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SOME GOOD SEQUENCES OF INTERPOLATORY POLYNOMIALS

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1. Introduction. In 1963, P. L. Butzer [**4**, p. 180] asked whether it was possible to prove Jackson's theorem by means of an operator which is "almost" interpolatory in the sense that it is based on the values of the approximee at a finite number of nodes. In answer to this question, G. Freud introduced [**4**] a sequence of operators which led to an independent proof of Jackson's theorem. Strictly speaking these operators are not interpolatory but they are "almost" interpolatory in the above sense.

The construction in [**4**] was based on the zeros of $T_n(x)$ ($= \cos n\theta$ when $x = \cos\theta$) and thereby uniform convergence was proved for $[-\frac{1}{2}, \frac{1}{2}]$. This result gave rise to an extensive literature. The same idea was applied by M. Sallay [**9**] to construct a Jackson type process where the nodes were taken to be the zeros of Legendre polynomials (or of orthogonal polynomials which are, in a well-defined sense, very similar to Legendre polynomials). Later Saxena [**10**] used the zeros of $(1 - x^2)U_n(x)$ (where $U_n(x)$ denotes the Tchebicheff polynomial of the second kind) as nodes and modified the construction of Freud to obtain the Jackson estimate on the whole interval $[-1, 1]$. P. Vertesi [**14**] showed that Saxena's result could also be obtained if the zeros of $U_n(x)$ are replaced by the zeros of $T_n(x)$ as nodes.

In Freud-Vertesi [**7**] it was proved that the process of Vertesi [**14**] leads also to an independent proof of A. F. Timan's approximation theorem, i.e., the error of approximation at any point x in $[-1, 1]$ does not exceed

$$c[\omega(1 - x^2)^{\frac{1}{2}}/n) + \omega(1/n^2)]$$

where $\omega(\delta)$ is the modulus of continuity of the approximee function. We shall call such an approximation process a Timan-type process. More recently Saxena [**12**] employed the zeros of $(1 - x^2)U_n(x)$ as nodes to obtain similar results with Timan-type estimates, while Mathur [**8**] obtained Jackson-type estimates on the zeros of $P_n^{(-\frac{1}{2},\frac{1}{2})}$.

An alternative approach was initiated in Vertesi-Kis [**15**]. Their approximation process is based on the $n + 1$ zeros of $(1 - x)P_n^{(-\frac{1}{2},\frac{1}{2})}(x)$ and their approximating polynomials are of degree $4n - 4$, slightly less than those in the earlier works mentioned above. But the essential improvement which they bring about is that their process is of Timan-type and is also interpolatory in the usual sense.

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In the present paper we construct interpolatory polynomials with Timan-type estimates, taking as nodes the zeros of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, with arbitrary $\alpha, \beta > -1$. Moreover we can considerably reduce the degree of the interpolatory polynomial. In fact for every $c > 0$, we can match our process so that at the nth step, it is based on $n + 2$ nodes and the interpolatory polynomial is of degree less than $n(1 + c)$. This property links our result to another classical problem of S. N. Bernstein [**1**], who constructed a sequence $\{A_n(f; x)\}$ of polynomial operators (depending on c) with the following properties:

(i) $A_n(f; x)$ is a linear operator mapping $C[-1, 1]$ into polynomials of degree less than $n(1 + c)$;

(ii) $A_n(f; x_{kn}) = f(x_{kn})$, $k = 1, \ldots, n$; and

(iii) for every $f \in C[-1, 1]$, $A_n(f; x)$ tends uniformly to $f(x)$ in $[-1, 1]$.

In Berstein's construction, the x_{kn}'s were taken to be the zeros of $T_n(x)$. Later P. Erdös [**3**] gave necessary and sufficient conditions which a triangular matrix X of nodes $\{x_{kn}; 1 \leqq k \leqq n, n = 1, 2, \ldots\}$ must satisfy in order that there exists for every fixed $c > 0$ a sequence $\{A_n(f; x)\}$ satisfying (i), (ii) and (iii). Suppose all the nodes lie in $(-1, 1)$ and $x_{kn} = \cos \theta_{kn}$. If $N_n(a_n, b_n)$ denotes the number of θ_{kn}'s in (a_n, b_n) and if $n(b_n - a_n) \to 0$, $0 \leqq a_n < b_n \leqq \pi$, then the Erdös conditions mentioned above are:

$$\overline{\lim_{n\to\infty}} \frac{N_n(a_n, b_n)}{n(b_n - a_n)} \leqq \frac{1}{\pi}; \tag{E_1}$$

$$\underline{\lim_{n\to\infty}}\, n(\theta_{in} - \theta_{i+1,n}) > 0, \quad i \text{ arbitrary}. \tag{E_2}$$

Later Freud [**5**] proved that for every triangular matrix X satisfying (E_1) and (E_2) there exists also, for every $c > 0$, a "good" approximating sequence $\{A_n(f; x)\}$ satisfying (i) and (ii) and the requirement that

(iv) for every $f \in C[-1, 1]$, we have for $-1 \leqq x \leqq 1$

$$|A_n(f; x) - f(x)| \leqq K_1(c) E_{n-1}(f), \tag{1.1}$$

where $E_{n-1}(f)$ is the error of best approximation to f by polynomials of degree $\leqq n - 1$ in the uniform norm. Clearly (iv) implies (iii) so that the conditions (E_1) and (E_2) are also necessary and sufficient for the existence of $\{A_n(f; x)\}$ with properties (i), (ii) and (iv) for every $c > 0$. Since the zeros of the Jacobi polynomials $P_n^{(\alpha,\beta)}$ can be shown to satisfy (E_1) and (E_2) for arbitrary fixed $\alpha, \beta > -1$, we may take these zeros as nodes of interpolation to form the sequence $\{A_n(f; x)\}$.

In the present paper we show (Theorem 2, §2) that if the nth row of the triangular matrix X consists of the zeros of $(1 - x^2) \cdot P_n^{(\alpha,\beta)}(x)$, $n = 1, 2, \ldots$ then for every $c > 0$, we can form a sequence $\{A_n^{(\alpha,\beta)}(f; x)\}$ satisfying (i), (ii) and the property that

(v) for every $f \in C[-1, 1]$, a Timan-type estimate holds. We do not know whether for every triangular matrix X satisfying (E_1) and (E_2) and for every $c > 0$, there exists a sequence $\{A_n(f; x)\}$ satisfying (i), (ii) and (v). Our method of constructing the sequence $\{A_n^{(\alpha,\beta)}(f;\, x)\}$ has close similarity to that of Freud [**4**] as modified by Saxena [**10**], but there are some essential differences as well. While our starting point is the Lagrange interpolation, that of Freud and Saxena is the Hermite interpolation. Our sequence is in fact interpolatory while the sequences of Freud and of Saxena [**11**; **12**] are not so. It remains an open problem to find if our process (respectively some other process) would give a linear approximating process satisfying (i), (ii) and the Teljakowski-Gopengauz† estimate.

2. Preliminaries and main results. Let $\{x_{kn}\}_1^n$ denote the zeros of $P_n^{(\alpha,\beta)}(x)$, $\alpha,\, \beta > -1$ and let $l_{kn}(x)$ denote the fundamental polynomials of Lagrange interpolation on these nodes. Then

$$l_{kn}(x) = \frac{P_n^{(\alpha,\beta)}(x)}{(x - x_{kn})P_n'^{(\alpha,\beta)}(x_{kn})}\,. \tag{2.1}$$

Let r be a given positive integer $\geqq 2$ and let $m = [n\rho]$ for some ρ, $0 < \rho < 1/2r$. Set

$$\Phi_m(x, y) = \frac{1}{m}\left[1 + 2\sum_{\nu=1}^{m} T_\nu(x)T_\nu(y)\right] \tag{2.2}$$

where $T_\nu(x) = \cos \nu\theta (x = \cos \theta)$ is the Tchebicheff polynomial of degree ν. Then by Lagrange interpolation, we have

$$\Phi_m^{2r}(x, y) = \sum_{k=1}^{n} \Phi_m^{2r}(x_{kn}, y)l_{kn}(x). \tag{2.3}$$

Hence

$$\Phi_m^{2r}(x, x) = \sum_{k=1}^{n} \phi_{kn}(x) \tag{2.4}$$

where we set

$$\phi_{kn}(x) = \Phi_m^{2r}(x_{kn}, x)l_{kn}(x). \tag{2.5}$$

If $f \in C[-1, 1]$, we define the linear operator $J_n^{(\alpha,\beta)}$:

$$J_n^{(\alpha,\beta)}(f;x) \equiv \lambda(x) + \sum_{k=1}^{n} \{f(x_{kn}) - \lambda(x)\}\,\phi_{kn}(x), \tag{2.6}$$

where

$$\lambda(x) = \frac{1+x}{2}f(1) + \frac{1-x}{2}f(-1). \tag{2.7}$$

We shall prove

THEOREM 1. *Let* $\{x_{kn}\}_1^n$ *denote the zeros of* $P_n^{(\alpha,\beta)}(x)$, α, $\beta > -1$ (*arbitrary but fixed*). *If* $f \in C[-1, 1]$ *with modulus of continuity* $\omega(\delta)$ *and if* $2r >$ max

†S. A. Teljakowski, Mat. Sb. *70* (1966), 252–265.
G. V. Gopengauz, Mat. Zametki *1* (1967), 163–172.

$(4, \alpha + 5/2, \beta + 5/2)$, *then*

$$|f(x) - J_n^{(\alpha,\beta)}(f; x)| \leqq c\left[\omega\left(\frac{(1-x^2)^{\frac{1}{2}}}{n}\right) + \omega\left(\frac{1}{n^2}\right)\right] \tag{2.8}$$

where c is a constant independent of n and x.

Observe that by an appropriate choice of ρ the degree of the polynomial $J_n^{(\alpha,\beta)}(f; x)$ is $n + 2rm - 1 < n(1 + 2r\rho) \leqq n(1 + c)$ for any fixed $c > 0$. $J_n^{(\alpha,\beta)}(f; x)$ does not have the interpolatory property. In order to make up for this, we set

$$A_n^{(\alpha,\beta)}(f; x) = \lambda(x) + \sum_{k=1}^{n} \{f(x_{kn}) - \lambda(x)\} \frac{\phi_{kn}(x)}{\Phi_m^{2r}(x_{kn}, x_{kn})}. \tag{2.9}$$

Then $A_n^{(\alpha,\beta)}(f; x_{kn}) = f(x_{kn})$, $(k = 1, 2, \ldots, n)$. In §5, we shall prove

THEOREM 2. *If* $f \in C[-1, 1]$ and $\omega(\delta)$ *denotes its modulus of continuity, then*

$$|f(x) - A_n^{(\alpha,\beta)}(f; x)| \leqq c\left[\omega\left(\frac{(1-x^2)^{\frac{1}{2}}}{n}\right) + \omega\left(\frac{1}{n^2}\right)\right] \tag{2.10}$$

where c is a constant independent on n and x.

It may be remarked that the operators $J_n^{(\alpha,\beta)}$ and $A_n^{(\alpha,\beta)}$ depend also on the parameters r and ρ but for the sake of simplicity of writing we do not use them in the notation.

In the sequel we shall need some results on the zeros of Jacobi polynomials, which we now formulate.

LEMMA 1 [**3**, Theorem 7.32.2]. *For α, β arbitrary and real and c a fixed positive constant, as $n \to \infty$ we have*

$$\begin{aligned} P_n^{(\alpha,\beta)}(\cos\theta) &= (\sin\theta)^{-\alpha-\frac{1}{2}}O(n^{-\frac{1}{2}}), \quad cn^{-1} \leqq \theta \leqq \pi/2 \\ &= O(n^{\alpha}), \quad 0 \leqq \theta \leqq cn^{-1}. \end{aligned} \tag{2.11}$$

Remark. For $\pi/2 < \theta \leqq \pi(1 - cn^{-1})$ and for $\pi(1 - cn^{-1}) \leqq \theta \leqq \pi$, replace α by β in the above.

Since

$$P_n'^{(\alpha,\beta)}(x) = \tfrac{1}{2}(n + \alpha + \beta + 1)P_{n-1}^{(\alpha+1,\beta+1)}(x)$$

it follows [**13**, Formula (8.9.2)] that for $\alpha, \beta > -1$,

$$\begin{aligned} P_n'^{(\alpha,\beta)}(\cos\theta_{kn}) &\sim n^{1/2}(\sin\theta_{kn})^{-\alpha-3/2}, \quad 0 \leqq \theta_{kn} \leqq \pi/2 \\ &\sim n^{1/2}(\sin\theta_{kn})^{-\beta-3/2}, \quad \pi/2 \leqq \theta_{kn} \leqq \pi. \end{aligned}$$

If $\pi/2 \leqq \theta_{kn} \leqq 3\pi/4$, then $1/\sqrt{2} \leqq \sin\theta_{kn} \leqq 1$, so that from (2.12) it follows that

$$P_n'^{(\alpha,\beta)}(\cos\theta_{kn}) \sim n^{+1/2}(\sin\theta_{kn})^{-\alpha-3/2}, \quad 0 < \theta_{kn} \leqq 3\pi/4. \tag{2.13}$$

LEMMA 2. *Let $\alpha, \beta > -1$ and let $x_{kn} = \cos\theta_{kn}$ with $0 < \theta_{ln} < \ldots < \theta_{nn} < \pi$ be the zeros of $P_n^{(\alpha,\beta)}(x)$. Let $\theta_{on} = 0$, $\theta_{n+1,n} = \pi$. Then*

$$c_1/n < \theta_{k+1,n} - \theta_{kn} < c_2/n, \quad k = 0, 1, \ldots, n; \quad n = 1, 2, \ldots \tag{2.14}$$

where c_1, c_2 are constants independent of n and depending only on α and β.

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Remark. For $-\frac{1}{2} \leqq \alpha, \beta \leqq \frac{1}{2}$, or for $\alpha, \beta > \frac{1}{2}$, this result is implicitly contained in Buell's result [**2**]. In the general case it was recently proved by G. I. Natanson (Izv. Vysš. Učebn. Zaved. Matematika 11 (66) 1–67, pp 67–74); our proof is different.

Proof. The function

$$u(\theta) = (\sin \theta/2)^{\alpha+\frac{1}{2}}(\cos \theta/2)^{\beta+\frac{1}{2}}P_n^{(\alpha,\beta)}(\cos \theta)$$

which has the zeros $\theta_{kn}(k = 1, \ldots, n)$ satisfies the differential equation [**13**, Formula (4.24.2)]

$$\frac{d^2u}{d\theta^2} + V(\theta)u = 0 \tag{2.15}$$

where

$$V(\theta) = \frac{\frac{1}{4} - \alpha^2}{4 \sin^2 \frac{\theta}{2}} + \frac{\frac{1}{4} - \beta^2}{4 \cos^2 \frac{\theta}{2}} + \left(n + \frac{\alpha + \beta + 1}{2}\right)^2. \tag{2.16}$$

By a proper choice of $c_3 = c_3(\alpha, \beta)$, we have for $n > n_0(\alpha, \beta)$

$$n^2/2 \leqq V(\theta) \leqq 2n^2, \quad \text{for} \quad c_3/n \leqq \theta \leqq \pi - c_3/n. \tag{2.17}$$

Hence by Sturm's theorem, if $\theta_{kn}, \theta_{k+1,n} \in [c_3/n, \pi - c_3/n]$, we have

$$c_5/n \leqq \theta_{k+1,n} - \theta_{kn} < c_5/n. \tag{2.18}$$

For the zeros with small indices, we apply the relation

$$\lim_{n\to\infty} n\theta_{\nu n} = j_\nu \tag{2.19}$$

where j_ν is the νth zero of the Bessel function $J_\nu(x)$ [**13**, Theorem 8.1.2]. We infer that the number of indices k for which $\theta_{kn} \in [0, (c_3 + c_4)/n]$ have a a bound M independent of n. By symmetry, this is also true for $[\pi - (c_3 + c_4)/n, \pi]$. Consequently (2.14) is valid for $n > n_1$. After replacing c_1 by a smaller constant (respectively c_2 by a greater constant), if necessary, (2.14) is valid for $n \geqq 1$.

LEMMA 3. *The following estimates* (2.20)–(2.23), *hold for* $\Phi_m(x, x)$ *and* $\Phi_m(x, y)$:

$$\tfrac{1}{2} < \Phi_m(x, x) \leqq 3 \tag{2.20}$$

$$|\Phi_m^2(x, x) - 1|(1 - x^2)^{\frac{1}{2}} \leqq 4/m \tag{2.21}$$

$$\Phi_m^2(x, y) \leqq 9. \tag{2.22}$$

Moreover, if $x = \cos \theta$, $x_{kn} = \cos \theta_{kn}$ *and if* $\theta_{in} < \theta \leqq \theta_{i+1,n}$ *for some* i $(0 \leqq i \leqq n)$, *then*

$$|\Phi_m(x_{kn}, x)| \leqslant \frac{2}{m}\left[\sin \frac{|\theta - \theta_{kn}|}{2}\right]^{-1}. \tag{2.23}$$

Remark. This lemma is essentially due to Vertesi [**14**] (see also Saxena [**10**]) who proved it for $m = n$. Here we outline the proof since the original is not easily accessible.

Proof. From (2.2), we have

$$\Phi_m(x, y) = \frac{1}{m} \cdot \frac{T_{m+1}(x)T_m(y) - T_{m+1}(y)T_m(x)}{x - y} \tag{2.24}$$

whence we easily have

$$\Phi_m(x, x) = \frac{1}{m}\left\{m + \frac{1}{2} + \frac{1}{2}\frac{\sin (2m+1)\theta}{\sin \theta}\right\}, \quad x = \cos\theta. \tag{2.25}$$

If $\cos \pi/(2m + 1) \leqq |x| \leqq 1$, i.e., $2m\pi/(2m + 1) \leqq \theta \leqq \pi$ or $0 \leqq \theta \leqq \pi/(2m + 1)$, then

$$\frac{\sin (2m+1)\theta}{\sin \theta} \geqq 0$$

and so $\Phi_m(x, x) \geqq 1$. On the other hand, if $|x| < \cos \pi/(2m + 1)$, i.e., $\pi/(2m + 1) < \theta < 2m\pi/(2m + 1)$, then

$$\Phi_m(x, x) \geqq \frac{1}{2m}\left(m + \frac{1}{2}\right) > \frac{1}{2},$$

since

$$\frac{\sin (2m+1)\theta}{\sin \theta} \geqq -\left[\sin \frac{\pi}{2m+1}\right]^{-1} \geqq -\left(m + \frac{1}{2}\right)$$

by the elementary inequality $\sin x \geqq (2/\pi)x$ for $0 \leqq x \leqq \pi/2$. This proves the left side of inequality (2.20). The right side is immediate from (2.25).

Using Schwarz inequality it follows from (2.2) that $\Phi_m^2(x, y) < \Phi_m(x, x)\Phi_m(y, y)$ whence from (2.20), we have (2.22).

From (2.25) it follows that

$$|\Phi_m(x, x) - 1|(1 - x^2)^{\frac{1}{2}} < 1/m$$

which combined with (2.20) yields (2.21).

In order to prove (2.23) we remark that for $0 \leqq \theta \leqq \pi$, we have

$$\sin \frac{\theta + \theta_{kn}}{2} < \sin \frac{\theta}{2} + \sin \frac{\theta_{kn}}{2} < 2 \sin \frac{\theta + \theta_{kn}}{2} \tag{2.26}$$

$$\frac{\sin \theta_{kn}}{|\cos \theta - \cos \theta_{kn}|} \leqslant \left[\sin \frac{|\theta - \theta_{kn}|}{2}\right]^{-1}. \tag{2.27}$$

Since

$$\Phi_m(x_{kn}, x) = \frac{1}{m}\left[\frac{\cos (m+1)\theta - \cos m\theta}{\cos \theta - \cos \theta_{kn}} \cos m\theta_{kn} + \frac{\cos m\theta_{kn} - \cos (m+1)\theta_{kn}}{\cos \theta - \cos \theta_{kn}} \cos m\theta\right]$$

it follows that

$$|\Phi_m(x_{kn}, x)| \leqq \frac{2}{m} \cdot \frac{\sin \frac{\theta}{2} + \sin \frac{\theta_{kn}}{2}}{|\cos \theta - \cos \theta_{kn}|} \tag{2.28}$$

and inequality (2.23) now follows on using (2.26). This completes the proof of Lemma 3.

3. Some estimates and lemmas. In the sequel c denotes a constant, not necessarily the same, independent of n. We shall prove

LEMMA 4. *Suppose* $0 \leqq \theta \leqq \pi/2$, $\theta_{in} \leqq \theta < \theta_{i+1,n}$ *for some* $i \geqq 0$ *and let* $f \in C[-1, 1]$ *with modulus of continuity* $\omega(\delta)$.

(A) *If* $\theta_{kn} > 3\pi/4$, *then*

$$|\phi_{kn}(x)| \cdot |f(x) - f(x_{kn})| \leqq cn^{-2r+3/2+\max(\alpha,-1/2)}\omega(1/n^2). \tag{3.1}$$

(B) *If* $0 \leqq \theta_{kn} \leqq 3\pi/4$, *and if* $0 \leqq \theta \leqq cn^{-1}$ *for some c sufficiently small, then*

$$\begin{aligned} &|\phi_{kn}(x)| \cdot |f(x) - f(x_{kn})| \\ &\leqslant c\left\{\frac{n^{\alpha+\frac{1}{2}}\theta_{kn}{}^{\alpha+\frac{1}{2}}}{k^{2r}}\,\omega\left(\frac{\sin\theta}{n}\right) + \frac{n^{\alpha+\frac{1}{2}}\theta_{kn}{}^{\alpha+\frac{1}{2}}}{k^{2r-1}}\,\omega\left(\frac{1}{n^2}\right)\right\}. \end{aligned} \tag{3.2}$$

(C) *If* $0 \leqq \theta_{kn} \leqq 3\pi/4$ *and if* $cn^{-1} \leqq \theta \leqq \pi/2$, *then*

$$\begin{aligned} &|\phi_{kn}(x)| \cdot |f(x) - f(x_{kn})| \\ &\leqslant \begin{cases} c\left[\dfrac{\theta_{kn}}{\theta}\right]^{\alpha+\frac{1}{2}}\left\{\dfrac{1}{(k-i)^{2r}}\,\omega\left(\dfrac{\sin\theta}{n}\right) + \dfrac{1}{|k-i|^{2r-1}}\,\omega\left(\dfrac{1}{n^2}\right)\right\}, & k \neq i, i+1 \\ c\left[\dfrac{\theta_{in}}{\theta}\right]^{\alpha+\frac{1}{2}}\left\{\omega\left(\dfrac{\sin\theta}{n}\right) + \omega\left(\dfrac{1}{n^2}\right)\right\}, & k = i, i+1. \end{cases} \end{aligned} \tag{3.3}$$

Proof. (A) In order to prove (3.1), we observe that for $0 \leqq \theta \leqq \pi/2$ and $3\pi/4 < \theta_{kn} < \pi$, we have $|x - x_{kn}| > c$ so that using (2.11), (2.12) and (2.28), yields

$$\begin{aligned} |\phi_{kn}(x)| &= |l_{kn}(x)| \cdot |\Phi_m{}^{2r}(x_{kn}, x)| \\ &\leqq \frac{c}{n^{2r}} \cdot \frac{n^{\max(\alpha,-\frac{1}{2})}}{n^{\frac{1}{2}}} (\sin \theta_{kn})^{\beta+3/2} \\ &\leqq \frac{c}{n^{2r}} \cdot \frac{n^{\max(\alpha,-\frac{1}{2})}}{n^{\frac{1}{2}}}, \textit{ since } \beta > -1. \end{aligned} \tag{3.4}$$

Since $|f(x) - f(x_{kn})| \leqq \omega(2) < cn^2\omega(1/n^2)$, the inequality (3.1) follows from (3.4).

(B) In this case if c is sufficiently small, and $0 \leqq \theta \leqq cn^{-1}$, then $\theta_{0n} \leqq \theta < \frac{1}{2}\theta_{1n}$. Hence using (2.11), (2.12) and (2.13), we have

$$\begin{aligned}|\phi_{kn}(x)| &\leqq \frac{cn^{\alpha}}{n^{\frac{1}{2}}(\sin\theta_{kn})^{-\alpha-\frac{1}{2}}}\frac{1}{n^{2r}}\left(\operatorname{cosec}\frac{|\theta-\theta_{kn}|}{2}\right)^{2r+1} \\ &\leqq c\,\frac{(n\theta_{kn})^{\alpha+\frac{1}{2}}}{k^{2r+1}}.\end{aligned} \tag{3.5}$$

Since $\sin\frac{1}{2}(\theta+\theta_{kn}) \leqq \sin\theta + \sin\frac{1}{2}|\theta-\theta_{kn}|$, it follows that

$$\begin{aligned}|f(x)-f(x_{kn})| &\leqq c\omega\left(\sin\frac{\theta+\theta_{kn}}{2}\sin\frac{|\theta-\theta_{kn}|}{2}\right) \\ &\leqq c\left\{k\omega\left(\frac{\sin\theta}{n}\right)+k^2\,\omega\left(\frac{1}{n^2}\right)\right\}.\end{aligned} \tag{3.6}$$

Combining (3.5) and (3.6) yields (3.2).

(C) Since $0 \leqq \theta_{kn} \leqq 3\pi/4$ and $cn^{-1} \leqq \theta \leqq \pi/2$, we use (2.11), (2.12) and (2.13) and for $k \neq i, i+1$, we have

$$\begin{aligned}|\phi_{kn}(x)| &\leqq \frac{\theta^{-\alpha-\frac{1}{2}}n^{-\frac{1}{2}}}{n^{\frac{1}{2}}(\sin\theta_{kn})^{-\alpha-\frac{1}{2}}}\cdot\frac{\sin\theta_{kn}}{|\cos\theta-\cos\theta_{kn}|}\left(\frac{2}{m}\cdot\left[\sin\frac{\theta-\theta_{kn}}{2}\right]^{-1}\right)^{2r} \\ &\leqq \frac{c}{n^{2r+1}}\left[\frac{\theta_{kn}}{\theta}\right]^{\alpha+\frac{1}{2}}\cdot\frac{1}{|\theta-\theta_{kn}|^{2r+1}} \\ &\leqq c\left[\frac{\theta_{kn}}{\theta}\right]^{\alpha+\frac{1}{2}}\cdot\frac{1}{|k-i|^{2r+1}}.\end{aligned} \tag{3.7}$$

For $k = i$ or $i+1$, we have

$$\begin{aligned}|\phi_{kn}(x)| &\leqq c\,\frac{P_n'^{(\alpha,\beta)}(\cos\theta^*)}{P_n'^{(\alpha,\beta)}(\cos\theta_{kn})}, \quad \theta^* \in [\theta_{kn},\theta] \subset [\theta_{kn},\theta_{k+1,n}] \\ &\leqq c\left[\frac{\theta_{kn}}{\theta^*}\right]^{\alpha+\frac{1}{2}} \\ &\leqq c.\end{aligned} \tag{3.8}$$

Also for $k \neq i, i+1$ we have as in (3.6),

$$|f(x)-f(x_{kn})| \leqq c\left[|k-i|\omega\left(\frac{\sin\theta}{n}\right)+(k-i)^2\omega\left(\frac{1}{n^2}\right)\right] \tag{3.9}$$

while for $k = i$ or $i+1$, we have

$$|f(x)-f(x_{kn})| \leqq c\left[\omega\left(\frac{\sin\theta}{n}\right)+\omega\left(\frac{1}{n^2}\right)\right]. \tag{3.10}$$

Hence using (3.7), (3.9), (3.8) and (3.10), we get (3.3). This completes the proof of the lemma.

LEMMA 5. *If* $0 \leqq \theta \leqq \pi/2$ *and if* $f \in C[-1, 1]$, *then*

$$(3.11) \qquad S_1 = \sum_{k=1}^{m} |\phi_{kn}(x)| \cdot |f(x) - f(x_{kn})| \leqq c\left[\omega\left(\frac{\sin\theta}{n}\right) + \omega\left(\frac{1}{n^2}\right)\right]$$

provided $2r > \max(4, \alpha + 5/2)$.

Proof. Since

$$S_1 = \sum_{0\leqq\theta_{kn}\leqq 3\pi/4} + \sum_{3\pi/4<\theta_{kn}\leqq\pi} |\phi_{kn}(x)| \cdot |f(x) - f(x_{kn})|$$
$$= S_1' + S_1''$$

we see from (3.1) that

$$S_1'' \leqq \omega\left(\frac{1}{n^2}\right) \sum n^{-2r+3/2+\max(\alpha,-1/2)}$$

$$(3.12) \qquad = O(1)\,\omega\left(\frac{1}{n^2}\right)$$

if $2r - 5/2 - \max(\alpha, -1/2) > 0$.

In order to estimate S_1', we first consider the case when $0 \leqq \theta \leqq cn^{-1}$ for c sufficiently small. Then from (3.2), we get (since $n\theta_{kn} \sim k$),

$$S_1 \leqq c\left[\omega\left(\frac{\sin\theta}{n}\right)\sigma_{2r} + \omega\left(\frac{1}{n^2}\right)\sigma_{2r-1}\right]$$

where

$$\sigma_\mu = \sum_{k=1}^{\infty} k^{\alpha+\frac{1}{2}-\mu}, \quad \mu = 2r, 2r-1.$$

If $2r - 1 - \alpha - \frac{1}{2} > 1$, $\sigma_\mu < \infty$ for $\mu = 2r$, $2r - 1$ and so

$$(3.13) \qquad S_1' \leqq c\left[\omega\left(\frac{\sin\theta}{n}\right) + \omega\left(\frac{1}{n^2}\right)\right], \quad 0 \leqq \theta \leqq cn^{-1}.$$

If on the other hand, $cn^{-1} \leqq \theta \leqq \pi/2$, (3.3) yields

$$S_1' \leqq c\omega\left(\frac{\sin\theta}{n}\right) \sum_{k\neq i,i+1} \left[\frac{\theta_{kn}}{\theta}\right]^{\alpha+\frac{1}{2}} \frac{1}{(k-i)^{2r}}$$
$$+ c\omega\left(\frac{1}{n^2}\right) \sum_{k\neq i,i+1} \left[\frac{\theta_{kn}}{\theta}\right]^{\alpha+\frac{1}{2}} \frac{1}{|k-i|^{2r-1}}$$
$$+ c\left(\frac{\theta_{in}}{\theta}\right)^{\alpha+\frac{1}{2}} \left\{\omega\left(\frac{\sin\theta}{n}\right) + \omega\left(\frac{1}{n^2}\right)\right\}.$$

Then

$$(3.13\text{a}) \quad S_1' \leqq c\left[\omega\left(\frac{\sin\theta}{n}\right)\tau_{2r} + \omega\left(\frac{1}{n^2}\right)\tau_{2r-1}\right] + c\left[\omega\left(\frac{\sin\theta}{n}\right) + \omega\left(\frac{1}{n^2}\right)\right],$$

where

$$\tau_\mu = \sum_{k\neq i,\, i+1} \left[\frac{\theta_{kn}}{\theta}\right]^{\alpha+\frac{1}{2}} \frac{1}{|k-i|^\mu}, \quad \mu = 2r,\, 2r-1.$$

We now consider two cases: (i) $\alpha \geqq -\frac{1}{2}$, (ii) $\alpha < -\frac{1}{2}$.

(i) In this case, since $\theta > ic/n$ and $\theta_{kn} < ck/n$, we have

$$\begin{aligned} \tau_\mu &\leqq c \sum_{k\neq i,\, i+1} \frac{k^{\alpha+\frac{1}{2}}}{i} \cdot \frac{1}{|k-i|^\mu}, \\ &\leqq c \sum_{k=1}^{\infty} k^{-\mu} = c \sum_{k\leqq 2i} k^{-\mu} + c \sum_{k>2i} k^{-\mu} \qquad (3.14)\\ &= c(\tau_\mu' + \tau_\mu''). \end{aligned}$$

If $k > 2i$, then $k - i > k/2$ so that for $\mu - \alpha - \frac{1}{2} > 1$, we have

$$\tau_\mu' \leqq c i^{-\alpha-\frac{1}{2}} \sum_{k>2i} k^{\alpha+\frac{1}{2}-\mu} = O(1) i^{-\mu+1}. \qquad (3.15)$$

From (3.14) it follows that $\tau_\mu = O(1)$ if $2r - 1 > 1$, i.e., $r > 1$ and (3.15) shows that $\tau_\mu = O(1)$ if $\mu = 2r - 1 > 1$, i.e., $r > 1$ and $2r - \alpha - 3/2 > 1$, i.e., $2r > \alpha + 5/2$.

(ii) If $\alpha < -\frac{1}{2}$, then from $\theta < c(i+1)/n$ and $\theta_{kn} > ck/n$, it follows that

$$\left[\frac{\theta}{\theta_{kn}}\right]^{-\alpha-\frac{1}{2}} < c\left[\frac{i+1}{k}\right]^{-\alpha-\frac{1}{2}},$$

so that

$$\begin{aligned} \tau_\mu &< c \sum_{k\neq i,\, i+1} \left[\frac{i+1}{k}\right]^{-\alpha-\frac{1}{2}} \cdot \frac{1}{|k-i|^\mu} \\ &= c(\tau_\mu' + \tau_\mu'') \end{aligned}$$

where the summation in τ_μ' is for $k < i/2$ and in τ_μ'' for $k \geqq i/2$.

If

$$k \geqq \frac{i}{2}, \qquad \left[\frac{i+1}{k}\right]^{-\alpha-\frac{1}{2}} = O(1) \quad \text{and} \quad \sum_{k\neq i,\, i+1} \frac{1}{|k-i|^\mu} < 2 \sum_{k=1} \frac{1}{k^\mu}.$$

For $\mu > 1$, i.e., $2r - 1 > 1$, we have $\tau_\mu'' = O(1)$. If $2k < i$, then $|k - i| > i/2$, so that

$$\tau_\mu' \leqq c i^{-\alpha-\frac{1}{2}-\mu} \sum_{k<i/2} \left[\frac{1}{k}\right]^{-\alpha-\frac{1}{2}} < c i^{-\alpha+\frac{1}{2}-\mu} = O(1), \qquad (3.16)$$

if $\mu > \alpha - \frac{1}{2}$, i.e., $\mu - 1 > +\alpha - \frac{3}{2}$, i.e., $2r - 2 > \alpha - \frac{3}{2}$. Since $-1 < \alpha < -\frac{1}{2}$, this condition is certainly satisfied for $r \geqq 2$.

The estimate (3.11) now follows from (3.13a) and (3.14), (3.15) and (3.16).

LEMMA 6. *If* $f \in C[-1, 1]$ *and if* $\alpha, \beta > -1$ *and* $2r > \max(4, \alpha + 5/2)$, *then for* $0 \leqq x \leqq 1$, *we have*

$$|f(x) - J_n^{(\alpha,\beta)}(f; x)| \leqq c\left[\omega\left(\frac{(1-x)^{\frac{1}{2}}}{n}\right) + \omega\left(\frac{1}{n^2}\right)\right]. \qquad (3.17)$$

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Proof. Using (2.6) and (2.7), we have

$$|f(x) - J_n^{(\alpha,\beta)}(f;x)| \leqq S_1 + S_2 + S_3 \tag{3.18}$$

where S_1 is given by (3.11) and

$$S_2 = \frac{1+x}{2} \cdot |f(x) - f(1)| \cdot \left|1 - \sum_{k=1}^{n} \phi_{kn}(x)\right|$$

$$S_3 = \frac{1-x}{2} \cdot |f(x) - f(-1)| \cdot \left|1 - \sum_{k=1}^{n} \phi_{kn}(x)\right|.$$

From (2.4) and Lemma 3, it follows that

$$\begin{aligned} S_2 &\leqq \frac{1+x}{2} \cdot \omega(|x-1|) \cdot |1 - \Phi_m^{2r}(x,x)| \\ &\leqq \frac{1+x}{2} \cdot \left(1 + n\frac{|x-1|}{\sin\theta}\right) \omega\left(\frac{\sin\theta}{n}\right) \cdot |1 - \Phi_m^{2r}(x,x)| \\ &\leqq c\omega\left(\frac{\sin\theta}{n}\right). \end{aligned} \tag{3.19}$$

Similarly,

$$S_3 \leqq c\omega\left(\frac{\sin\theta}{n}\right). \tag{3.20}$$

Hence (3.18), (3.19) and (3.20) together with (3.11) yield (3.17).

4. Proof of Theorem 1. Since $J_n^{(\alpha,\beta)}(f;x)$ defined by (2.6) depends on the zeros of $P_n^{(\alpha,\beta)}(x)$ and since

$$P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x) \tag{4.1}$$

it follows that

$$J_n^{(\beta,\alpha)}(f(-t);x) = J_n^{(\alpha,\beta)}(f(t);-x). \tag{4.2}$$

By Lemma 6, for $0 \leqq x \leqq 1$, we have

$$|f(x) - J_n^{(\beta,\alpha)}(f(-t);x)| \leqq c\left\{\omega\left(\frac{(1-x)^{\frac{1}{2}}}{n}\right) + \omega\left(\frac{1}{n^2}\right)\right\} \tag{4.3}$$

if $2r > \max(4, \beta + 5/2)$. Hence for $-1 \leqq x \leqq 0$

$$|f(x) - J_n^{(\alpha,\beta)}(f(t);x)| \leqq c\left\{\omega\left(\frac{(1+x)^{\frac{1}{2}}}{n}\right) + \omega\left(\frac{1}{n^2}\right)\right\} \tag{4.4}$$

so that combining (4.4) with (3.17), we have (2.8), provided $2r > \max(4, \alpha + 5/2, \beta + 5/2)$. This completes the proof of Theorem 1.

5. Proof of Theorem 2. For the proof of Theorem 2, we shall need an estimate for

$$S_4(x) \equiv 1 - \sum_{k=1}^{n} \frac{\phi_{kn}(x)}{\phi_m^{2r}(x_{kn}, x_{kn})}. \tag{5.1}$$

For this purpose we prove

LEMMA 7. *If* $S_4(x)$ *is given by* (5.1) *then for* $0 \leqq x \leqq 1$, *we have*

$$|S_4(x)|(1 - x^2)^{\frac{1}{2}} \leqq c/m. \tag{5.2}$$

Proof. It is easy to see that

$$|S_4(x)| \leqq |S_4'| + |S_4''| \tag{5.3}$$

where

$$S_4' = 1 - \sum_{k=1}^{n} \phi_{kn}(x)$$

$$S_4'' = \sum_{k=1}^{n} \phi_{kn}(x) \frac{\Phi_m^{2r}(x_{kn}, x_{kn}) - 1}{\Phi_m^{2r}(x_{kn}, x_{kn})}.$$

Applying (2.4) and Lemma 3, we have

$$|S_4'|(1 - x^2)^{\frac{1}{2}} \leqq c/m \tag{5.4}$$

$$|S_4''|(1 - x^2)^{\frac{1}{2}} \leqq \frac{c}{m} \sum_{k=1}^{n} |\phi_{kn}(x)| \cdot \frac{(1 - x^2)^{\frac{1}{2}}}{(1 - x_{kn}^2)^{\frac{1}{2}}}. \tag{5.5}$$

In order to prove (5.2) it is enough to show, in view of (5.3), (5.4) and (5.5), that

$$(1 - x)^{\frac{1}{2}} \sum_{k=1}^{n} \frac{|\phi_{kn}(x)|}{(1 - x_{kn}^2)^{\frac{1}{2}}} = O(1). \tag{5.6}$$

If $3\pi/4 \leqq \theta_{kn} \leqq \pi$, it follows from (3.1) that

$$(1 - x^2)^{\frac{1}{2}} \sum_{3\pi/4 \leqq \theta_{kn} \leqq \pi} \frac{|\phi_{kn}(x)|}{(1 - x_{kn}^2)^{\frac{1}{2}}} \leqq \sum_{3\pi/4 \leqq \theta_{kn} \leqq \pi} |\phi_{kn}(x)| = O(1). \tag{5.7}$$

If $0 < \theta_{kn} < 3\pi/4$, the reasoning of Lemma 5 applies *mutatis mutandis* and we have

$$(1 - x^2)^{\frac{1}{2}} \sum_{0 \leqq \theta_{kn} \leqq 3\pi/4} \frac{|\phi_{kn}(x)|}{(1 - x_{kn}^2)^{\frac{1}{2}}} = O(1). \tag{5.8}$$

Hence (5.6) follows from (5.7) and (5.8). This completes the proof of Lemma 7.

Proof of Theorem 2. As in the proof of Theorem 1, we have

$$|f(x) - A_n^{(\alpha,\beta)}(f; x)|$$

$$\leqq \frac{1 + x}{2} \cdot |f(x) - f(1)| \cdot |S_4(x)| + \frac{1 - x}{2} \cdot |f(x) - f(-1)| \cdot |S_4(x)| + |S_5(x)|,$$

where $S_4(x)$ is given by (5.1) and

$$S_5(x) = \sum_{k=1}^{n} \frac{|f(x) - f(x_{kn})| \cdot |\phi_{kn}(x)|}{\Phi_m^{2r}(x_{kn}, x_{kn})}.$$

By Lemma 3, $\Phi_m(x, x)$ is bounded above by 3 and below by 1/2, so that by Lemma 5, we have for $0 \leqq \theta \leqq \pi/2$,

$$|S_5(x)| \leqq c\left\{\omega\left(\frac{\sin\theta}{n}\right) + \omega\left(\frac{1}{n^2}\right)\right\}. \tag{5.9}$$

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By Lemma 7,

$$\frac{1+x}{2} \cdot |f(x) - f(1)| \cdot |S_4(x)| \leqq c\omega\left(\frac{\sin\theta}{n}\right) \tag{5.10}$$

$$\frac{1-x}{2} \cdot |f(x) - f(-1)| \cdot |S_4(x)| \leqq c\omega\left(\frac{\sin\theta}{n}\right). \tag{5.11}$$

Combining (5.9), (5.10) and (5.11), we are able to prove Theorem 2 for $0 \leqq x \leqq 1$. The proof for $-1 \leqq x \leqq 1$, can be completed as in §4 since because of (4.1) we have

$$A_n^{(\beta,\alpha)}(f(-t);x) = A_n^{(\alpha,\beta)}(f(t);x).$$

We omit the rest of the details.

	Nodes	# of Nodes	Interval of Convergence	Degree of Poly.	Nature of Process	Type of Estimate
Freud [4]	$Z(T_n(x))$ in $\lvert x\rvert \leqq \frac{1}{2}$	$\sim \frac{1}{3}n$	$\lvert x\rvert \leqq \frac{1}{2}$	$4n-3$	almost interpolatory	Jackson
M. Sallay [9]	$Z(P_n(x))$ in $\lvert x\rvert \leqq \frac{1}{2}$		$\lvert x\rvert \leqq \frac{1}{2}$	$4n-3$	"	Jackson
Saxena [10]	$Z(U_n(x)(1-x^2))$	$n+2$	$\lvert x\rvert \leqq 1$	$4n-2$	"	Jackson
Vertesi [14]	$Z(T_n(x)(1-x^2))$	$n+2$	$\lvert x\rvert \leqq 1$	$4n-2$	"	Jackson
Mathur [8]	$Z(P_n^{(-\frac{1}{2},\frac{1}{2})}(1-x^2))$	$n+2$	$\lvert x\rvert \leqq 1$	$4n-2$	"	Jackson
Freud-Vertesi [7]	$Z(T_n(x)(1-x^2))$	$n+2$	$\lvert x\rvert \leqq 1$	$4n-2$	"	Timan
Vertesi-Kis	$Z((1-x)P_n^{(-\frac{1}{2},\frac{1}{2})})$	$n+1$	$\lvert x\rvert \leqq 1$	$4n-4$	interpolatory	Timan
Saxena [11]	$Z((1-x^2)U_n(x))$	$n+2$	$\lvert x\rvert \leqq 1$	$4n-2$	almost interpolatory	Teljakowski-Gopengauz
Saxena [12]	$Z((1-x^2)T_n(x))$	$n+2$	$\lvert x\rvert \leqq 1$	$4n-2$	"	Teljakowski-Gopengauz
Theorem I (§ 2)	$Z(P_n^{(\alpha,\beta)}(x))(1-x^2)$	$n+2$	$\lvert x\rvert \leqq 1$	$n(1+c)$ arbitrary	almost interpolatory (Theorem 1)	Timan
Theorem II (§ 2)	$Z(P_n^{(\alpha,\beta)}(x)(1-x^2))$	$n+2$	$\lvert x\rvert \leqq 1$	"	interpolatory	Timan

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Addendum. It may be remarked that whereas the foregoing results deal with interpolatory operators, R. DeVore has answered Butzer's question using convolution type operators (J. Approximation Theory *1* (1968), 607–615). See also R. Bojanic (A note on the degree of approximation to continuous functions).

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