AN APPROXIMATION TO $\{X, Y\}$

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(received June 21, 1968)

1. Notation and Terminology. All homology and cohomology groups are reduced. $\mathcal{C}(\hat{p})$ is the class of finite abelian groups whose orders are prime to p. $\mathcal{C} < a$ is the class of abelian groups whose orders are products of primes less than a. If G is a finitely generated abelian group, G_p is the quotient of G by the subgroup of G made up of all elements whose orders are prime to p. If X and Y are finite C-W complexes and $\mathcal{C}$ a class of abelian groups, $\mathcal{C}$ stem $\{X, Y\}$ = $\max\{i \mid H^i(X;Z) \notin \mathcal{C}\} - \min\{j \mid H_j(Y;Z) \notin \mathcal{C}\}$.

If θ is a stable cohomology operation of type (k, Z_p, Z_p), $D(\theta)$ is the stable homology operation of type (k, Z_p, Z_p) which for any finite C-W complex X and n-dual X^* of X makes the following diagram commutative:

$$\begin{array}{ccc} H^p(X;Z_p) & \xrightarrow{\ \theta\ } & H^{p+k}(X;Z_p) \\ \big\downarrow D_n & & \big\downarrow D_n \\ H_{n-p}(X^*;Z_p) & \xrightarrow{\ D(\theta)\ } & H_{n-p-k}(X^*;Z_p) \end{array}$$

where the vertical maps are the duality isomorphisms. Note that "n-dual" is used in the sense of [1]. D is well defined, and is an isomorphism from the group of stable cohomology operations of type (k, Z_p, Z_p) to the group of stable homology operations of type (k, Z_p, Z_p) (see [2]).

δ is the Z_p cohomology Bockstein and β the Z_p homology Bockstein. Let $h : Z \to Z_p$ be the canonical projection; $h_{\#} : H_i(Y;Z) \to H_i(Y;Z_p)$ is the induced homomorphism and

$h_* : H^j(X;H_i(Y;Z))_p \to H^j(X;H_i(Y;Z_p))$ is the homomorphism induced by $h_\#$.

2. Statement of Results.

THEOREM C. X and Y are finite C-W complexes, p a prime, $p \neq 2$, $s = ¢(\hat{p})$ stem $\{X, Y\}$. Let $a(q) = 2q(p-1)-1$. Then the spectral sequence with

$$E_1^{r,o} = \sum_t H^t(X;H_{t-r}(Y;Z))_p$$

$$E_1^{r,q} = \sum_t H^t(X;Z_p) \otimes H_{t-a(q)-r}(Y;Z_p) \qquad q > 0$$

$$E_1^{r,q} = 0 \qquad q < 0$$

and $d_1^{r,q} : E_1^{r,q} \to E_1^{r+1,q+1}$ defined, for $q = 0$, by $d_1^{r,o} = (p^1 \otimes 1 + 1 \otimes D(p^1))h_*$ and, for $q > 0$, by

$$d_1^{r,q}\Big|_{H^t(X;Z_p) \otimes H_{t-a(q)-r}(Y;Z_p)} =$$

$$((q+1)p^1\delta - q\delta p^1) \otimes 1 + \delta \otimes D(p^1) + (-1)^t p^1 \otimes \beta$$

$$+ (-1)^t \otimes ((q+1)D(p^1)\beta - q\beta D(p^1))$$

converges to $\{X, S^rY\}_p$ when $r \geq s - 2p(p-1) + 2$. That is, we can compute $\{X, Y\}_p$ if $p \geq \frac{1+\sqrt{2s+5}}{2}$.

COROLLARY 1. Let s = stem $\{X, Y\}$.

$$\{X, Y\} \cong \sum_t H^t(X;H_t(Y;Z)) \quad \text{mod } ¢ < \frac{s+4}{2}.$$

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COROLLARY 2. *Let* $s = \text{stem}\ \{X, Y\}$.

Suppose

$$A : H^{2i}(X; Z_p) = H_{2i}(Y; Z_p) = 0 \qquad \textit{for all } i;$$

or

$$B : H^{2i+1}(X; Z_p) = H_{2i+1}(Y; Z_p) = 0 \quad \textit{for all } i;$$

then if $p \geq \frac{1+\sqrt{2s+5}}{2}$,

$$\{X, Y\}_p \cong \sum_t H^t(X; H_t(Y; Z))_p .$$

If either A *or* B *holds for all* p *such that*

$$\frac{1+\sqrt{2s+5}}{2} \leq p \leq \frac{s+3}{2} \qquad \textit{then}$$

$$\{X, Y\} \cong \sum_t H^t(X; H_t(Y; Z)) \qquad \text{mod } \mathcal{C} < \frac{1+\sqrt{2s+5}}{2} .$$

COROLLARY 3. $\{CP(n), CP(m)\}$ *is isomorphic* mod $\mathcal{C} < \frac{1+\sqrt{4n+1}}{2}$ *to the free abelian group on* min (m, n) *generators*.

3. *Proof of Theorem C.* Y is a finite C-W complex so, for some n, it has an n-dual Y'. $\mathcal{C}(\hat{p})$ stem $\{X, Y\} = \mathcal{C}(\hat{p})$ stem $\{X \# Y', S^n\}$. Theorem B of [4] is a special case of Theorem C and is valid when the image space is a sphere. Thus we have a spectral sequence with $\bar{E}_1^{r,o} = H^{r+n}(X \# Y'; Z)_p$; $\bar{E}_1^{r,q} = H^{r+n+a(q)}(X \# Y'; Z_p)$, for $q > o$; $\bar{E}_1^{r,q} = 0$ for $q < 0$; $\bar{d}_1^{r,o} = p'h_{\#}$; $\bar{d}_1^{r,q} = (q+1)p'\delta - q\delta p'$ and this spectral sequence converges to $\{X \# Y', S^{r+n}\}_p$ when $r \geq s - 2p(p-1) + 2$.

Now we will describe an isomorphism from this spectral sequence to the one described in the statement of Theorem C. Let $\aleph$ be the appropriate Kunneth isomorphism, λ the diagonal map in the Steenrod Algebra and θ_q the primary operation $(q+1)p'\delta - q\delta p'$. Then the following two diagrams are commutative and the vertical maps are isomorphisms.

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$$
\begin{array}{ccc}
\bar{E}_1^{r,o} & \xrightarrow{\quad \bar{d}_1^{r,o} \quad} & E_1^{r+1,1} \\
\| & & \| \\
H^{n+r}(X \# Y';Z)_p & \xrightarrow{\quad p_1 h_{\#} \quad} & H^{r+1+n+a(1)}(X \# Y';Z_p) \\
\downarrow \aleph & & \downarrow \aleph \\
\sum_t H^t(X;H^{n+r-t}(Y';Z))_p & \xrightarrow{\quad \lambda(p')h_* \quad} & \sum_t H^t(X;Z_p) \otimes H^{r+1+n+a(1)-t}(Y';Z_p) \\
\downarrow (D_n)_* & & \downarrow 1 \otimes D_n \\
\sum_t H^t(X;H_{t-r}(Y;Z))_p & \xrightarrow{\quad (1 \otimes D)\lambda(p')h_* \quad} & \sum_t H^t(X;Z_p) \otimes H_{t-(a(1)+r+1)}(Y;Z_p) \\
\| & & \| \\
E_1^{r,o} & \xrightarrow[\quad d_1^{r,o} \quad]{} & E_1^{r+1,1}
\end{array}
$$

$$
\begin{array}{ccc}
\bar{E}_1^{r,q} & \xrightarrow{\quad \bar{d}_1^{r,q} \quad} & \bar{E}_1^{r+1,1,q+1} \\
\| & & \| \\
H^{n+r+a(q)}(X \# Y';Z_p) & \xrightarrow{\quad \theta_q \quad} & H^{n+r+1+a(q+1)}(X \# Y';Z_p) \\
\downarrow \aleph & & \downarrow \aleph \\
\sum_t H^t(X;Z_p) \otimes H^{n+r+a(q)-t}(Y',Z_p) & \xrightarrow{\quad \lambda(\theta_q) \quad} & \sum_t H^t(X;Z_p) \otimes H^{n+r+1+a(q+1)-t}(Y';Z_p) \\
\downarrow 1 \otimes D_n & & \downarrow 1 \otimes D_n \\
\sum_t H^t(X;Z_p) \otimes H_{t-r-a(q)}(Y;Z_p) & \xrightarrow{\quad (1 \otimes D)\lambda(\theta_q) \quad} & \sum_t H^t(X;Z_p) \otimes H_{t-(r+1)-a(q+1)}(Y;Z_p) \\
\| & & \| \\
E_1^{r,q} & \xrightarrow[\quad d_1^{r,q} \quad]{} & E_1^{r+1,q+1}
\end{array}
$$

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So the spectral sequences $\bar{E}$ and E both converge to $\{X \# Y', S^{r+n}\}_p$ when $r \geq s - 2p(p-1) + 2$. But $\{X \# Y', S^{r+n}\} \approx \{X, S^r Y\}$ (cf. [1]) and the proof of Theorem C is complete.

In each of the three corollaries the hypothesis assures us that $E_1^{o,q} = 0$ except when $q = 0$ and that $d_1^{o,o} = 0$. Consequently $E_\infty^{o,o} = E_1^{o,o}$ and $E_\infty^{o,q} = 0$ $(q \neq 0)$. The conclusions then follow easily.

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