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ON THE LAW OF THE ITERATED LOGARITHM IN THE INFINITE VARIANCE CASE

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Abstract

The main purpose of the paper is to give necessary and sufficient conditions for the almost sure boundedness of $(S_n-\alpha_n)/B(n)$, where $S_n = X_1+X_2+\ldots+X_n$, X_i being independent and identically distributed random variables, and α_n and $B(n)$ being centering and norming constants. The conditions take the form of the convergence or divergence of a series of a geometric subsequence of the sequence $P(S_n-\alpha_n > a\,B(n))$, where a is a constant. The theorem is distinguished from previous similar results by the comparative weakness of the subsidiary conditions and the simplicity of the calculations. As an application, a law of the iterated logarithm general enough to include a result of Feller is derived.

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1. Results

Let X_i, $i \geqslant 1$, be independent random variables with distribution F, and let $S_n = X_1+X_2+\ldots+X_n$. Many papers have been devoted to the extension of the classical law of the iterated logarithm to the case when the variances of the X_i are infinite, and the definitive result was obtained by Kesten (1972), Theorem 6, who showed that there are sequences $\gamma(n) \to +\infty$, $\delta(n)$, for which

$$-\infty < \liminf_{n\to+\infty}[S_n-\delta(n)]/\gamma(n) < \limsup_{n\to+\infty}[S_n-\delta(n)]/\gamma(n) < +\infty \quad \text{a.s.}$$

if and only if F is in the domain of partial attraction of the normal distribution.

This theorem, although completely general, gives little information on the properties of $\gamma(n)$, and in Theorem 7 of the same paper, Kesten gave a restricted result with a more explicit form for the norming sequence. This result is similar to one of Feller (1968), Theorem 1, who showed (although only for symmetric X_i) that if a_n is a sequence satisfying $a_n^2 = 2nV(a_n)\log\log a_n$, $n^{-\frac{1}{2}}a_n$ is nondecreasing and

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$\limsup_{n\to+\infty} a_{2n}/a_n < +\infty$, and

$$I = \int_3^\infty \{V(x)\log\log x\}^{-1}\, dV(x) < +\infty,$$

then

$$\lim_{n\to+\infty} \binom{\sup}{\inf} (S_n - nEX_1)/a_n = \pm 1 \quad \text{a.s.},$$

where

$$V(x) = \int_{-x}^{x} u^2\, dF(u) - \left[\int_{-x}^{x} u\, dF(u)\right]^2.$$

Martikainen and Petrov (1977) give very general criteria for deciding when $\limsup S_n/a_n = 1$ a.s. in terms of the convergence of divergence of a series of the subsequence of probabilities $P(S_{k_n} - S_{k_{n-1}} > xa_{k_n})$; they do not even require that the X_i be identically distributed. However, in order to give a more easily applicable result for the case when k_n is a geometric subsequence, which is an important special case, they assume that the nondecreasing sequence a_n satisfies $\lim_{n\to+\infty} a_{c^n}/a_{c^{n-1}} = l(c)$ for each $c > 1$. This implies in particular that $\limsup_{n\to+\infty} a_{2n}/a_n < +\infty$, which is the same as Feller's restriction. Martikainen and Petrov's results under this assumption can be compared with those of Baum, Katz and Stratton (1971).

The main aim of the present paper is to give, in Theorem 1, necessary and sufficient conditions for the almost sure boundedness of $(S_n - \alpha_n)/B(n)$, where α_n and $B(n)$ are centering and norming constants. These take the form of the convergence or divergence of a series of a geometric subsequence of the sequence $P(S_n - \alpha_n > aB(n))$, where a is a constant. Results like this have been used in the proof of the law of the iterated logarithm from the beginning, of course, and Theorem 1 is only new by virtue of the comparative weakness of the subsidiary conditions imposed. The symmetry of the X_i is not required, and our approach avoids any intricate calculations with subsequences (see the working of Feller (1968) and the concluding remark of Kesten (1972)). Our restrictions on the norming sequence are very mild and in particular $B(2n)/B(n)$ is not required to be bounded above to obtain $\limsup_{n\to+\infty} (S_n - \alpha_n)/B(n) = 1$ a.s. This is important since in many cases no such upper bound applies (for example, the last example of Feller (1968)). On the other hand, the restriction of Theorem 1 to geometric subsequences seems of minor concern since even the most general result (Kesten's Theorem 6) is proved with them.

In practice, results like Theorem 1 must be applied to obtain generalized laws of the iterated logarithm which involve only conditions on the distribution of the X_i. As an example of such a result, in Theorem 2 we derive from Theorem 1 a law of the iterated logarithm general enough to include Feller's result as a special case. That this is so, we prove in a corollary, and in a further corollary, we deduce a result related to Kesten's. Following these, we discuss the limitations of our approach and mention some related work.

If $\lambda > 1$ let $\lambda_j = [\lambda^j]$, where $[x]$ denotes the integer part of x.

THEOREM 1. *Suppose $(S_n - \alpha_n)/B(n) \to_p 0$ for constants α_n, $B(n)$, $B(n)$ being positive, nondecreasing and satisfying*

$$\liminf_{n\to+\infty} B[n\mu]/B(n) \geqslant b_-(\mu) \quad \textit{for } \mu \geqslant 1,$$

where $b_-(+\infty) = +\infty$. Let $0 < a < +\infty$. If $\sum P\{S_{\lambda_j} - \alpha_{\lambda_j} \geqslant aB(\lambda_j)\}$ diverges for every $\lambda \geqslant \lambda_0$ for some $\lambda_0 > 1$, then

$$\limsup_{n\to+\infty} (S_n - \alpha_n)/B(n) \geqslant a \quad \text{a.s.}$$

If the same series converges for every $\lambda \in (1, \lambda_0)$ for some $\lambda_0 > 1$, then

$$\limsup_{n\to+\infty} (S_n - \alpha_n)/B(n) \leqslant a \quad \text{a.s.}$$

The condition $(S_n - \alpha_n)/B(n) \to_p 0$ in Theorem 1 is imposed essentially in lieu of a symmetry assumption. It is a very mild restriction in the context of Theorem 1 since the almost sure boundedness of $\limsup_{n\to+\infty} |S_n - \alpha_n|/B(n)$ (for some α_n) ensures $(S_n - \alpha_n)/B(n) \to_p 0$ (for certain α_n) by Theorem 7 of Kesten (1972). Other conditions for $(S_n - \alpha_n)/B(n) \to_p 0$ are well known (see also the end of this section).

The assumption $\lim_{\mu\to+\infty} \liminf_{n\to+\infty} B[n\mu]/B(n) = +\infty$ in Theorem 1 is a mild condition on the rate of increase of $B(n)$, and is satisfied if $B(n)$ is regularly varying with positive index or if $n^{-\delta} B(n)$ is nondecreasing for some $\delta > 0$; it is also satisfied by the sequence of Kesten's Theorem 6.

In order to replace the conditions of Theorem 1 with conditions involving only the distribution of X_i, we need only test for the convergence or divergence of $\sum P\{S_{\lambda_j} - \alpha_{\lambda_j} \geqslant aB(\lambda_j)\}$. The time-honoured procedure for doing this is to use a bound on the convergence of S_n, when suitably normed and centered, to normality. By applying a nonuniform bound for such convergence due to Nagaev (1965), we obtain:

THEOREM 2. *Suppose a_n is a nondecreasing sequence for which $\sum P(|X_1| > a_n)$ converges, and suppose $\liminf_{n\to+\infty} B(n)/a_n > 0$, where $B^2(n) = 2nV(a_n)\log\log n$. Then*

$$\lim_{n\to+\infty} \binom{\sup}{\inf} (S_n - \alpha_n)/B(n) = \pm 1 \quad \text{a.s.},$$

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where

$$\alpha_n = n \int_{-a_n}^{a_n} u\,dF(u).$$

If in addition $\limsup_{n\to+\infty} B[n\lambda]/B(n) \leqslant b_+(\lambda)$ *for* $\lambda \geqslant 1$, *where* $b_+(1+) = 1$, *then* $(S_n - \alpha_n)/B(n)$ *has as its set of almost sure limit points precisely the interval* $[-1, 1]$.

The case $a_n = n^{\frac{1}{2}}$ in Theorem 2 gives the classical law of the iterated logarithm when $EX_1^2 < +\infty$. We now give two corollaries which relate Theorem 2 to the previously mentioned results of Feller and Kesten.

COROLLARY 1. *Suppose* a_n *is a nondecreasing sequence satisfying*

$$a_n^2 \sim 2nV(a_n)\log\log a_n, \text{ and } \limsup_{n\to+\infty} a_{n+1}/a_n < +\infty.$$

Then if $I < +\infty$,

$$\lim_{n\to+\infty} \binom{\sup}{\inf} (S_n - nEX_i)/a_n = \pm 1 \quad \text{a.s.}$$

If in addition $\limsup_{n\to+\infty} a_{[n\lambda]}/a_n \leqslant a_+(\lambda)$ *for* $\lambda \geqslant 1$, *where* $a_+(1+) = 1$, *then* $(S_n - n\,EX_1)/a_n$ *has as its set of almost sure limit points precisely the interval* $[-1, 1]$.

COROLLARY 2. *Suppose* $\gamma(n)$ *is a nondecreasing sequence for which* $\Sigma P(|X_1| > \gamma(n)) < +\infty$, *and suppose*

$$\gamma^2(n)/nV(\gamma(n))\log\log\{nV(\gamma(n))\} \to \tfrac{1}{2}.$$

Suppose also that $\limsup_{n\to+\infty} \gamma(n+1)/\gamma(\mathrm{n}) < +\infty$. *Then*

$$\lim_{n\to+\infty} \binom{\sup}{\inf} (S_n - nEX_1)/\gamma(n) = \pm 1 \quad \text{a.s.}$$

If an addition $\limsup_{n\to+\infty} \gamma[n\lambda]/\gamma(n) \leqslant \gamma_+(\lambda)$ *for* $\lambda \geqslant 1$, *where* $\gamma_+(1+) = 1$, *then* $(S_n - nEX_1)/\gamma(n)$ *has as its set of almost sure limit points precisely the interval* $[-1, 1]$.

The hypotheses of Corollary 1 are considerably less restrictive than those of the corresponding result of Feller. We do not require symmetry, or that a_n be a root of $a_n^2 = 2nV(a_n)\log\log a_n$, and the restrictions on the rate of increase of a_n are weaker. Comparing Corollary 2 with the relevant part of Theorem 7 of Kesten, it will be seen that we impose the extra condition $\limsup_{n\to+\infty} \gamma(n+1)/\gamma(\mathrm{n}) < +\infty$, under which the index ρ defined by Kesten is equal to 1. It may be that Kesten's result can be derived in full by methods similar to ours. A condition like $\limsup_{n\to+\infty} \gamma[n\lambda]/\gamma(n) \leqslant \gamma_+(\lambda)$ for $\lambda \geqslant 1$, where $\gamma_+(1+) = 1$, is satisfied when γ is regularly varying.

The simplest example of a distribution in the domain of partial attraction of the normal distribution for which the variance is infinite, is $P(|X| > x) = x^{-2}, x \geqslant 1$. It is surprising, but easily verified, that neither Theorem 2, Feller's theorem, nor Kesten's original result, apply to this case. (The correct norming constants for this distribution are given at the end of Feller (1968).) Clearly the arguments of Theorem 2, and like results, lack a great deal of generality.

A distribution to which all three results apply is given by a symmetric X for which $P(|X_1| > x) = (x^2 \log x)^{-1}$ for large x. Then $V(x) \sim \log\log x$ and the conditions of Theorem 2 are fulfilled by taking $a_n^2 = 2n(\log\log n)^2 \sim B^2(n)$. We could just as well take $a_n^2 = 2n(\log\log n)^2 \log n$, in which case $B(n)/a_n \to 0$ although the other conditions of the Theorem hold. Thus $\liminf B(n)/a_n > 0$ is not necessary in Theorem 2.

It can be shown that if $(S_n - nEX_1)/a_n$ converges to normality and $\Sigma P(|X_1| > a_n)$ is finite then EX_1^2 is finite. It can also be shown that if $n^{-\frac{1}{2}} a_n \uparrow +\infty$ and $\Sigma P(|X_1| > a_n)$ is finite then $(S_n - \alpha_n)/a_n \to_p 0$, where

$$\alpha_n = n \int_{-a_n}^{a_n} u\, dF(u);$$

see Theorem 1 of Klass and Teicher (1977). We omit the proofs.

The results of the present paper are intended to apply to the case when $\limsup_{n\to+\infty} (S_n - \alpha_n)/B(n)$ and $\liminf_{n\to+\infty} (S_n - \alpha_n)/B(n)$ are finite. Recent papers of Klass (1976, 1977) and Klass and Teicher (1977) are concerned with one-sided generalizations of the law of the iterated logarithm, and use methods not closely related to ours. A paper to appear by Pruitt (1980), seen after the present paper was submitted, generalizes these results and those of Kesten.

2. Proofs

PROOF OF THEOREM 1. Suppose the divergence of the series and let n_j be the sequence $\sum_{k=0}^{j} \lambda_k$. Since $n_j - n_{j-1} = \lambda_j$, $\Sigma P\{S_{n_j} - \alpha_{\lambda_j} - S_{n_{j-1}} \geqslant aB(\lambda_j)\} = +\infty$, so by independence and the Borel–Cantelli lemma, $P\{S_{n_j} - \alpha_{\lambda_j} - S_{n_{j-1}} \geqslant aB(\lambda_j)\, io\} = 1$. If $\lambda \geqslant 2$, $n_{j-1} \leqslant \lambda_j/(\lambda-1) \leqslant \lambda_j$, so $(S_{n_{j-1}} - \alpha_{n_{j-1}})/B(\lambda_j) \to_p 0$, and from Lemma 1 of Baum, Katz and Stratton (1971) or Lemma 3.2 of Klass (1976), together with the Hewitt–Savage 0–1 law, we obtain if $\varepsilon > 0$,

$$\begin{aligned} 1 = P\{S_{n_{j-1}} - \alpha_{n_{j-1}} \geqslant -\varepsilon B(\lambda_j), S_{n_j} - \alpha_{\lambda_j} - S_{n_{j-1}} \geqslant aB(\lambda_j)\, io\} \\ \leqslant P\{S_{n_j} - \alpha_{\lambda_j} - \alpha_{n_{j-1}} \geqslant (\alpha - \varepsilon) B(\lambda_j)\, io\}. \end{aligned}$$

This means $\limsup (S_{n_j} - \alpha_{\lambda_j} - \alpha_{n_{j-1}})/B(\lambda_j) \geqslant a$ a.s. Since

$$(S_{n_j} - \alpha_{\lambda_j} - \alpha_{n_{j-1}})/B(\lambda_j) = (S_{\lambda_j} - \alpha_{\lambda_j})/B(\lambda_j) + \left(\sum_{i=\lambda_j+1}^{n_j} X_i - \alpha_{n_{j-1}} \right) \Big/ B(\lambda_j)$$

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and since $\sum_{i=\lambda_j+1}^{n_j} X_i$ has the same probabilistic structure as $S_{n_j-\lambda_j} = S_{n_{j-1}}$, the second term above has lim sup equal to

$$\limsup (S_{n_{j-1}} - \alpha_{n_{j-1}})/B(\lambda_j) \leqslant a \limsup B(n_{j-1})/B(\lambda_j) \leqslant a/b_-(\lambda - 1) \quad \text{a.s.}$$

Here we assumed lim sup $(S_n - \alpha_n)/B(n) < a$ a.s. (if lim sup $(S_n - \alpha_n)/B(n) \geqslant a$ a.s. there is nothing to prove), and used the inequality $n_{j-1} \leqslant \lambda_j/(\lambda - 1)$. We thus deduce that lim sup $(S_{\lambda_j} - \alpha_{\lambda_j})/B(\lambda_j) \geqslant a - a/b_-(\lambda - 1)$, so lim sup $(S_n - \alpha_n)/B(n) \geqslant a - a/b_-(\lambda - 1)$, and the required result follows by letting $\lambda \to +\infty$, since $b_-(+\infty) = +\infty$.

We now prove the second part of Theorem 1. Suppose lim sup $(S_n - \alpha_n)/B(n) = b > a$ a.s., where $b \geqslant 0$ because $(S_n - \alpha_n)/B(n) \to_p 0$, and possibly $b = +\infty$. We first show that lim sup $(S_{[n\lambda]} - \alpha_{[n\lambda]})/B[n\lambda] = b$ a.s. for every $\lambda \in (1, 2)$. If this were not true we would have lim sup $(S_{[n\lambda]} - \alpha_{[n\lambda]})/B[n\lambda] < b$ a.s. Then given $j > 1$ we could choose $n(j)$ so that $[\lambda(n-1)] < j+1 \leqslant [\lambda n]$. Since $[\lambda n] \leqslant \lambda n < [\lambda(n-1)] + 3$ when $\lambda \in [1, 2]$, the possible values of j would be $[\lambda(n-1)]$ or $[\lambda(n-1)] + 1$. Since $(S_n - \alpha_n)/B(n) \to_p 0$ we can easily see that $(\alpha_{n+1} - \alpha_n)/B(n+1) \to 0$, while by hypothesis $B(n)$ is nondecreasing. Suppose we had lim sup $X_n/B(n) \leqslant 0$ a.s. These would mean

$$\begin{aligned} \limsup (S_{[\lambda(n-1)]+1} - \alpha_{[\lambda(n-1)]+1})/B([\lambda(n-1)]+1) & \\ \leqslant \limsup (S_{[\lambda(n-1)]} - \alpha_{[\lambda(n-1)]})/B[\lambda(n-1)] & \\ < b \quad \text{a.s.,} & \end{aligned}$$

so that

$$\begin{aligned} \limsup_j (S_j - \alpha_j)/B(j) \leqslant \limsup_j \max \{ & S_{[\lambda(n-1)]} - \alpha_{[\lambda(n-1)]})/B[\lambda(n-1)], \\ & (S_{[\lambda(n-1)]+1} - \alpha_{[\lambda(n-1)]+1})/B[\lambda(n-1)+1)\} \\ & < b \quad \text{a.s.} \end{aligned}$$

This contradiction shows that indeed lim sup $(S_{[n\lambda]} - \alpha_{[n\lambda]})/B[n\lambda] = b$ a.s. Next note that $S_{[n\lambda]} - S_n$ has the same probabilistic structure as $S_{[n\lambda]-n}$, and this is the same as $S_{[n(\lambda-1)]}$ if n is sufficiently large. By Lemma 2 of Maller (1979),

$$\alpha_{[n\lambda]} - \alpha_n - \alpha_{[n\lambda]-n} = o(B[n\lambda]),$$

so we have

$$\begin{aligned} \limsup & \{(S_{[n\lambda]} - \alpha_{[n\lambda]}) - (S_n - \alpha_n)\}/B[n\lambda] \\ & = \limsup (S_{[n(\lambda-1)]} - \alpha_{[n(\lambda-1)]})/B[n\lambda] \\ & \leqslant \{\limsup (S_n - \alpha_n)/B[n\lambda]\} \limsup B[[n(\lambda-1)]\lambda]/B[n\lambda] \\ & \leqslant \{\limsup (S_n - \alpha_n)/B[n\lambda]\}/b_-(\lambda-1)^{-1} \end{aligned}$$

because $\liminf B[n\mu]/B(n) \geqslant b_-(\mu)$ if $\mu > 1$. It follows, using the inequality

$$b = \limsup (S_{[n\lambda]} - \alpha_{[n\lambda]})/B[n\lambda] \leqslant \limsup (S_n - \alpha_n)/B[n\lambda]$$
$$+ \limsup \{(S_{[n\lambda]} - \alpha_{[n\lambda]}) - (S_n - \alpha_n)\}/B[n\lambda]$$

that $\limsup (S_n - \alpha_n)/B[n\lambda] \geqslant b/(1 + 1/b_-(\lambda - 1)^{-1})$.

So far we have not used the convergence of the series. An application of a version of Lévy's inequality (Maller (1979), Lemma 2) shows that, if j_0 is large enough and $\varepsilon > 0$,

$$\sum_{i \geqslant j_0} P\{\max_{\lambda_{j-1} < n \leqslant \lambda_j} (S_n - \alpha_n) \geqslant (a + \varepsilon) B(\lambda_j)\} \leqslant \Sigma P\{S_{\lambda_j} - \alpha_{\lambda_j} \geqslant aB(\lambda_j)\} < +\infty$$

so by the Borel–Cantelli lemma,

$$\limsup_j \max_{\lambda_{j-1} < n \leqslant \lambda_j} (S_n - \alpha_n)/B(\lambda_j) \leqslant a \quad \text{a.s.}$$

Given $n > \lambda_2$ choose $j = j(n)$ so that $\lambda_{j-1} < n \leqslant \lambda_j$; this means $n \geqslant \lambda_{j-1} + 1 = [\lambda^{j-1}] + 1 > \lambda^{j-1}$, so $n\lambda > \lambda^j$ and $[n\lambda] \geqslant \lambda_j$. Hence, since $B(n)$ is monotone,

$$\limsup_n (S_n - \alpha_n)/B[n\lambda] \leqslant \limsup_j \max_{\lambda_{j-1} < n \leqslant \lambda_j} (S_n - \alpha_n)/B(\lambda_j) \leqslant a \quad \text{a.s.,}$$

whereas we showed that the lefthand side is $\geqslant b/(1 + 1/b_-(\lambda - 1)^{-1})$. Since $b_-(+\infty) = +\infty$, letting $\lambda \to 1+$ gives a contradiction.

There is one gap remaining: to show that

$$\limsup (S_{[n\lambda]} - \alpha_{[n\lambda]})/B[n\lambda] < b < +\infty \quad \text{a.s.}$$
$$\text{implies } \limsup X_n/B(n) \leqslant 0 \quad \text{a.s.}$$

If $1 < \lambda < 2$ it is easy to see that $[n\lambda] - [(n-1)\lambda] = 2$ or 1 according as $n\lambda$ is an integer or not (abbreviated $n\lambda = i$ or $n\lambda \neq i$). Suppose there is a $c > b$ for which $\Sigma P(X_1 \geqslant cB[n\lambda])$ diverges. Then

$$\sum_{n \geqslant 1} P\{S_{[n\lambda]} - S_{[(n-1)\lambda]} \geqslant cB[n\lambda]\}$$
$$= \sum_{n\lambda = i} P(X_{[n\lambda]} + X_{[n\lambda]-1} \geqslant cB[n\lambda]) + \sum_{n\lambda \neq i} P(X_{[n\lambda]} \geqslant cB[n\lambda])$$
$$\geqslant P(X_{[n\lambda]-1} > 0) \sum_{n\lambda = i} P(X_{[n\lambda]} \geqslant cB[n\lambda]) + \sum_{n\lambda \neq i} P(X_{[n\lambda]} \geqslant cB[n\lambda])$$
$$\geqslant P(X_1 > 0) \sum_{n \geqslant 1} P(X_1 \geqslant cB[n\lambda])$$

by independence and stationarity. Thus $\Sigma P\{S_{[n\lambda]} - S_{[(n-1)\lambda]} \geqslant cB[n\lambda]\}$ diverges unless $X_1 \leqslant 0$, in which case certainly $\limsup X_n/B(n) \leqslant 0$ a.s. In the former case, from Lemma 1 of Baum, Katz and Stratton (1971) we have, if $\delta > 0$ is such that $c > b + \delta$,

$$1 = P\{S_{[(n-1)\lambda]} - \alpha_{[(n-1)\lambda]} \geqslant -\delta B[n\lambda],\ S_{[n\lambda]} - S_{[(n-1)\lambda]} \geqslant cB[n\lambda] io\}$$
$$\leqslant P\{S_{[n\lambda]} - \alpha_{[(n-1)\lambda]} \geqslant (c-\delta)\, B[n\lambda]\, io\}$$

which is impossible since (clearly) $(\alpha_{[n\lambda]} - \alpha_{[(n-1)\lambda]})/B[n\lambda] \to 0$ and lim sup $(S_{[n\lambda]} - \alpha_{[n\lambda]})/B[n\lambda] < b$ a.s. Thus $\Sigma P(X_1 \geqslant cB[n\lambda])$ converges, and since $\liminf B(n\mu)/B(n) > 1$ for some $\mu > 1$, from Theorem 3.3. of Klass (1976) $\Sigma P(X_1 \geqslant \varepsilon B(n))$ converges for every $\varepsilon > 0$ and so $\limsup X_n/B(n) \leqslant 0$ a.s.

Proof of Theorem 2. Define the truncated rv's $X_i^j = X_i$ if $|X_i| \leqslant a_{\lambda_j}$, 0 otherwise, and let $S_n^j = X_1^j + X_2^j + \ldots + X_n^j$. A standard argument shows that $\Sigma P\{S_{\lambda_j} - \alpha_{\lambda_j} \geqslant x\}$ converges or diverges with $\Sigma P\{S_{\lambda_j}^j - \alpha_{\lambda_j} \geqslant x\}$, since $\Sigma \lambda_j P(|X_1| > a_{\lambda_j})$ converges as a simple consequence of the convergence of $\Sigma P(|X_1| > a_n)$. Here

$$\alpha_n = n \int_{-a_n}^{a_n} u\, dF(u).$$

Noting that

$$\{X_i^j - \int_{-a_{\lambda_j}}^{a_{\lambda_j}} u\, dF(u)\}/V^{\frac{1}{2}}(a_{\lambda_j})$$

has, for each j, mean 0 and variance 1, it is easy to deduce from Nagaev (1965), Theorem 2, that if

$$\Phi(x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{x} e^{-\frac{1}{2}u^2}\, du,$$

$$\sum |P\{S_{\lambda_j}^j - \alpha_{\lambda_j} \geqslant a(2\lambda_j V(a_{\lambda_j}) \log\log \lambda_j)^{\frac{1}{2}}\} - \{1 - \Phi(2 \log\log \lambda_j)^{\frac{1}{2}} a)\}|$$
$$\leqslant L \sum \lambda_j^{-\frac{1}{2}} E|X_1^j|^3 V^{-3/2}(a_{\lambda_j})(2 \log j)^{-3/2}$$

where L is an absolute constant and $a > 0$. The last series is

$$\leqslant L' \sum_{j \geqslant 1} \lambda_j^{-\frac{1}{2}} V^{-3/2}(a_{\lambda_j})(\log j)^{-3/2} \sum_{k=1}^{j} \int_{a_{j_{k-1}}}^{a_{\lambda_k}} u^3 |dP(|X_1| > u)|$$
$$\leqslant L' \sum_{k \geqslant 1} a_{\lambda_k}^3 \{P(|X_1| > a_{\lambda_{k-1}}) - P(|X_1| > a_{\lambda_k})\} V^{-3/2}$$
$$\times (a_{\lambda_k})(\log k)^{-3/2} \sum_{j \geqslant k} \lambda_j^{-\frac{1}{2}}$$
$$\leqslant L' \sum \lambda_k \{P(|X_1| > a_{\lambda_{k-1}}) - P(|X_1| > a_{\lambda_k})\}$$
$$\leqslant L' \sum \lambda_k P(|X_1| > a_{\lambda_k}) < +\infty,$$

using the fact that $\limsup a_n^2/nV(a_n) \log\log n = \limsup a_n^2/B^2(n) < +\infty$ (L' are positive constants). Thus we immediately conclude that $\sum P\{S_{\lambda_j}-\alpha_{\lambda_j} \geqslant aB(\lambda_j)\}$ converges for $a > 1$ and diverges for $a < 1$, so by Theorem 1, $\limsup (S_n-\alpha_n)/B(n) = 1$ a.s., where we note that $\liminf_{n\to+\infty} B[n\mu]/B(n) \geqslant \mu^{\frac{1}{2}}$ for $\mu \geqslant 1$ since $n^{-\frac{1}{2}} B(n)$ is nondecreasing, while $(S_n-\alpha_n)/B(n) \to_p 0$ by the remark at the end of Section 1. Replacing X_i by $-X_i$ we obtain $\liminf(S_n-\alpha_n)/B(n) = -1$ a.s., while the fact that $(S_n-\alpha_n)/B(n)$ has as its limit points the interval $[-1, 1]$ under the extra condition on $B(n)$ follows from a straightforward application of Lemma 1 of Maller (1979).

Proof of Corollary 1. If $EX_1^2 = +\infty$, it is easy to see that

$$V(x) \sim V^*(x) = \int_{-x}^{x} u^2\, dF(u),$$

and that $|dV(x)-dV^*(x)| = o(x^2\, dV(x))$, which means that V and V^* may be used interchangeably in the definition of I. Hence

$$\begin{aligned}
&\sum n\{P(|X_1| > a_{n-1})-P(|X_1| > a_n)\}\\
&\quad = \sum n \int_{a_{n-1}}^{a_n} |dP(|X_1| > u)|\\
&\quad \leqslant \sum n a_{n-1}^{-2} \int_{a_{n-1}}^{a_n} dV^*(u)\\
&\quad \leqslant c \sum n a_n^{-2} \int_{a_{n-1}}^{a_n} dV^*(u)\\
&\quad = c \sum \{V^*(a_n)\log\log a_n\}^{-1} \int_{a_{n-1}}^{a_n} dV^*(u)\\
&\quad \leqslant c \int_3^{\infty} \{V^*(u)\log\log u\}^{-1}\, dV^*(u) < +\infty,
\end{aligned}$$

since $a_n^2 \leqslant c a_{n-1}^2$. Thus $\sum P(|X_1| > a_n) < +\infty$, and this also holds in the case $EX_1^2 < +\infty$, since then, $a_n \geqslant n^{\frac{1}{2}}$ for n large. A simple consequence of $I < +\infty$ is $V(x) \leqslant V^*(x) \leqslant (\log x)^{\varepsilon}$ for $\varepsilon > 0$ if $x \geqslant x_0(\varepsilon)$, so $n^{\frac{1}{2}} \leqslant a_n \leqslant n^{\frac{1}{2}+\varepsilon}$ for $\varepsilon > 0$ and n large. Defining $B^2(n) = 2nV(a_n)\log\log n$, we then have $B^2(n) \sim 2nV(a_n)\log\log a_n \sim a_n^2$, so Corollary 1 follows directly from Theorem 1. (That α_n can be replaced by nEX_1 is easily seen.) Also, it is easy to reverse the above working to show that $I < +\infty$ if $\sum P(|X_1| > a_n) < +\infty$ (again $\limsup a_{n+1}/a_n < +\infty$ is required), so that, if $I = +\infty$, $\limsup (S_n-\alpha_n)/a_n = +\infty$ a.s. for any α_n.

Proof of Corollary 2. Putting $a_n = \gamma(n)$ and $B^2(n) = 2nV(\gamma(n))\log\log n$ the proof will follow from Theorem 1 if we can show that $B(n)/\gamma(n) \to 1$. For this we need only show that $\log\log\{nV(\gamma(n))\} \sim \log\log n$, and this will clearly follow from

$\limsup \log\log \{nV^*(\gamma(n))\}/\log\log n \leqslant 1$. A proof like that of Corollary 1, using the fact that $\sum P(|X_1| > \gamma(n)) < +\infty$, shows that $V_*(\gamma(n)) \leqslant (\log n)^c$ for some $c > 0$. The required result is an easy consequence of this.

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