24

The quark model

The first understanding of the underlying structure and "periodic table" of the hadrons came from the quark model of Gell-Mann and Zweig [Ge64, Zw65]. We now know that quarks form the underlying fermionic degrees of freedom for QCD and a field theory of the strong interactions. Solution to the dynamics of strong-coupling QCD presents formidable problems. It is often useful to make simple dynamical models that emphasize one or another aspect of QCD and that provide physical insight and guidance for further work [Bh88, Wa95]. Models build on three features of QCD:

- Baryons have the *quantum numbers* of (qqq) systems and mesons of $(\bar{q}q)$ systems where the flavor quantum numbers of the quarks q are given in Table 24.1;
- Color and the strong color forces are *confined* to the interior of the hadrons. Quarks come in three colors (R, G, B). Lattice gauge theory calculations indicate that confinement arises from the strong nonlinear couplings of the gauge fields at large distances;
- QCD is *asymptotically free*; at short distances the renormalized coupling constant goes to zero. One can do perturbation theory at short distances.

One approach to model building is that of the M.I.T. bag which provides an extreme picture of each of the three items listed above [Ch74, Ch74a, De75, Ja76]. For baryons, three massless non-interacting quarks (correct quantum numbers), with the one-gluon-exchange interaction treated as a perturbation (asymptotic freedom), are placed inside a vacuum *bubble* of radius R (confinement). It is assumed that it takes a positive amount of internal energy density to create this bubble in the vacuum. The Dirac equation is then solved within this scalar bubble, wave functions for the

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Table 24.1. Flavor quantum numbers of the lightest quarks: isospin, third component of isospin, baryon number, strangeness, charm, and electric charge, respectively.

Quark/field	T	T_3	B	S	C	$Q = T_3 + (B + S + C)/2$
u	1/2	1/2	1/3	0	0	2/3
d	1/2	−1/2	1/3	0	0	−1/3
s	0	0	1/3	−1	0	−1/3
c	0	0	1/3	0	1	2/3

nucleon are constructed, and its properties calculated. The M.I.T. bag model is discussed in detail in [Wa95].

Another approach is the non-relativistic quark model [Bh88] whose most extensive application is due to Isgur and Karl [Is77, Is80, Is85]. Here "constituent quarks" with masses of $\approx m/3$ move non-relativistically in a confining potential.[1] The confining potential is most simply taken to be that of a harmonic oscillator, which has the distinct advantage that the center-of-mass motion of the three-quark system can be treated exactly (appendix B).

Let us confine the discussion to the nuclear domain where only the lightest (u, d) quarks and their antiquarks are retained. The quark field is thus approximated by

$$\psi \doteq \begin{pmatrix} u \\ d \end{pmatrix} \qquad ; \text{ nuclear domain} \tag{24.1}$$

To do a calculation one needs the (qqq) wave functions, including all the quantum numbers. We make an independent-quark shell model of hadrons and start with the simple case of non-relativistic quarks in a potential (where the spin and spatial wave functions decouple). In this case one can write the one-quark wave function as

$$\psi = \underbrace{\psi_{nlm_l}(\mathbf{r})}_{\text{space}}\ \underbrace{\chi_{m_s}}_{\text{spin}}\ \underbrace{\eta_{m_t}}_{\text{isospin}}\ \underbrace{\rho_{\alpha}}_{\text{color}}\ ; \qquad \begin{aligned} m_s &= \pm 1/2 \\ m_t &= \pm 1/2 \\ \alpha &= (R, G, B) \end{aligned} \tag{24.2}$$

Consider the *color wave function* for the (qqq) system. The observed hadrons are color singlets. Hence the color wave function in this case is just the completely antisymmetric combination (a Slater determinant with respect to color)

$$\Psi_{\text{color}}(1,2,3) = \frac{1}{\sqrt{6}} \begin{vmatrix} \rho_R(1) & \rho_G(1) & \rho_B(1) \\ \rho_R(2) & \rho_G(2) & \rho_B(2) \\ \rho_R(3) & \rho_G(3) & \rho_B(3) \end{vmatrix} \qquad ; \text{ antisymmetric} \tag{24.3}$$

[1] The masses of these constituent quarks are presumably generated by spontaneously broken chiral symmetry in QCD.

If G_α^{color} with $\alpha = 1, \ldots, 8$ are the generators of the color transformation among the quarks, then all of the generators annihilate this wave function[2]

$$G_\alpha^{\text{color}} \Psi_{\text{color}} = 0 \qquad ; \ \alpha = 1, \ldots, 8 \tag{24.4}$$

Since the total wave function must be antisymmetric in the interchange of any two fermions, the remaining space-spin-isospin wave function must be *symmetric*.

For the ground state in this shell model, the spatial wave functions $\psi_{n00}(\mathbf{r})$ will all be the same, all $1s$, and hence the spatial part of the wave function is totally symmetric

$$\Psi_{\text{space}}(1,2,3) = \psi_{1s}(\mathbf{r}_1)\psi_{1s}(\mathbf{r}_2)\psi_{1s}(\mathbf{r}_3) \qquad ; \text{ symmetric} \tag{24.5}$$

The spin-isospin wave function must thus be *totally symmetric*. Start with isospin. One is faced with the problem of coupling three angular momenta; however, the procedure follows immediately from the discussion of 6-j symbols in quantum mechanics [Ed74]. An eigenstate of total angular momentum can be formed as follows

$$\begin{aligned} |(j_1 j_2) j_{12} j_3 j m\rangle \ &= \sum_{m_1 m_2 m_3 m_{12}} \langle j_1 m_1 j_2 m_2 | j_1 j_2 j_{12} m_{12}\rangle \\ &\quad \times \langle j_{12} m_{12} j_3 m_3 | j_{12} j_3 j m\rangle |j_1 m_1\rangle |j_2 m_2\rangle |j_3 m_3\rangle \end{aligned} \tag{24.6}$$

These states form a complete orthonormal basis for given (j_1, j_2, j_3). The states formed by coupling in the other order $|j_1(j_2 j_3) j_{23} j m\rangle$ are linear combinations of these with 6-j symbols as coefficients.

For *isospin* in the nuclear domain all the $t_i = 1/2$, thus there are a total of $2 \times 2 \times 2 = 8$ basis states. Consider first the states with total $T = 3/2$. Here the only possible intermediate value is $t_{12} = 1$. The state with $T_3 = 3/2$ is readily constructed from the above as $\alpha(1)\alpha(2)\alpha(3)$. Now apply the total lowering operator $T_- = t(1)_- + t(2)_- + t(3)_-$ and use $t_-\alpha = \beta$, $t_-\beta = 0$. The set of states with $T = 3/2$ follows immediately

$$\begin{aligned} \Phi\left[\left(\frac{1}{2}\frac{1}{2}\right)1\frac{1}{2}\frac{3}{2}\frac{3}{2}\right] &= \alpha(1)\alpha(2)\alpha(3) \\ \Phi\left[\left(\frac{1}{2}\frac{1}{2}\right)1\frac{1}{2}\frac{3}{2}\frac{1}{2}\right] &= \frac{1}{\sqrt{3}}[\beta(1)\alpha(2)\alpha(3) + \alpha(1)\beta(2)\alpha(3) + \alpha(1)\alpha(2)\beta(3)] \\ \Phi\left[\left(\frac{1}{2}\frac{1}{2}\right)1\frac{1}{2}\frac{3}{2}-\frac{1}{2}\right] &= \frac{1}{\sqrt{3}}[\beta(1)\beta(2)\alpha(3) + \beta(1)\alpha(2)\beta(3) + \alpha(1)\beta(2)\beta(3)] \\ \Phi\left[\left(\frac{1}{2}\frac{1}{2}\right)1\frac{1}{2}\frac{3}{2}-\frac{3}{2}\right] &= \beta(1)\beta(2)\beta(3) \qquad ; \text{ 4 symmetric states} \end{aligned} \tag{24.7}$$

There are four symmetric states with $T = 3/2$.

[2] Just as the fully occupied Slater determinant of spins has $S = 0$, or of j-shells has $J = 0$.

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Consider next the states with total $T = 1/2$. Here there are two possible intermediate values in the above, $t_{12} = 0, 1$. For the first of these values one finds

$$\Phi^{\rho}\left[\left(\frac{1}{2}\frac{1}{2}\right)0\frac{1}{2}\frac{1}{2}\frac{1}{2}\right] = \frac{1}{\sqrt{2}}\left[\alpha(1)\beta(2)-\alpha(2)\beta(1)\right]\alpha(3) \tag{24.8}$$

$$\Phi^{\rho}\left[\left(\frac{1}{2}\frac{1}{2}\right)0\frac{1}{2}\frac{1}{2}-\frac{1}{2}\right] = \frac{1}{\sqrt{2}}\left[\alpha(1)\beta(2)-\alpha(2)\beta(1)\right]\beta(3) \quad ;\ 2\text{ states}$$

These two states have *mixed symmetry*; they are antisymmetric in the interchange of particles $(1 \leftrightarrow 2)$.

The second value $t_{12} = 1$ yields

$$\Phi^{\lambda}\left[\left(\frac{1}{2}\frac{1}{2}\right)1\frac{1}{2}\frac{1}{2}\frac{1}{2}\right] = \frac{1}{\sqrt{6}}\left[2\alpha(1)\alpha(2)\beta(3)-\alpha(1)\beta(2)\alpha(3)-\beta(1)\alpha(2)\alpha(3)\right]$$

$$\Phi^{\lambda}\left[\left(\frac{1}{2}\frac{1}{2}\right)1\frac{1}{2}\frac{1}{2}-\frac{1}{2}\right] = \tag{24.9}$$

$$-\frac{1}{\sqrt{6}}\left[2\beta(1)\beta(2)\alpha(3)-\beta(1)\alpha(2)\beta(3)-\alpha(1)\beta(2)\beta(3)\right] \quad ;\ 2\text{ states}$$

These two states also have mixed symmetry; they are symmetric in the interchange of particles $(1 \leftrightarrow 2)$.

Now look at the *spin* wave functions. The analysis is exactly the same! We have a set of spin states Ξ identical to those above.

For the overall spin-isospin wave function, we must take a product of these wave functions and make the result totally symmetric. Recall first from quantum mechanics how one makes a wave function totally antisymmetric. Introduce the antisymmetrizing operator

$$\mathscr{A} = N\sum_{(P)}(-1)^{p}P \tag{24.10}$$

Here the sum goes over all permutations, produced by the operator P, of a complete set of coordinates for each particle. The signature of the permutation is $(-1)^p$, and $N = 1/\sqrt{N_P}$ where N_P is the total number of permutations.

Similarly, to make a wave function totally symmetric introduce the (unnormalized) *symmetrizing operator*

$$\mathscr{S} = N\sum_{(P)}P \tag{24.11}$$

Note that if a wave function is antisymmetric under the interchange of any two particles, the application of $\mathscr{S}$ will give zero. This result is established as follows. Use

$$P_{12}\mathscr{S} = \mathscr{S}P_{12} \tag{24.12}$$

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Table 24.2. Totally symmetric spin-isospin states for three non-relativistic quarks.

T	S	Number of states
3/2	3/2	16
1/2	1/2	4
		20

This follows since as P goes over all permutations, so does $P_{12}P$ or PP_{12}

$$\sum_{(P)} P_{12}P = \sum_{(P)} P = \sum_{(P)} PP_{12} \tag{24.13}$$

It follows that

$$P_{12}\mathscr{S}\psi = \mathscr{S}\psi = \mathscr{S}P_{12}\psi = -\mathscr{S}\psi = 0 \tag{24.14}$$

This is the stated result.

Note further that if the operator $\mathscr{S}$ is applied to the product of the totally symmetric 3/2 state and either of the 1/2 states with mixed symmetry, the result will vanish. The proof is as follows. Since $\mathscr{S}\Phi_{3/2} = \Phi_{3/2}\mathscr{S}$, one just needs to show that

$$\mathscr{S}[A\Phi^{\rho} + B\Phi^{\lambda}] = 0 \tag{24.15}$$

The first term gives zero since Φ^{ρ} is antisymmetric in the interchange of the first pair of particles. The second vanishes because of the nature of the sums in Eqs. (24.9) and the fact that $\mathscr{S}$ produces an identical result when applied to each term in the sum

$$\mathscr{S}(\alpha\alpha\beta) = \mathscr{S}(\alpha\beta\alpha) = \mathscr{S}(\beta\alpha\alpha) \tag{24.16}$$

It is a consequence of these two observations that *the only non-zero totally symmetric wave function will be obtained by combining the spin and isospin wave functions of the same symmetry*. Thus one must combine the two totally symmetric spin and isospin states and the other two pairs of states with the same mixed symmetry; in the latter case there is only one totally symmetric linear combination (this is proven in appendix J of [Wa95]). This leads to the set of totally symmetric spin-isospin states shown in Table 24.2 and given by

$$\begin{aligned} &\Phi_{\frac{3}{2}m_t}\Xi_{\frac{3}{2}m_s} \\ &\frac{1}{\sqrt{2}}\left(\Phi^{\lambda}_{\frac{1}{2}m_t}\Xi^{\lambda}_{\frac{1}{2}m_s} + \Phi^{\rho}_{\frac{1}{2}m_t}\Xi^{\rho}_{\frac{1}{2}m_s}\right) \end{aligned} \tag{24.17}$$

These are all the baryons one can make in this model. Since all these states are degenerate in the model as presently formulated, one has a

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supermultiplet of baryons. The present calculation predicts the spins and isospins of the members of this supermultiplet.[3]

These arguments can be extended to the situation in the M.I.T. bag model where, in contrast to massive, non-relativistic constituents, one has massless relativistic quarks. The problem is more complicated since the space–spin parts of the wave functions are now coupled; however, if the quarks occupy a common lowest positive energy $\psi_{1s_{1/2}m_j}(\mathbf{r})$ ground state, the problem is greatly simplified. Make the following replacement in the space–spin wave functions discussed above

$$\psi_{1s}(\mathbf{r})\chi_{m_s} \rightarrow \psi_{1s_{1/2}m_j}(\mathbf{r}) \tag{24.18}$$

Instead of the spin $\mathbf{S}$, now talk about the total angular momentum $\mathbf{J}$; the angular momentum and symmetry arguments are then *exactly the same as before.*

Let us investigate some consequences of the quark model. Consider the nucleon (N) ground-state expectation value of the following operator

$$O = \sum_{i=1}^{3} O_i(\mathbf{r}_i, \boldsymbol{\sigma}_i) I_i(\boldsymbol{\tau}_i) \tag{24.19}$$

Assume that the isospin factor is diagonal $I_i = (1, \tau_3)_i$. Since the wave function is totally symmetric, it follows that one need evaluate the matrix element only for the third particle.[4]

$$\langle \Psi_N | \sum_{i=1}^{3} O_i I_i | \Psi_N \rangle = 3\langle \Psi_N | O_3 I_3 | \Psi_N \rangle \tag{24.20}$$

Substitution of Eq. (24.17) then yields[5] for the state of total $m_j = 1/2$

$$\begin{aligned} 3\langle \Psi_N | O_3 I_3 | \Psi_N \rangle &= \frac{3}{2}\langle \Phi^\rho | I_3 | \Phi^\rho \rangle \langle \frac{1}{2}(3) | O_3 | \frac{1}{2}(3) \rangle \\ &+ \frac{3}{2}\langle \Phi^\lambda | I_3 | \Phi^\lambda \rangle \frac{1}{6} \left\{ 4\langle -\frac{1}{2}(3) | O_3 | -\frac{1}{2}(3) \rangle + 2\langle \frac{1}{2}(3) | O_3 | \frac{1}{2}(3) \rangle \right\} \end{aligned} \tag{24.21}$$

[3] Define $\zeta_i \equiv \chi_{m_s}\eta_{m_t}$ with $(m_s, m_t) = (\pm 1/2, \pm 1/2)$. Then in a non-relativistic quark model with spin-independent interactions one has an internal global $SU(4)$ (flavor) symmetry — this is just Wigner's supermultiplet theory [Wi37]. Here the baryons belong to the totally symmetric irreducible representation one gets from $4 \bigotimes 4 \bigotimes 4$; this is the [20] dimensional representation with spin-isospin content worked out in the text and shown in Table 24.2.

[4] Assume the operators form the identity with respect to color; the color wave function then goes right through the matrix element, and it is normalized.

[5] Use $\langle \Phi^\rho | I_3 | \Phi^\lambda \rangle = 0$ if I_3 is diagonal; this follows immediately from the form of Eqs. (24.8) and the orthogonality of the mixed-symmetry wave functions.

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Here the remaining labels on the single-particle matrix elements of O_3 are $|m_j, (\text{particle number})\rangle$. The result is

$$\langle \Psi^N_{m_t \frac{1}{2}} | \sum_{i=1}^{3} O_i I_i | \Psi^N_{m_t \frac{1}{2}} \rangle = \langle \frac{1}{2} | O | \frac{1}{2} \rangle \left[\frac{3}{2} \langle \Phi^\rho | I_3 | \Phi^\rho \rangle + \frac{1}{2} \langle \Phi^\lambda | I_3 | \Phi^\lambda \rangle \right] + \langle -\frac{1}{2} | O | -\frac{1}{2} \rangle \left[\langle \Phi^\lambda | I_3 | \Phi^\lambda \rangle \right] \tag{24.22}$$

This result is for total $m_j = 1/2$; the remaining isospin operator I_3 acts only on the third particle. For an *isoscalar* operator with $I_3 = 1$ this expression reduces to

$$\langle \Psi^N_{m_t \frac{1}{2}} | \sum_{i=1}^{3} O_i | \Psi^N_{m_t \frac{1}{2}} \rangle = 2 \langle \frac{1}{2} | O | \frac{1}{2} \rangle + \langle -\frac{1}{2} | O | -\frac{1}{2} \rangle \tag{24.23}$$

This is now just a sum of single-particle matrix elements. For an *isovector* operator with $I_3 = \tau_3$, the required isospin matrix elements for the proton with $m_t = 1/2$ follow from Eqs. (24.8) and (24.9)

$$\begin{aligned} \langle \Phi^\rho | \tau_3(3) | \Phi^\rho \rangle &= 1 \\ \langle \Phi^\lambda | \tau_3(3) | \Phi^\lambda \rangle &= \frac{1}{6}(-4 + 1 + 1) = -\frac{1}{3} \qquad ; \text{ proton } m_t = \frac{1}{2} \end{aligned} \tag{24.24}$$

For a neutron with $m_t = -1/2$, these isovector matrix elements simply change sign. It follows that

$$\begin{aligned} \langle \Psi^N_{\frac{1}{2}\frac{1}{2}} | \sum_{i=1}^{3} O_i \tau_3(i) | \Psi^N_{\frac{1}{2}\frac{1}{2}} \rangle &= \frac{4}{3} \langle \frac{1}{2} | O | \frac{1}{2} \rangle - \frac{1}{3} \langle -\frac{1}{2} | O | -\frac{1}{2} \rangle \\ \langle \Psi^N_{-\frac{1}{2}\frac{1}{2}} | \sum_{i=1}^{3} O_i \tau_3(i) | \Psi^N_{-\frac{1}{2}\frac{1}{2}} \rangle &= -\frac{4}{3} \langle \frac{1}{2} | O | \frac{1}{2} \rangle + \frac{1}{3} \langle -\frac{1}{2} | O | -\frac{1}{2} \rangle \end{aligned} \tag{24.25}$$

The notation here is $\Psi^N_{m_t, m_j}$.

In the nuclear domain with only (u, d) quarks the electric charge is given by

$$e_i = \left[\frac{1}{6} + \frac{1}{2} \tau_3(i) \right] e_p \tag{24.26}$$

Hence the expectation value of an operator proportional to the charge in the composite three-quark proton and neutron ground state

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is given by

$$
\begin{aligned}
\langle p|\sum_{i=1}^{3} O_i e_i|p\rangle &= e_p\left[\frac{1}{6}(2O_{1/2}+O_{-1/2})+\frac{1}{2}(\frac{4}{3}O_{1/2}-\frac{1}{3}O_{-1/2})\right] \\
&= e_p\langle\frac{1}{2}|O|\frac{1}{2}\rangle \\
\langle n|\sum_{i=1}^{3} O_i e_i|n\rangle &= e_p\left[\frac{1}{6}(2O_{1/2}+O_{-1/2})+\frac{1}{2}(-\frac{4}{3}O_{1/2}+\frac{1}{3}O_{-1/2})\right] \\
&= -\frac{e_p}{3}\langle\frac{1}{2}|O|\frac{1}{2}\rangle+\frac{e_p}{3}\langle-\frac{1}{2}|O|-\frac{1}{2}\rangle
\end{aligned}
\tag{24.27}
$$

Let us apply this result to compute the magnetic moment of the ground state of the nucleon in the non-relativistic quark model using for the expectation value of the single quark matrix element the Dirac magnetic moment of a point quark of mass m_q

$$\langle\frac{1}{2}|O|\frac{1}{2}\rangle=\frac{1}{2m_q} \tag{24.28}$$

Since the magnetic moment is a vector operator, its expectation value in the state $m_j=-1/2$ must simply change sign $\langle-\frac{1}{2}|O|-\frac{1}{2}\rangle=-1/2m_q$. This yields

$$\mu_p=\frac{e_p}{2m_q} \qquad\qquad \mu_n=-\frac{2\mu_p}{3} \tag{24.29}$$

The experimental results are

$$\mu_p=+2.79\,\text{n.m.} \qquad\qquad \mu_n=-1.91\,\text{n.m.} \tag{24.30}$$

The calculated ratio is quite impressive, and the absolute value can be fitted in the first relation with a constituent quark mass of $m_q=m/2.79$, which is certainly in the right ballpark.

Suppose that instead of just the static magnetic moment, one wanted the matrix element of the transverse magnetic dipole operator at all momentum transfer in the constituent quark model, how would the calculation change? From Eq. (9.16) one has

$$\hat{T}_{1M}^{\text{mag}}(\kappa)=\int d^3x\left\{j_1(\kappa x)\mathscr{Y}_{111}^{M}\cdot\hat{\mathbf{J}}_c(\mathbf{x})+[\nabla\times j_1(\kappa x)\mathscr{Y}_{111}^{M}]\cdot\hat{\boldsymbol{\mu}}(\mathbf{x})\right\} \tag{24.31}$$

There is no convection current in a $1s$ state, so the first term does not contribute. For the second term use the general relation [Ed74]

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$$\nabla \times [j_J(\kappa x)\mathscr{Y}^M_{JJ1}] = -i\kappa \left[j_{J+1}(\kappa x) \left(\frac{J}{2J+1}\right)^{1/2} \mathscr{Y}^M_{J,J+1,1} - j_{J-1}(\kappa x) \left(\frac{J+1}{2J+1}\right)^{1/2} \mathscr{Y}^M_{J,J-1,1} \right] \quad (24.32)$$

Since there is no orbital angular momentum in the initial and final states, the first term does not contribute; retention of just the second leads to

$$\begin{aligned} \hat{T}^{\text{mag}}_{1M}(\kappa) &\doteq i\kappa \left(\frac{2}{3}\right)^{1/2} \int d^3x\, j_0(\kappa x)\mathscr{Y}^M_{101} \cdot \hat{\boldsymbol{\mu}}(\mathbf{x}) \\ &= i\kappa \left(\frac{1}{6\pi}\right)^{1/2} \int d^3x\, j_0(\kappa x)\hat{\boldsymbol{\mu}}(\mathbf{x})_{1M} \end{aligned} \quad (24.33)$$

The spatial distribution of the magnetization is that of a $1s$ harmonic oscillator wave function, and from the discussion is chapter 20 we know that

$$\langle 1s|j_0(\kappa x)|1s\rangle = e^{-y} \qquad ; \; y = \left(\frac{\kappa\, b_{\text{osc}}}{2}\right)^2 \quad (24.34)$$

Hence, for the nucleon

$$e_p\langle N\tfrac{1}{2}|\hat{T}^{\text{mag}}_{10}(\kappa x)|N\tfrac{1}{2}\rangle = i\left(\frac{1}{6\pi}\right)^{1/2} \kappa\,\mu_N\, e^{-y} \quad (24.35)$$

The C-M motion for particles in a harmonic oscillator is now treated as in appendix B.

Consider the transition magnetic dipole moment between the ground state (N) and the excited state (Δ) formed from the product of the totally symmetric isospin state and totally symmetric space–spin state. Since only different m_j states are involved in the latter, we are in a position to calculate this matrix element. The wave functions are given by

$$\begin{aligned} \Psi^N_{\frac{1}{2}\frac{1}{2}} &= \frac{1}{\sqrt{2}}\left[\Phi^\lambda_{\frac{1}{2}\frac{1}{2}}\Xi^\lambda_{\frac{1}{2}\frac{1}{2}} + \Phi^\rho_{\frac{1}{2}\frac{1}{2}}\Xi^\rho_{\frac{1}{2}\frac{1}{2}}\right] \\ \Psi^\Delta_{\frac{1}{2}\frac{1}{2}} &= \Phi_{\frac{3}{2}\frac{1}{2}}\Xi_{\frac{3}{2}\frac{1}{2}} \end{aligned} \quad (24.36)$$

The subscripts on the left are (m_t, m_j) and those of the right (Tm_t, Jm_j); in detail, these wave functions are

$$\Phi_{\frac{3}{2}\frac{1}{2}} = \quad (24.37)$$
$$\frac{1}{\sqrt{3}}\left[\phi_{-\frac{1}{2}}(1)\phi_{\frac{1}{2}}(2)\phi_{\frac{1}{2}}(3) + \phi_{\frac{1}{2}}(1)\phi_{-\frac{1}{2}}(2)\phi_{\frac{1}{2}}(3) + \phi_{\frac{1}{2}}(1)\phi_{\frac{1}{2}}(2)\phi_{-\frac{1}{2}}(3)\right]$$

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A similar expression holds for $\Xi_{\frac{3}{2}\frac{1}{2}}$. The transition magnetic dipole moment is now given by

$$\mu^* = \langle\Psi^{\Delta}_{\frac{1}{2}\frac{1}{2}}|\sum_{i=1}^{3}\mu(i)\frac{1}{2}\tau_3(i)e_p|\Psi^{N}_{\frac{1}{2}\frac{1}{2}}\rangle \;=\; \frac{3}{2}e_p\langle\Psi^{\Delta}_{\frac{1}{2}\frac{1}{2}}|\mu(3)\tau_3(3)|\Psi^{N}_{\frac{1}{2}\frac{1}{2}}\rangle \quad (24.38)$$

Here it has been observed that only the isovector part of the magnetic dipole operator can contribute to the transition and the total symmetry of the states has been used. It now follows from Eq. (24.37) and the previous results that

$$\begin{aligned}
\langle\Phi_{\frac{3}{2}\frac{1}{2}}|\tau_3(3)|\Phi^{\rho}_{\frac{1}{2}\frac{1}{2}}\rangle &= 0 \qquad (24.39)\\
\langle\Phi_{\frac{3}{2}\frac{1}{2}}|\tau_3(3)|\Phi^{\lambda}_{\frac{1}{2}\frac{1}{2}}\rangle &= \frac{1}{\sqrt{18}}\left[2\langle-\frac{1}{2}|\tau_3|-\frac{1}{2}\rangle - 2\langle\frac{1}{2}|\tau_3|\frac{1}{2}\rangle\right] = -\frac{4}{\sqrt{18}}\\
\langle\Xi_{\frac{3}{2}\frac{1}{2}}|\mu(3)|\Xi^{\lambda}_{\frac{1}{2}\frac{1}{2}}\rangle &= \frac{1}{\sqrt{18}}\left[2\langle-\frac{1}{2}|\mu|-\frac{1}{2}\rangle - 2\langle\frac{1}{2}|\mu|\frac{1}{2}\rangle\right] = -\frac{4}{\sqrt{18}}\langle\frac{1}{2}|\mu|\frac{1}{2}\rangle
\end{aligned}$$

Use of Eqs. (24.27) allows the final result for μ^* to be expressed in terms of the ground-state magnetic moment of the proton

$$\mu^* \;=\; \frac{3}{2}\frac{1}{\sqrt{2}}\frac{16}{18}\mu_p \;=\; \frac{4}{3\sqrt{2}}\mu_p \qquad (24.40)$$

This is the matrix element for $(m_j, m_t) = (\frac{1}{2}\frac{1}{2}) \to (\frac{1}{2}\frac{1}{2})$; other components follow from the Wigner–Eckart theorem. This result agrees to about 30% with experimental observations of the transition magnetic dipole matrix element obtained from electroproduction of the first nucleon resonance [Ka83].

Since only the spin is flipped in the constituent quark model, and the radial 1s wave functions are unchanged in the $N \to \Delta$ transition, one can simply read off from Eq. (24.35) that the transition matrix element of the transverse magnetic dipole operator is given by

$$\begin{aligned}
e_p\langle\Delta^+\,\frac{1}{2}|\hat{T}^{\text{mag}}_{10}(\kappa x)|p\frac{1}{2}\rangle &= i\left(\frac{1}{6\pi}\right)^{1/2}\kappa\left(\frac{4}{3\sqrt{2}}\mu_p\right)e^{-y}\\
&= i\frac{2}{3\sqrt{3\pi}}\,\kappa\,\mu_p\,e^{-y} \qquad (24.41)
\end{aligned}$$

Particularly simple is then the *ratio* of the transition to the static matrix elements of the transverse magnetic dipole operator

$$\frac{\langle\Delta^+\,\frac{1}{2}|\hat{T}^{\text{mag}}_{10}(\kappa x)|p\frac{1}{2}\rangle}{\langle p\frac{1}{2}|\hat{T}^{\text{mag}}_{10}(\kappa x)|p\frac{1}{2}\rangle} = \frac{2\sqrt{2}}{3} \qquad (24.42)$$

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Note that this ratio is a numerical constant independent of κ in the constituent quark model. This result is also independent of the detailed form of the single-quark wave function since the form factor cancels in the ratio.[6]

The Coulomb monopole moment for the proton simply reflects the 1s radial wave function of each quark, and, as in chapter 20, the elastic scattering form factor for the proton is given by

$$\langle p\frac{1}{2}|M_{00}(\kappa x)|p\frac{1}{2}\rangle = \frac{1}{\sqrt{4\pi}}\,e^{-y} \tag{24.43}$$

The transition magnetic dipole form factor is thus proportional to the elastic form factor of the proton in this model.

$$\frac{e_p\langle\Delta^+\,\frac{1}{2}|\hat{T}^{\text{mag}}_{10}(\kappa x)|p\frac{1}{2}\rangle}{\langle p\frac{1}{2}|M_{00}(\kappa x)|p\frac{1}{2}\rangle} \quad = \quad i\frac{4}{3\sqrt{3}}\kappa\mu_p \tag{24.44}$$

This result is again independent of the form of the single-quark radial wave functions since the form factor cancels in this ratio. Since there is no orbital angular momentum in either the ground or excited state, the transition matrix elements of the Coulomb and transverse electric quadrupole operators vanish here.

To the extent that the cross section is dominated by the transverse interaction and $q_\mu^2 \approx \mathbf{q}^2 \equiv \kappa^2$, the constancy of the ratio in Eq. (24.42) is indeed manifest by the experimental data shown in Fig. 12.9. Of course, the experimental elastic form factor itself falls off as a dipole [Eq. (22.5)] and not the gaussian of the simple-harmonic oscillator model, and it is certainly inconsistent to use a non-relativistic model for $\kappa \geq m_q$.

The $N \to \Delta$ transition is particularly simple in the constituent quark model. Higher excitations of the nucleon can be constructed by promoting one of the quarks to a higher oscillator state and then constructing totally symmetric space-spin-isospin wave functions for the nucleon. Similarly, the hyperfine splitting coming from (asymptotically-free) one-gluon exchange can be readily included in the model. We refer the reader to the literature for these developments [Is77, Bh88].

[6] The present treatment of the C-M motion, however, only holds in the simple harmonic oscillator model (appendix B).