \left(1 - \frac{t}{\nu}\right) \tilde{F}(t) \sin xt \, dt$$

(see for example [5, p. 26]), where $\nu > 0$ and $x \geq 0$ are arbitrary real numbers.

The usage of $\sigma_\nu(f)$ and $\tilde{\sigma}_\nu(f)$ is analogous to the summability of series by the method of the first arithmetic mean, which is too well-known to need much discussion here. (See for example [6, Vol. 1, Ch. 3].)

From now on, we also assume that F and $\tilde{F}$ are locally absolutely continuous (in abbreviation: LAC) over $[0, \infty)$. In particular, this is the case if both $f(x)$ and $xf(x)$ belong to $L^1[0, \infty)$. In fact, then the cosine Fourier transform F is differentiable and

(1.4)
$$F'(t) = -\tilde{F}_1(t) \quad (t \geq 0),$$

where $\tilde{F}_1$ is the sine Fourier transform of $f_1(x) := xf(x)$.

The last statement is a common place. But for the reader's convenience, we prove it briefly. The function

$$f(x)\frac{\cos(t+h)x - \cos tx}{h} = f(x) \cos tx \frac{\cos hx - 1}{h} - xf(x) \sin tx \frac{\sin hx}{hx}$$

is majorized in absolute value by the integrable function $|xf(x)|$ independent of h, and tends to the limit $-xf(x)\sin tx$ as $h \to 0$. By the Lebesgue dominated convergence theorem,

$$\lim_{h\to 0}\frac{F(t+h)-F(t)}{h} = -\tilde{F}_1(t) \qquad (t \geq 0),$$

which is (1.4). (In case $t = 0$, we consider $h > 0$ only.)

Making use of definition (1.2) and Fubini's theorem results in the following

$$\int_0^t F'(u)\,du = -\int_0^t\int_0^\infty xf(x)\sin ux\,dx du = -\int_0^\infty xf(x)\int_0^t \sin ux\,du dx$$

$$= \int_0^\infty f(x)(\cos tx - 1)\,dx = F(t) - F(0) \quad (t \geq 0).$$

This shows that F is indeed LAC over $[0, \infty)$.

The statement that $\tilde{F}$ is also LAC over $[0, \infty)$ can be proved analogously. In particular, we have

$$\tilde{F}'(t) = F_1(t) \quad (t \geq 0),$$

where F_1 is the cosine Fourier transform of $f_1(x)$.

Next, we introduce the auxiliary expressions

$$\tau_v(f, \lambda, x) := \frac{\lambda}{\lambda - 1}\int_v^{\lambda v}\left(1 - \frac{t}{\lambda v}\right)F'(t)\frac{\sin xt}{x}\,dt,$$

$$\tilde{\tau}_v(f, \lambda, x) := -\frac{\lambda}{\lambda - 1}\int_v^{\lambda v}\left(1 - \frac{t}{\lambda v}\right)\tilde{F}'(t)\frac{\cos xt}{x}\,dt,$$

where $v > 0$, $\lambda > 1$ and $x > 0$ are arbitrary real numbers. They will play a decisive role in our results on L^1-convergence.

By integration by parts, we obtain

$$s_v(f, x) = F(v)\frac{\sin vx}{x} - \int_0^v F'(t)\frac{\sin xt}{x}\,dt.$$

This motivates the introduction of the modified partial integral defined by

(1.5) $$u_v(f, x) := s_v(f, x) - F(v)\frac{\sin vx}{x} = -\int_0^v F'(t)\frac{\sin xt}{x}\,dt,$$

and analogously,

(1.6) $$\tilde{u}_v(f, x) := \tilde{s}_v(f, x) + \tilde{F}(v)\frac{\cos vx}{x} = \int_0^v \tilde{F}'(t)\frac{\cos xt}{x}\,dt.$$

2. Main results

We characterize the $L^1[0, \infty)$-convergence of $u_\nu(f)$ and $\tilde{u}_\nu(f)$ to f as $\nu \to \infty$.

THEOREM 1. *If $f \in L^1[0, \infty)$ and F is LAC over $[0, \infty)$, then*

$$(2.1) \qquad \lim_{\nu \to \infty} \int_0^\infty |u_\nu(f, x) - f(x)| \, dx = 0$$

if and only if

$$(2.2) \qquad \lim_{\lambda \downarrow 1} \limsup_{\nu \to \infty} \int_0^\infty |\tau_\nu(f, \lambda, x)| \, dx = 0.$$

THEOREM 2. *If $f \in L^1[0, \infty)$ and $\tilde{F}$ is LAC over $[0, \infty)$, then*

$$(2.3) \qquad \lim_{\nu \to \infty} \int_{1/\nu}^\infty |\tilde{u}_\nu(f, x) - f(x)| \, dx = 0$$

if and only if

$$(2.4) \qquad \lim_{\lambda \downarrow 1} \limsup_{\nu \to \infty} \int_{1/\nu}^\infty |\tilde{\tau}_\nu(f, \lambda, x)| \, dx = 0.$$

It is an easy consequence that, under the conditions of Theorems 1 and 2, we cannot expect the $L^1[0, \infty)$-convergence of $s_\nu(f)$ and $\tilde{s}_\nu(f)$ as $\nu \to \infty$, respectively.

COROLLARY 1. *If $f \in L^1[0, \infty)$, F is LAC over $[0, \infty)$, and condition (2.2) is satisfied, then $s_\nu(f) \notin L^1[0, \infty)$ for each ν such that $F(\nu) \neq 0$.*

COROLLARY 2. *If $f \in L^1[0, \infty)$, $\tilde{F}$ is LAC over $[0, \infty)$, and condition (2.4) is satisfied, then $\tilde{s}_\nu(f) \notin L^1[0, \infty)$ for each ν such that $\tilde{F}(\nu) \neq 0$.*

On the other hand, we are able to characterize the L^1-convergence of $s_\nu(f)$ and $\tilde{s}_\nu(f)$ to f over any finite interval $[0, X]$ as $\nu \to \infty$, where $0 < X < \infty$ is arbitrary.

COROLLARY 3. *If $f \in L^1[0, \infty)$, F is LAC over $[0, \infty)$, and condition (2.2) is satisfied, then for any $0 < X < \infty$*

$$(2.5) \quad \lim_{\nu \to \infty} \int_0^X |s_\nu(f, x) - f(x)| \, dx = 0 \text{ if and only if } \lim_{\nu \to \infty} F(\nu) \ln \nu = 0.$$

COROLLARY 4. *If* $f \in L^1[0, \infty)$, $\tilde{F}$ *is* LAC *over* $[0, \infty)$, *and condition* (2.4) *is satisfied, then for any* $0 < X < \infty$

$$(2.6) \quad \lim_{v \to \infty} \int_0^X |\tilde{s}_v(f, x) - f(x)| \, dx = 0 \text{ if and only if } \lim_{v \to \infty} \tilde{F}(v) \ln v = 0.$$

The following simple example shows that there exists a function $f \in L^1[0, \infty)$ such that its cosine Fourier transform F is absolutely continuous over $[0, \infty)$, $F(v) \ln v \to 0$ as $v \to \infty$, but $s_v(f) \notin L^1[0, \infty)$ for all $v > 0$. Consequently, it makes no sense to speak of the $L^1[0, \infty)$-convergence of $s_v(f)$ as $v \to \infty$.

EXAMPLE. Let $F(t) := 2/\pi(t + 1)$ for $t \geq 0$,

$$f(x) := \lim_{v \to \infty} \int_0^v F(t) \cos xt \, dt \qquad (x > 0).$$

It is plain that

$$\int_0^\infty (\text{ess sup}_{u \leq t \leq 2u} |F'(t)|) \, du < \infty.$$

By [3, Theorem 1], it follows that $f \in L^1[0, \infty)$. Keeping the inversion formula (1.3) in mind, by Corollary 5 below, we conclude (2.1). By (1.5), the difference $s_v(f) - u_v(f)$, and a fortiori $s_v(f)$, does not belong to $L^1[0, \infty)$ for any $v > 0$.

We note that an easy computation gives that

$$f(x) = \frac{2}{\pi} \int_0^\infty \frac{\sin xt}{(1 + t)^2 x} \, dt,$$

$$u_v(f, x) - f(x) = -\frac{2}{\pi} \int_v^\infty \frac{\sin xt}{(1 + t)^2 x} \, dt,$$

$$s_v(f, x) - f(x) = -\frac{2}{\pi} \int_v^\infty \frac{\sin xt}{(1 + t)^2 x} \, dt + \frac{2 \sin vx}{\pi(1 + v)x},$$

whence the above statements can be checked directly, too.

PROBLEM 1. It is an open problem how to deduce sufficient or/and necessary conditions for the $L^1[0, \infty)$-convergence of $s_v(f)$ and $\tilde{s}_v(f)$ to f as $v \to \infty$? Since the differences $u_v(f) - s_v(f)$ and $\tilde{u}_v(f) - \tilde{s}_v(f)$ do not belong to $L^1[0, \infty)$ (cf. (1.5) and (1.6)), one should avoid the usage of $u_v(f)$ and $\tilde{u}_v(f)$ during the solution of this problem.

According to Theorems 1, 2 and Corollaries 3, 4, the question of L^1-convergence of $u_v(f)$, $\tilde{u}_v(f)$, $s_v(f)$, and $\tilde{s}_v(f)$ reduces to finding sufficient conditions for the fulfillment of (2.2) and (2.4), respectively.

THEOREM 3. *If* $f \in L^1[0, \infty)$, F *is* LAC *over* $[0, \infty)$, *and for some* $p > 1$ *we have*

$$(2.7) \qquad \lim_{\lambda \downarrow 1} \limsup_{\nu \to \infty} \int_\nu^{\lambda\nu} t^{p-1} |F'(t)|^p \, dt < \infty,$$

then condition (2.2) *is satisfied.*

COROLLARY 5. *If* $f \in L^1[0, \infty)$, F *is* LAC *over* $[0, \infty)$, *and*

$$(2.8) \qquad t F'(t) = \mathcal{O}(1) \quad as \quad t \to \infty,$$

then condition (2.2) *is satisfied.*

THEOREM 4. *If* $f \in L^1[0, \infty)$, $\tilde{F}$ *is* LAC *over* $[0, \infty)$, *and for some* $p > 1$ *we have*

$$(2.9) \qquad \lim_{\lambda \downarrow 1} \limsup_{\nu \to \infty} \int_\nu^{\lambda\nu} t^{p-1} |\tilde{F}'(t)|^p \, dt < \infty,$$

then condition (2.4) *is satisfied.*

COROLLARY 6. *If* $f \in L^1[0, \infty)$, $\tilde{F}$ *is* LAC *over* $[0, \infty)$, *and*

$$(2.10) \qquad t \tilde{F}'(t) = \mathcal{O}(1) \quad as \quad t \to \infty,$$

then condition (2.4) *is satisfied.*

Also, conditions (2.7), or in particular (2.8), are sufficient for the fulfillment of both (2.1) and (2.5). Likewise, conditions (2.9), or in particular (2.10), are sufficient for the fulfillment of both (2.3) and (2.6).

A few comments are appropriate here.

(i) Theorems 3 and 4 are stronger when p is closer to 1. By Hölder's inequality, if $0 < p_1 < p_2 < \infty$, then

$$\left(\int_\nu^{\lambda\nu} t^{p_1-1} |F'(t)|^{p_1} \, dt \right)^{1/p_1} \le (\ln\lambda)^{(1/p_1)-(1/p_2)} \left(\int_\nu^{\lambda\nu} t^{p_2-1} |F'(t)|^{p_2} \, dt \right)^{1/p_2}.$$

Thus, without loss of generality we may assume that for some $1 < p \le 2$ we actually have

$$(2.7') \qquad \lim_{\lambda \downarrow 1} \limsup_{\nu \to \infty} \int_\nu^{\lambda\nu} t^{p-1} |F'(t)|^p \, dt = 0.$$

A similar observation pertains to condition (2.9). That is, we may assume that for some $1 < p \leq 2$ we actually have

(2.9′)
$$\lim_{\lambda \downarrow 1} \limsup_{\nu \to \infty} \int_\nu^{\lambda \nu} t^{p-1} |\tilde{F}'(t)|^p \, dt = 0.$$

(ii) The counterparts of conditions (2.7′) and (2.9′) are well-known in the theory of Fourier series and called Tauberian conditions of Hardy-Karamata kind (see, for example [1, 2, 4]).

(iii) In the special case $p = 1$, condition (2.7′) is of the form

(2.11)
$$\lim_{\lambda \downarrow 1} \limsup_{\nu \to \infty} \int_\nu^{\lambda \nu} |F'(t)| \, dt = 0.$$

PROBLEM 2. We guess that condition (2.11) is not enough to ensure L^1-convergence. However, we are unable to construct a function $f \in L^1[0, \infty)$ such that its cosine Fourier transform F is LAC over $[0, \infty)$, condition (2.11) is satisfied, and relation (2.1) is not satisfied.

We claim an analogous conjecture in connection with condition (2.9′).

3. Auxiliary results

We present four lemmas.

LEMMA 1. (See [5, p. 11].) *If* $f \in L^1[0, \infty)$, *then* $F(t)$ *and* $\tilde{F}(t)$ *are continuous over* $[0, \infty)$ *and*

$$\lim_{t \to \infty} F(t) = \lim_{t \to \infty} \tilde{F}(t) = 0.$$

LEMMA 2. (See [5, p. 35].) *If* $f \in L^1[0, \infty)$, *then*

$$\lim_{\nu \to \infty} \int_0^\infty |\sigma_\nu(f, x) - f(x)| \, dx = \lim_{\nu \to \infty} \int_0^\infty |\tilde{\sigma}_\nu(f, x) - f(x)| \, dx = 0.$$

The next Tauberian type results are interesting in themselves.

LEMMA 3. *If* $f \in L^1[0, \infty)$, *F and* $\tilde{F}$ *are* LAC *over* $[0, \infty)$, $v > 0$, *and* $\lambda > 1$, *then*

(3.1)
$$u_v(f, x) - \sigma_v(f, x) = \frac{\lambda}{\lambda - 1}[\sigma_{\lambda v}(f, x) - \sigma_v(f, x)]$$
$$+ \tau_v(f, \lambda, x) - \frac{1}{(\lambda - 1)v} \int_v^{\lambda v} F(t) \frac{\sin xt}{x} dt,$$

(3.2)
$$\tilde{u}_v(f, x) - \tilde{\sigma}_v(f, x) = \frac{\lambda}{\lambda - 1}[\tilde{\sigma}_{\lambda v}(f, x) - \tilde{\sigma}_v(f, x)]$$
$$+ \tilde{\tau}_v(f, \lambda, x) + \frac{1}{(\lambda - 1)v} \int_v^{\lambda v} \tilde{F}(t) \frac{\cos xt}{x} dt.$$

PROOF. By definition,

$$s_v(f, x) - \sigma_v(f, x) = \int_0^v \frac{t}{v} F(t) \cos xt \, dt,$$

$$\sigma_{\lambda v}(f, x) - \sigma_v(f, x) = \frac{\lambda - 1}{\lambda} \int_0^v \frac{t}{v} F(t) \cos xt \, dt$$
$$+ \int_v^{\lambda v} \left(1 - \frac{t}{\lambda v}\right) F(t) \cos xt \, dt.$$

Hence it follows that

(3.3)
$$s_v(f, x) - \sigma_v(f, x) = \frac{\lambda}{\lambda - 1}[\sigma_{\lambda v}(f, x) - \sigma_v(f, x)]$$
$$- \frac{\lambda}{\lambda - 1} \int_v^{\lambda v} \left(1 - \frac{t}{\lambda v}\right) F(t) \cos xt \, dt.$$

By integration by parts, we get

$$\int_v^{\lambda v} \left(1 - \frac{t}{\lambda v}\right) F(t) \cos xt \, dt = -\left(1 - \frac{1}{\lambda}\right) F(v) \frac{\sin vx}{x} + \frac{1}{\lambda v} \int_v^{\lambda v} F(t) \frac{\sin xt}{x} dt$$
$$- \int_v^{\lambda v} \left(1 - \frac{t}{\lambda v}\right) F'(t) \frac{\sin xt}{x} dt.$$

Combining this with (3.3) gives

$$s_v(f, x) - \sigma_v(f, x) = \frac{\lambda}{\lambda - 1}[\sigma_{\lambda v}(f, x) - \sigma_v(f, x)] + F(v) \frac{\sin vx}{x}$$
$$- \frac{1}{(\lambda - 1)v} \int_v^{\lambda v} F(t) \frac{\sin xt}{x} dt + \tau_v(f, \lambda, x).$$

Hence (3.1) follows by (1.5).

The proof of (3.2) runs along the same lines. We omit it.

The next two Sidon type inequalities are crucial in the proofs of Theorems 1 and 2 below.

LEMMA 4. (See [3].) *If f is locally L^p-integrable over $[0, \infty)$ for some $1 < p \leq 2$, then for any $v > 0$ we have*

$$(3.4) \qquad \int_0^\infty \left| \frac{1}{v} \int_v^{2v} f(t) \frac{\sin xt}{x} dt \right| dx \leq C_p \left(\frac{1}{v} \int_v^{2v} |f(t)|^p dt \right)^{1/p},$$

$$(3.5) \qquad \int_{1/v}^\infty \left| \frac{1}{v} \int_v^{2v} f(t) \frac{\cos xt}{x} dt \right| dx \leq C_p \left(\frac{1}{v} \int_v^{2v} |f(t)|^p dt \right)^{1/p}.$$

Here and in the sequel, by C_p we denote a constant depending only on p, whose value may be different at different occurrencies.

4. Proofs of the results in Section 2

PROOF OF THEOREM 1. By (3.1), we can estimate as follows

$$\left| \int_0^\infty |u_v(f, x) - \sigma_v(f, x)| \, dx - \int_0^\infty |\tau_v(f, \lambda, x)| \, dx \right|$$

$$(4.1) \qquad \leq \frac{\lambda}{\lambda - 1} \int_0^\infty |\sigma_{\lambda v}(f, x) - \sigma_v(f, x)| \, dx$$

$$+ \frac{1}{\lambda - 1} \int_0^\infty \left| \frac{1}{v} \int_v^{\lambda v} F(t) \frac{\sin xt}{x} dt \right| dx =: J_1 + J_2,$$

say. By Lemma 2, for any fixed $\lambda > 1$ we have

$$(4.2) \qquad \lim_{v \to \infty} J_1 = 0.$$

We may assume that $1 < \lambda \leq 2$. By (3.4),

$$J_2 \leq \frac{C_p}{\lambda - 1} \left(\frac{1}{v} \int_v^{\lambda v} |F(t)|^p dt \right)^{1/p}.$$

Making use of Lemma 1, for any fixed $\lambda > 1$ we conclude

$$(4.3) \qquad \lim_{v \to \infty} J_2 \leq \lim_{v \to \infty} \frac{C_p}{(\lambda - 1)^{1/q}} \sup_{t \geq v} |F(t)| = 0,$$

where $1/p + 1/q = 1$. Combining (4.1)-(4.3), we see that

$$(4.4) \qquad \lim_{v \to \infty} \int_0^\infty |u_v(f, x) - \sigma_v(f, x)| \, dx = 0$$

if and only if condition (2.2) is satisfied.

It remains to take into account that relations (2.1) and (4.4) are equivalent, thanks to Lemma 2. This completes the proof of the equivalence of (2.1) and (2.2).

PROOF OF THEOREM 2. In overall outline, it follows the proof of Theorem 1. This time, we have to use inequality (3.5) instead of (3.4). This is the reason that in (2.3) the integration interval is $[1/\nu, \infty)$ instead of $[0, \infty)$.

PROOF OF COROLLARY 1. It follows immediately from Theorem 1, Definition (1.5), and the fact that $(\sin \nu x)/x$ does not belong to $L^1[0, \infty)$ for any $\nu > 0$.

PROOF OF COROLLARY 2. It follows immediately from Theorem 2, Definition (1.6), and the fact that $(\cos \nu x)/x$ does not belong to $L^1[1/\nu, \infty)$ for any $\nu > 0$.

PROOF OF COROLLARY 3. By (1.5) and (2.2),

(4.5)
$$\left| \int_0^X |s_\nu(f, x) - f(x)| \, dx - |F(\nu)| \int_0^X \left| \frac{\sin \nu x}{x} \right| dx \right|$$
$$\leq \int_0^X |u_\nu(f, x) - f(x)| \, dx \to 0 \quad \text{as } \nu \to \infty,$$

thanks to Theorem 1. It is easy to check that

(4.6) $$C \ln(\nu X) \leq \int_{1/\nu}^X \left| \frac{\sin \nu x}{x} \right| dx \leq \int_0^X \left| \frac{\sin \nu x}{x} \right| dx \leq 1 + \ln(\nu X) \quad (\nu X \geq 1)$$

with a constant $C > 0$, not depending on ν and X. From (4.5) and (4.6) we conclude (2.3).

PROOF OF COROLLARY 4. By (1.6) and (2.4),

$$\left| \int_{1/\nu}^X |\tilde{s}_\nu(f, x) - f(x)| \, dx - |\tilde{F}(\nu)| \int_{1/\nu}^X \left| \frac{\cos \nu x}{x} \right| dx \right|$$
$$\leq \int_{1/\nu}^X |\tilde{u}_\nu(f, x) - f(x)| \, dx \to 0 \quad \text{as } \nu \to \infty,$$

thanks to Theorem 2. Since

(4.7) $$C \ln(\nu X) \leq \int_{1/\nu}^X \left| \frac{\cos \nu x}{x} \right| dx \leq \ln(\nu X) \qquad (\nu X \geq 1),$$

it follows that

$$\lim_{v \to \infty} \int_{1/v}^{X} |\tilde{s}_v(f, x) - f(x)| \, dx = 0 \text{ if and only if } \lim_{v \to \infty} \tilde{F}(v) \ln v = 0.$$

It remains to take into account that

$$\lim_{v \to \infty} \int_0^{1/v} |f(x)| \, dx = 0,$$

and by definition,

$$\lim_{v \to \infty} \int_0^{1/v} |\tilde{s}_v(f, x)| \, dx \leq \lim_{v \to \infty} \int_0^{1/v} \int_0^{v} |\tilde{F}(t)| \, dt \, dx$$

$$= \lim_{v \to \infty} \frac{1}{v} \int_0^{v} |\tilde{F}(t)| \, dt = 0,$$

due to Lemma 1. This completes the proof of (2.6).

PROOF OF THEOREM 3. By (3.4),

$$\int_0^{\infty} |\tau_v(f, \lambda, x)| \, dx \leq \frac{\lambda v C_p}{\lambda - 1} \left(\frac{1}{v} \int_v^{\lambda v} \left(1 - \frac{t}{\lambda v} \right)^p |F'(t)|^p \, dt \right)^{1/p}$$

$$\leq C_p \left(v^{p-1} \int_v^{\lambda v} |F'(t)|^p \, dt \right)^{1/p}$$

$$\leq C_p \left(\int_v^{\lambda v} t^{p-1} |F'(t)|^p \, dt \right)^{1/p}.$$

According to Remark (i) made in Section 2 (cf. (2.7')), (2.2) follows.

PROOF OF COROLLARY 5. By (2.8), there exists a constant C such that

$$\int_v^{\lambda v} t^{p-1} |F'(t)|^p \, dt \leq C \int_v^{\lambda v} \frac{dt}{t} = C \ln \lambda \to 0 \quad \text{as } \lambda \downarrow 1.$$

PROOF OF THEOREM 4. It is similar to the proof of Theorem 3.

PROOF OF COROLLARY 6. It is the same as the proof of Corollary 5.

5. Extension to complex Fourier transform

In the sequel, we consider a complex-valued function g that is Lebesgue integrable over $(-\infty, \infty)$, in symbols : $g \in L^1(-\infty, \infty)$. The complex Fourier transform of g is defined by

$$G(t) := \frac{1}{2\pi} \int_{-\infty}^{\infty} g(x) e^{-itx} \, dx \qquad (-\infty < x < \infty).$$

This time the inversion formula is of the form

$$g(x) = \lim_{\nu \to \infty} \int_{-\nu}^{\nu} G(t) e^{ixt} \, dt$$

(see for example [5, p. 42]). Accordingly, we introduce the following notations:

$$s_\nu(g, x) := \int_{-\nu}^{\nu} G(t) e^{ixt} \, dt,$$

$$\sigma_\nu(g, x) := \int_{-\nu}^{\nu} \left(1 - \frac{|t|}{\nu}\right) G(t) e^{ixt} \, dt,$$

$$\tau_\nu(g, \lambda, x) := \frac{\lambda}{\lambda - 1} \int_{\nu \le |t| \le \lambda \nu} \left(1 - \frac{|t|}{\lambda \nu}\right) G'(t) \frac{e^{ixt}}{ix} \, dt,$$

provided that in the last case G is LAC over $(-\infty, \infty)$, where $\nu > 0$, $\lambda > 1$, and $x \ne 0$ are real numbers. By integration by parts, we obtain

$$s_\nu(g, x) = G(\nu) \frac{e^{i\nu x}}{ix} - G(-\nu) \frac{e^{-i\nu x}}{ix} - \int_{-\nu}^{\nu} G'(t) \frac{e^{ixt}}{ix} \, dt.$$

This motivates the introduction of the modified partial integral defined by

$$(5.1) \qquad u_\nu(g, x) := s_\nu(g, x) - G(\nu) \frac{e^{i\nu x}}{ix} + G(-\nu) \frac{e^{-i\nu x}}{ix}.$$

Analogously to Theorems 1 and 2, the following is true.

THEOREM 5. *If $g \in L^1(-\infty, \infty)$ and G is LAC over $(-\infty, \infty)$, then*

$$(5.2) \qquad \lim_{\nu \to \infty} \int_{|x| \ge 1/\nu} |u_\nu(g, x) - g(x)| \, dx = 0$$

if and only if

$$(5.3) \qquad \lim_{\lambda \downarrow 1} \limsup_{\nu \to \infty} \int_{|x| \ge 1/\nu} |\tau_\nu(g, \lambda, x)| \, dx = 0.$$

The counterpart of Corollaries 1 and 2 reads as follows.

COROLLARY 7. *If* $g \in L^1(-\infty, \infty)$, G *is* LAC *over* $(-\infty, \infty)$, *and condition* (5.3) *is satisfied, then* $s_\nu(g) \notin L^1((-\infty, -1/\nu) \cup (1/\nu, \infty))$ *for each* ν *such that* $|G(\nu)| + |G(-\nu)| \neq 0$.

The counterpart of Corollaries 3 and 4 reads as follows.

COROLLARY 8. *If* $g \in L^1(-\infty, \infty)$, G *is* LAC *over* $(-\infty, \infty)$, *and condition* (5.3) *is satisfied, then for any* $0 < X < \infty$

$$(5.4) \qquad \lim_{\nu \to \infty} \int_{-X}^{X} |s_\nu(g, x) - g(x)| \, dx = 0 \quad \text{if and only if} \ \lim_{|\nu| \to \infty} G(\nu) \ln |\nu| = 0.$$

Analogously to Theorems 3 and 4, one can prove the following.

THEOREM 6. *If* $g \in L^1(-\infty, \infty)$, G *is* LAC *over* $(-\infty, \infty)$, *and for some* $p > 1$ *we have*

$$(5.5) \qquad \lim_{\lambda \downarrow 1} \limsup_{\nu \to \infty} \int_{\nu \leq |t| \leq \lambda \nu} |t|^{p-1} |G'(t)|^p \, dt < \infty,$$

then condition (5.3) *is satisfied.*

COROLLARY 9. *If* $g \in L^1(-\infty, \infty)$, G *is* LAC *over* $(-\infty, \infty)$, *and*

$$(5.6) \qquad\qquad\qquad t G'(t) = \mathcal{O}(1) \quad as \quad |t| \to \infty,$$

then condition (5.5) *is satisfied.*

Also, conditions (5.5), or in particular (5.6), are sufficient for the fulfillment of both (5.2) and (5.4).

6. Proofs of the results in Section 5

First, we formulate the auxiliary results corresponding to Lemmas 1–4.

LEMMA 5. (See [5, p. 1].) *If* $g \in L^1(-\infty, \infty)$, *then* $G(t)$ *is continuous over* $(-\infty, \infty)$ *and*

$$\lim_{|t| \to \infty} G(t) = 0.$$

LEMMA 6. (See [5, p. 35].) *If $g \in L^1(-\infty, \infty)$, then*

$$\lim_{\nu \to \infty} \int_{-\infty}^{\infty} |\sigma_\nu(g, x) - g(x)|\, dx = 0.$$

LEMMA 7. *If $g \in L^1(-\infty, \infty)$, G is LAC over $(-\infty, \infty)$, $\nu > 0$, and $\lambda > 1$, then*

$$u_\nu(g, x) - \sigma_\nu(g, x) = \frac{\lambda}{\lambda - 1}[\sigma_{\lambda\nu}(g, x) - \sigma_\nu(g, x)] + \tau_\nu(g, \lambda, x)$$

$$+ \frac{i}{(\lambda - 1)\nu}\left[\int_\nu^{\lambda\nu} G(t)\frac{e^{ixt}}{x}\, dt - \int_{-\lambda\nu}^{-\nu} G(t)\frac{e^{ixt}}{x}\, dt\right].$$

LEMMA 8. (See [3].) *If g is locally L^p-integrable over $(-\infty, \infty)$ for some $1 < p \le 2$, then for any $\nu > 0$ we have*

$$\int_{|x| \ge 1/\nu} \left|\frac{1}{\nu}\int_\nu^{\lambda\nu} g(t)\frac{e^{ixt}}{x}\, dt\right| dx \le C_p \left(\frac{1}{\nu}\int_\nu^{\lambda\nu} |g(t)|^p\, dt\right)^{1/p},$$

and an analogous inequality holds when $\int_{-\lambda\nu}^{-\nu}$ is substituted for $\int_\nu^{\lambda\nu}$ on both sides.

Our last lemma is interesting in its own right.

LEMMA 9. *If G is LAC over $(-\infty, \infty)$, then for any $0 < X < \infty$,*

(6.1) $$\lim_{\nu \to \infty} \int_{1/\nu \le |x| \le X} \left|G(\nu)\frac{e^{i\nu x}}{x} - G(-\nu)\frac{e^{-i\nu x}}{x}\right| dx = 0$$

if and only if

(6.2) $$\lim_{|\nu| \to \infty} G(\nu)\ln|\nu| = 0.$$

PROOF. *Sufficiency.* Relation (6.1) follows immediately from (4.6), (4.7), and (6.2). *Necessity.* An elementary reasoning and (4.6) yields

$$I(\nu, X) := \int_{1/\nu \le |x| \le X} \left|G(\nu)\frac{e^{i\nu x}}{x} - G(-\nu)\frac{e^{-i\nu x}}{x}\right| dx$$

$$= \int_{1/\nu}^{X}\left[\left|G(\nu)\frac{e^{i\nu x}}{x} - G(-\nu)\frac{e^{-i\nu x}}{x}\right|\right.$$

(6.3)

$$\left. + \left|G(\nu)\frac{e^{-i\nu x}}{x} - G(-\nu)\frac{e^{i\nu x}}{x}\right|\right] dx$$

$$\ge |G(\nu) + G(-\nu)| \int_{1/\nu}^{X} \left|\frac{e^{i\nu x} - e^{-i\nu x}}{x}\right| dx$$

$$\ge 2C|G(\nu) + G(-\nu)|\ln(\nu X).$$

Thus, from (6.1) it follows that

(6.4) $$\lim_{v \to \infty} |G(v) + G(-v)| \ln v = 0.$$

However, in (6.3) we can estimate in a different way, too. Namely, by (4.7),

(6.5)
$$I(v, X) \geq |G(v) - G(-v)| \int_{1/v}^{X} \left| \frac{e^{ivx} + e^{-ivx}}{x} \right| dx$$
$$\geq 2C|G(v) - G(-v)| \ln(vX),$$

whence

(6.6) $$\lim_{v \to \infty} |G(v) - G(-v)| \ln v = 0.$$

Combining (6.4) and (6.6) gives (6.2).

After these preliminaries, the proofs of Theorems 5, 6 and Corollary 9 follow the same patterns as those of the corresponding results in Section 4.

Only the proofs of Corollaries 7 and 8 need some explanation.

PROOF OF COROLLARY 7. By Theorem 5, $u_v(g) \in L^1(-\infty, \infty)$. Taking into account definition (5.1), it is enough to show that

$$G(v)\frac{e^{ivx}}{x} - G(-v)\frac{e^{-ivx}}{x} \notin L^1((-\infty, -1/v) \cup (1/v, \infty))$$

if v is such that at least one of the values $G(v)$ and $G(-v)$ differs from zero. But this is the case since, similarly to (6.3) and (6.5), we have

$$I(v, \infty) \geq 2|G(v) + G(-v)| \int_{1/v}^{\infty} \left| \frac{\cos vx}{x} \right| dx,$$
$$I(v, \infty) \geq 2|G(v) - G(-v)| \int_{1/v}^{\infty} \left| \frac{\sin vx}{x} \right| dx.$$

Consequently, $s_v(g) \notin L^1((-\infty, -1/v) \cup (1/v, \infty))$.

PROOF OF COROLLARY 8. By Theorem 5, we have (5.2). Keeping Definition (5.1) in mind, we conclude that, for any $0 < X < \infty$,

$$\lim_{v \to \infty} \int_{1/v \leq |x| \leq X} |s_v(g, x) - g(x)| \, dx = 0$$

if and only if condition (6.1) is satisfied. By Lemma 9, the latter is equivalent to condition (6.2).

It remains to take into account that

$$\lim_{\nu \to \infty} \int_{-1/\nu}^{1/\nu} |g(x)| \, dx = 0$$

and by Lemma 5,

$$\lim_{\nu \to \infty} \int_{-1/\nu}^{1/\nu} |s_\nu(g, x)| \, dx \le \lim_{\nu \to \infty} \int_{-1/\nu}^{1/\nu} \int_{-\nu}^{\nu} |G(t)| \, dt \, dx$$

$$= \lim_{\nu \to \infty} \frac{2}{\nu} \int_{-\nu}^{\nu} |G(t)| \, dt = 0.$$

This completes the proof of (5.4).

References

[1] W. O. Bray and Č. V. Stanojević, 'Tauberian L^1-convergence classes of Fourier series II', *Math. Ann.* **269** (1984), 469–486.

[2] Chang-Pao Chen, 'L^1-convergence of Fourier series', *J. Austral. Math. Soc. (Series A)* **41** (1986), 376–390.

[3] Dăng Vũ Giang and F. Móricz, 'Lebesgue integrability of Fourier transforms', *Acta. Sci. Math. (Szeged)* **60** (1995), 329–343.

[4] Č. V. Stanojević, 'Structure of Fourier and Fourier-Stieltjes coefficients of series with slowly varying convergence moduli', *Bull. Amer. Math. Soc. New Ser.* **19** (1988), 283–286.

[5] E. C. Titchmarsh, *Introduction to the theory of Fourier integrals* (Clarendon Press, Oxford, 1937).

[6] A. Zygmund, *Trigonometric series* (Cambridge University Press, Cambridge, 1959).

Institute of Mathematics
University of Veszprém
Egyetem u. 10.
8201 Veszprém
Hungary

Bolyai Institute
University of Szeged
Aradi vértanúk tere 1
6720 Szeged
Hungary