3
The reggeized gluon

A particle of mass M and spin J is said to 'reggeize' if the amplitude, $\mathcal{A}$, for a process involving the exchange in the t-channel of the quantum numbers of that particle behaves asymptotically in s as

$$\mathcal{A} \propto s^{\alpha(t)}$$

where $\alpha(t)$ is the trajectory and $\alpha(M^2) = J$, so that the particle itself lies on the trajectory.

The idea that particles should reggeize has a long history. It was first proposed by Gell-Mann *et al.* (1962, 1964a,b) and by Polkinghorne (1964). Mandelstam (1965) gave general conditions for reggeization to occur and this was developed by several authors (Abers & Teplitz (1967), Abers *et al.* (1970), Dicus & Teplitz (1971), Grisaru, Schnitzer, & Tsao (1973)). Calculations in Quantum Electrodynamics (QED) were carried out by Frolov, Gribov & Lipatov (1970, 1971) and by Cheng & Wu (1965, 1969a–c, 1970a,b), who showed that the photon had a fixed cut singularity (as opposed to a Regge pole). On the other hand McCoy & Wu (1976a–f) established that the fermion does indeed reggeize in QED. This was extended to non-abelian gauge theories by Mason (1976a,b) and Sen (1983). The demonstration of reggeization of the gluon was first shown to two-loop order by Tyburski (1976), Frankfurt & Sherman (1976), and Lipatov (1976) and to three loops by Cheng & Lo (1976). The reggeization to all orders in perturbation theory has been established by several authors using somewhat different techniques. Mason (1977) worked in Coulomb gauge and used time ordered perturbation theory to establish that the amplitude factorized in such a way that the reggeization must follow. Cheng & Lo (1977) developed a recursion relation for going to higher orders in perturbation theory.

The method that we shall follow in this chapter is that of Fadin,

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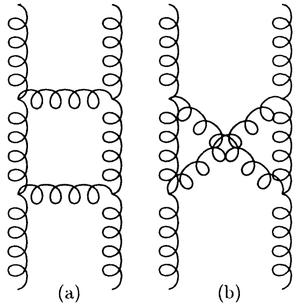

Fig. 3.1. Section of uncrossed and crossed gluon ladder diagrams.

Kuraev & Lipatov (1976), who used dispersive techniques developed in the preceding chapter. We feel that this is the most transparent derivation of reggeization and lends itself most easily to the discussion of the Pomeron in the next chapter.

In the preceding chapter we showed that in a ϕ^3 theory the amplitude for elastic scattering of scalar 'quarks' was dominated in the leading $\ln s$ approximation by uncrossed ladder diagrams. In particular, it was shown that a crossed rung gives rise to a hard denominator and is suppressed by $\sim \rho_i/\rho_{i-1}$. In QCD this does not work. A section of a ladder shown in Fig. 3.1(a) does not dominate over the crossed-rung section shown in Fig. 3.1(b). The reason for this is that the triple gluon vertices carry the momenta of the gluons in the numerators and in Fig. 3.1(b) the scalar product of these momenta between the top left and bottom right (or *vice versa*) vertices produces a term which is enhanced compared with the corresponding scalar product in Fig. 3.1(a). This enhancement compensates for the denominator suppression due to the hard propagator in the crossed-rung diagram.

Nevertheless we shall show that it is possible to organize high energy scattering amplitudes into 'effective' ladder-type diagrams. The vertices will not be the usual triple gluon vertices, but, rather, a non-local effective vertex, which we shall discuss below. Also the vertical lines of the ladder are not bare gluons whose propagators are given (in Feynman gauge) by

$$D_{\mu\nu}(q^2) = -i\frac{g_{\mu\nu}}{q^2}$$

but, rather, they are 'reggeized' gluons whose propagator (in Feynman gauge) is

$$\tilde{D}_{\mu\nu}(\hat{s}, q^2) = -i\frac{g_{\mu\nu}}{q^2}\left(\frac{\hat{s}}{\mathbf{k}^2}\right)^{\epsilon_G(q^2)}, \tag{3.1}$$

where $\sqrt{\hat{s}}$ is the centre-of-mass energy of the particles between which the 'reggeized' gluon is exchanged and $\alpha_G(q^2) = 1 + \epsilon_G(q^2)$ is the Regge trajectory of the gluon.†

In order to show that gluons reggeize in this way (and to determine the Regge trajectory) we need to calculate to all orders in the perturbation series but keeping only the leading $\ln s$ terms at each order. We need to select those diagrams in which the exchanged quantum numbers (in the t-channel) are those of the gluon, i.e. spin-1 and colour octet. As discussed in Chapter 1, the amplitude in which a single particle of spin J is exchanged has a large s behaviour proportional to s^J, so we are interested in the contributions to the amplitude which at order α_s^n are proportional to $s\,\alpha_s^n \ln^{n-1} s$ and we shall drop sub-leading logarithm terms. We shall begin by discussing the first three orders of perturbation theory and then generalize to all orders.

3.1 Leading order calculation

The QCD process we consider is the scattering of two quarks with different flavours due to colour octet exchange and within the Regge limit ($s \gg -t$). We neglect the masses of the quarks and assume that their incoming momenta p_1 and p_2 lie along the z-axis, i.e.

$$\begin{aligned} p_1 &= \frac{\sqrt{s}}{2}(1, 1, \mathbf{0}), \\ p_2 &= \frac{\sqrt{s}}{2}(1, -1, \mathbf{0}). \end{aligned}$$

The tree diagram contribution to this amplitude is shown in Fig. 3.2.‡ It is very important to realize that all the components of the momentum of the exchanged gluon, q^μ, are much smaller than $\sqrt{s}$. This is true because we are interested in the region $|q^2| = |t| \ll s$

† As in the preceding chapter $\mathbf{k}^2$ represents a typical transverse momentum.

‡ We use the Feynman rules for QCD given in the appendix at the end of the book.

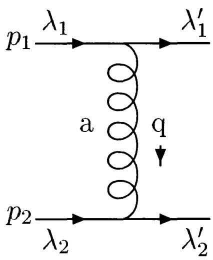

Fig. 3.2. Tree level amplitude.

and because the outgoing quarks are on mass-shell (i.e. $(p_1 - q)^2 = 0$ and $(p_2 + q)^2 = 0$).

3.1.1 The eikonal approximation

The eikonal approximation is an extremely important ingredient in building the 'reggeized' gluon and subsequently the QCD Pomeron.

The upper line of the diagram in Fig. 3.2 gives the factor

$$-ig\bar{u}(\lambda'_1, p_1 - q)\gamma^\mu u(\lambda_1, p_1)\tau^a_{ij}$$

(where λ_1, λ'_1 are the helicities of the incoming and outgoing quarks respectively and the τ^a are the generators of the colour group in the fundamental representation). Since all the components of q^μ are small we may replace this by

$$-ig\bar{u}(\lambda'_1, p_1)\gamma^\mu u(\lambda_1, p_1)\tau^a_{ij}.$$

For spinors normalized such that $u^\dagger(\lambda'_1, p_1)u(\lambda_1, p_1) = 2E_{p_1}\delta_{\lambda'_1\lambda_1}$ we have

$$-ig\bar{u}(\lambda'_1, p_1)\gamma^\mu u(\lambda_1, p_1)\tau^a_{ij} = -2igp_1^\mu\delta_{\lambda'_1\lambda_1}\tau^a_{ij}.$$

This is called the **eikonal approximation** and it is valid whenever the gauge particle exchanged is 'soft' (i.e. all its components are small compared with the momentum of the emitting quark).

Remarkably, the eikonal approximation works not only for spin-$\frac{1}{2}$ quarks but for particles with any spin. If we had a scalar particle instead of a quark in Fig. 3.2 the upper vertex would be

$$-ig(2p_1 - q)^\mu\tau^a_{ij},$$

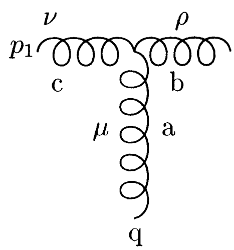

Fig. 3.3. Soft gluon emitted from a hard gluon.

which we approximate by $-2igp_1^\mu \tau_{ij}^a$. More importantly *it may be a gluon itself*, in which case the triple gluon vertex (see Fig. 3.3) is

$$ig\left[g^{\nu\rho}(2p_1 - q)^\mu + g^{\rho\mu}(2q - p_1)^\nu - g^{\mu\nu}(q + p_1)^\rho\right] T_{bc}^a$$

($T_{bc}^a = -if_{abc}$, where the f_{abc} are the structure constants of the gauge group, which we shall leave as $SU(N)$ so that the colour factor can be easily identified). Now neglecting q^μ and noting further that the incoming and outgoing gluons are on shell and therefore *transverse* (so that we may drop terms proportional to p_1^ν and $(p_1 - q)^\rho$) we once again end up with

$$2igp_1^\mu g^{\nu\rho} T_{bc}^a.$$

Thus, at lowest order, the amplitude for quark–quark scattering due to octet exchange is given by

$$\mathcal{A}_0^{(8)} = g^2 2p_1^\mu \frac{g_{\mu\nu}}{q^2} 2p_2^\nu \delta_{\lambda_1'\lambda_1} \delta_{\lambda_2'\lambda_2} G_0^{(8)} = 8\pi\alpha_s \frac{s}{t} \delta_{\lambda_1'\lambda_1} \delta_{\lambda_2'\lambda_2} G_0^{(8)}, \tag{3.2}$$

where $\alpha_s = g^2/4\pi$ and $G_0^{(8)}$ is the colour factor for colour octet exchange, $\tau_{ij}^a \tau_{kl}^a$, which we shall subsequently write as $\tau^a \otimes \tau^a$.

We find it convenient to work in Feynman gauge although the amplitude is gauge invariant (the reader who tries to check this should remember that it will only work up to corrections of order t/s, since we have assumed that we may use the eikonal approximation).

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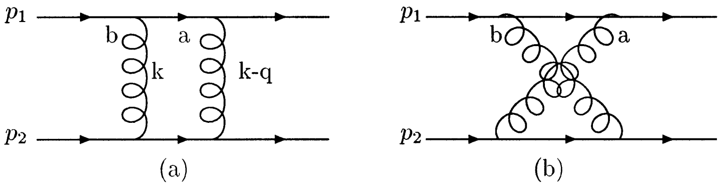

Fig. 3.4. Box and crossed box graphs.

3.2 Order α_s corrections

As explained in the preceding chapter, in leading $\ln s$ approximation we do *not* get contributions from one-loop graphs which contain corrections to propagators or to vertices, but only from the 'box' and 'crossed box' diagrams shown in Fig. 3.4 in which the loop integral depends on the centre-of-mass energy $\sqrt{s}$.†

Once again the contribution from Fig. 3.4(b) can be obtained from the contribution to Fig. 3.4(a) by crossing. However, in this case we not only have to interchange s and u (which introduces a minus sign since, in the Regge limit, $u \approx -s$) but also take into account a different colour factor. The colour factor for Fig. 3.4(a) is given by

$$G_a = (\tau^a\tau^b) \otimes (\tau^a\tau^b),$$

whereas the colour factor from Fig. 3.4(b) is

$$G_b = (\tau^a\tau^b) \otimes (\tau^b\tau^a).$$

Because crossing introduces a minus sign the total colour factor for octet exchange at the one-loop level is the difference between these two, i.e.

$$\begin{aligned} G_a - G_b &= (\tau^a\tau^b) \otimes [\tau^a, \tau^b] \\ &= i\frac{f_{abc}}{2}[\tau^a, \tau^b] \otimes \tau^c \\ &= \frac{if_{abc}if_{abd}}{2}\tau^d \otimes \tau^c = -\frac{N}{2}G_0^{(8)}, \end{aligned} \qquad (3.3)$$

† This argument only works in a covariant gauge. In Coulomb or axial gauge in which an external vector is introduced, it is possible that vertex or self-energy corrections on upper (momentum p_1) lines can give rise to terms proportional to s through scalar products with the external vector which can have a component proportional to p_2. We confine ourselves to covariant gauges in this book.

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where we have used the relation $f_{abc}f_{abd} = N\delta_{cd}$ for $SU(N)$, and $G_0^{(8)}$ is the colour factor for the tree diagram.

As in the preceding chapter, only Fig. 3.4(a) has an imaginary part so the imaginary part of the octet exchange amplitude at the one-loop level can be obtained using the Cutkosky rules and the tree level amplitude calculated in the preceding section (Eq.(3.2)), i.e.

$$\Im\mathrm{m}\mathcal{A}_{3.4a}^{(8)} = \frac{64\pi^2\alpha_s^2}{2}\int d(P.S.^2)\left(\frac{s}{k^2}\right)\left(\frac{s}{(k-q)^2}\right)\delta_{\lambda_1'\lambda_1}\delta_{\lambda_2'\lambda_2}G_a.$$

In terms of the Sudakov variables $\rho, \lambda, \mathbf{k}$ of the momentum k^μ the two-body phase-space integration element may be written

$$d(P.S.^2) = \frac{s}{8\pi^2}d\rho\, d\lambda\, d^2\mathbf{k}\,\delta(-\lambda s - \mathbf{k}^2)\,\delta(\rho s - \mathbf{k}^2), \qquad (3.4)$$

where we have already made use of the inequalities $\rho, |\lambda| \ll 1$. In this approximation for which $-\rho\lambda s \ll \mathbf{k}^2$ we have

$$k^2 \approx -\mathbf{k}^2$$

and similarly

$$(k-q)^2 \approx -(\mathbf{k}-\mathbf{q})^2, \quad (t = -\mathbf{q}^2),$$

hence

$$\Im\mathrm{m}\mathcal{A}_{3.4a}^{(8)} = 8\pi\alpha_s\frac{s}{t}\delta_{\lambda_1'\lambda_1}\delta_{\lambda_2'\lambda_2}G_a\frac{\alpha_s}{2\pi}\int d^2\mathbf{k}\frac{-\mathbf{q}^2}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2}. \qquad (3.5)$$

Using $\ln(-s) = \ln s - i\pi$, this means that the real part is given by

$$\Re\mathrm{e}\mathcal{A}_{3.4a}^{(8)} = -8\pi\alpha_s\frac{s}{t}\delta_{\lambda_1'\lambda_1}\delta_{\lambda_2'\lambda_2}\ln(s/\mathbf{k}^2)G_a\frac{\alpha_s}{2\pi^2}\int d^2\mathbf{k}\frac{-\mathbf{q}^2}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2}. \qquad (3.6)$$

Similarly the amplitude from Fig. 3.4(b) is

$$\Re\mathrm{e}\mathcal{A}_{3.4b}^{(8)} = -8\pi\alpha_s\frac{u}{t}\delta_{\lambda_1'\lambda_1}\delta_{\lambda_2'\lambda_2}\ln(-u/\mathbf{k}^2)G_b\frac{\alpha_s}{2\pi^2}\int d^2\mathbf{k}\frac{-\mathbf{q}^2}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2} \qquad (3.7)$$

(note that it is only the *sum* of these two which is actually octet exchange). Using $u \approx -s$ when $|t| \ll s$ and Eq.(3.3) we find that the complete one-loop amplitude in leading $\ln s$ approximation is given by

$$\mathcal{A}_1^{(8)} = \mathcal{A}_0^{(8)}\epsilon_G(t)\ln(s/\mathbf{k}^2), \qquad (3.8)$$

where

$$\epsilon_G(t) = \frac{N\alpha_s}{4\pi^2}\int d^2\mathbf{k}\frac{t}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2} \quad (t=-\mathbf{q}^2). \tag{3.9}$$

The reader should notice that the integral on the right hand side of Eq.(3.9) is infra-red divergent. In the original work by Fadin, Kuraev & Lipatov (1976, 1977) and by Cheng & Lo (1976), great care was taken to regularize this divergence by breaking the gauge group spontaneously and including contributions from graphs in which there are Higgs bosons. For our purposes such rigour is not necessary. The infra-red divergence arises because the external quarks are on mass-shell. In the 'real world' this is not the case: the quarks are bound inside hadrons and off shell typically by an amount of the order of their average transverse momenta. Such an off-shellness provides a cut-off for the infra-red divergent integrals. Furthermore, it will turn out that the integral equation for the perturbative Pomeron is free from infra-red divergences. Therefore it is sufficient for us to leave ϵ_G in the form of Eq.(3.9), and it is to be understood that the infra-red divergence is to be regularized in some convenient way, introducing a scale which is expected to be of order Λ_{QCD}.

3.3 Order α_s^2 corrections

The two-loop corrections were performed independently by Tyburski (1976), Frankfurt & Sherman (1976) and by Lipatov (1976). We follow Lipatov's calculation closely.

As explained in the preceding chapter we do not get any contributions proportional to $\alpha_s^2 \ln^2 s$ from graphs which consist of vertex or self-energy insertions on the one-loop graphs considered in the last section. In order to obtain the imaginary part of the contribution in this order (in the leading $\ln s$ approximation) we need to consider the amplitude for a quark with momentum p_1 and a quark with momentum p_2 to scatter into a quark with momentum $p_1 - k_1$, a quark with momentum $p_2 + k_2$ and a gluon with momentum $k_1 - k_2$. Using Sudakov variables to parametrize the momenta k_1 and k_2:

$$k_i^\mu = \rho_i p_1^\mu + \lambda_i p_2^\mu + k_{i\perp}^\mu \quad (i = 1, 2),$$

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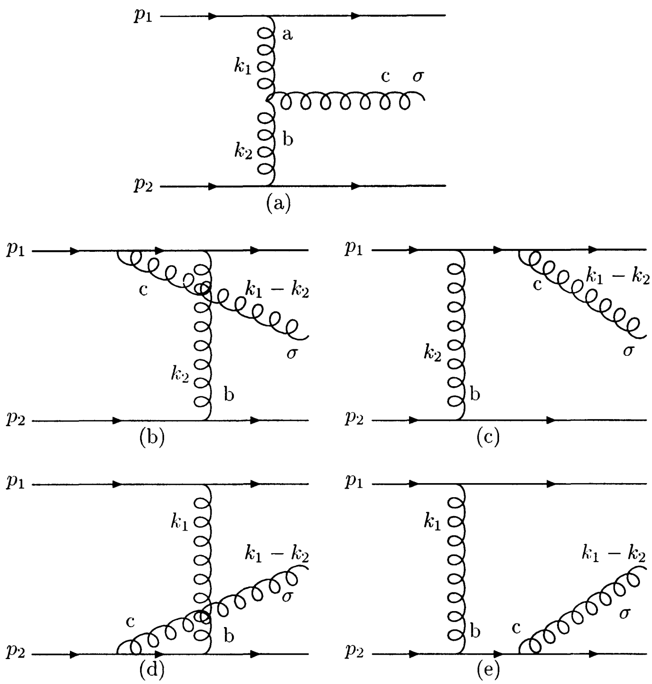

Fig. 3.5. Diagrams for the process qq $\to$ qq+g.

the leading logarithm contribution again comes from the region

$$\begin{aligned} 1 &\gg \rho_1 \gg \rho_2 \\ 1 &\gg |\lambda_2| \gg |\lambda_1| \end{aligned}$$

and the on-shell condition for the outgoing gluon becomes (in this approximation)

$$\rho_1 \lambda_2 s = -(\mathbf{k_1} - \mathbf{k_2})^2,$$

so that $k_1^2 \approx k_{1\perp}^2 = -\mathbf{k_1^2}$ and $k_2^2 \approx k_{2\perp}^2 = -\mathbf{k_2^2}$. Once again the transverse momenta are both of the same magnitude ($\mathbf{k_1^2}, \mathbf{k_2^2}$ are both of order $\mathbf{k}^2$). The graphs for this process are shown in Fig. 3.5. We need these amplitudes in order to compute the 25 (two-loop) diagrams using the s-channel cutting rules.

The contribution from Fig. 3.5(a) (in Feynman gauge) in the

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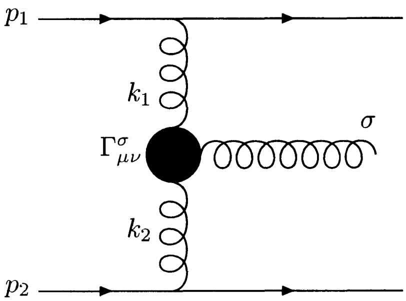

Fig. 3.6. The effective non-local vertex.

relevant kinematic regime is

$$-ig^3 2s\left[\rho_1 p_1^\sigma + \lambda_2 p_2^\sigma - (k_1+k_2)_\perp^\sigma\right]\frac{\delta_{\lambda_1'\lambda_1}\delta_{\lambda_2'\lambda_2}}{\mathbf{k_1^2 k_2^2}} f_{abc}\tau^a \otimes \tau^b.$$

The contributions from Fig. 3.5(b) and (c) in this regime are

$$-g^3 2s\frac{1}{\mathbf{k_2^2}}2p_1^\sigma\left[\frac{\tau^b\tau^c}{(p_1-k_1+k_2)^2}+\frac{\tau^c\tau^b}{(p_1-k_2)^2}\right]\otimes\tau^b\delta_{\lambda_1'\lambda_1}\delta_{\lambda_2'\lambda_2}.$$

Now $(p_1-k_1+k_2)^2 \approx s\lambda_2$ $(s \gg \mathbf{k}^2)$ and similarly $(p_1-k_2)^2 \approx -s\lambda_2$, so this contribution becomes

$$\begin{aligned} & -g^3 2s\frac{2p_1^\sigma}{\mathbf{k_2^2}\lambda_2 s}[\tau^b,\tau^c]\otimes\tau^b\delta_{\lambda_1'\lambda_1}\delta_{\lambda_2'\lambda_2} \\ = \quad & -ig^3 2s\frac{2p_1^\sigma}{\mathbf{k_2^2}\lambda_2 s}f_{abc}\tau^a\otimes\tau^b\delta_{\lambda_1'\lambda_1}\delta_{\lambda_2'\lambda_2}. \end{aligned}$$

Similarly the contributions from Fig. 3.5(d) and (e) are given by

$$-ig^3 2s\frac{2p_2^\sigma}{\mathbf{k_1^2}\rho_1 s}f_{abc}\tau^a\otimes\tau^b\delta_{\lambda_1'\lambda_1}\delta_{\lambda_2'\lambda_2}.$$

Although the contributions from Fig. 3.5(b) and (c) do not contain the denominator $\mathbf{k_1^2}$ and likewise the contributions from Fig. 3.5(d) and (e) do not contain the denominator $\mathbf{k_2^2}$, it is convenient to write all these contributions as though they all contained both of these denominators (multiplying by $\mathbf{k_1^2}$ or $\mathbf{k_2^2}$ where necessary) so that they may combined into an effective (left half of a) ladder, shown in Fig. 3.6.

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The complete amplitude is

$$\mathcal{A}^{(8)\sigma}_{2\to 3} = -\frac{2ig^3 2p_1^\mu p_2^\nu}{\mathbf{k}_1^2\mathbf{k}_2^2}\delta_{\lambda_1'\lambda_1}\delta_{\lambda_2'\lambda_2} f_{abc}\tau^a \otimes \tau^b \Gamma^\sigma_{\mu\nu}(k_1, k_2), \tag{3.10}$$

where $\Gamma^\sigma_{\mu\nu}(k_1, k_2)$ is an effective (non-local) vertex given by

$$\Gamma^\sigma_{\mu\nu}(k_1, k_2) = \frac{2p_{2\mu}p_{1\nu}}{s}\left[\left(\rho_1 + \frac{2\mathbf{k}_1^2}{\lambda_2 s}\right)p_1^\sigma \quad + \quad \left(\lambda_2 + \frac{2\mathbf{k}_2^2}{\rho_1 s}\right)p_2^\sigma \right.$$
$$\left. - \, (k_1 + k_2)^\sigma_\perp\right]. \tag{3.11}$$

This vertex is said to be 'non-local' since it encodes the denominators of the propagators of Fig. 3.5(b–e). The dark blob in Fig. 3.6 represents the effective vertex.

We have been working in the Feynman gauge. Nevertheless the effective vertex $\Gamma^\sigma_{\mu\nu}(k_1, k_2)$ is gauge invariant. It can easily be shown to obey the Ward identity†

$$(k_1 - k_2)_\sigma \Gamma^\sigma_{\mu\nu}(k_1, k_2) = 0.$$

Individual graphs in Fig. 3.5 are gauge dependent, but the sum is gauge invariant.

It is fun to notice (and will be useful later when we consider higher order graphs) that we can exploit the gauge invariance in such a way that only the genuine ladder-type graph (Fig. 3.5(a)) contributes in leading logarithm order. If we remove the lower quark line from the graphs in Fig. 3.5 and write the amplitude as $\mathcal{M}^\sigma_\tau(k_1, k_2)$ (see Fig. 3.7), then since all but the bottom gluon are on mass-shell we have the Ward identity

$$k_2^\tau \mathcal{M}^\sigma_\tau(k_1, k_2) = 0. \tag{3.12}$$

Now since the component of momenta proportional to p_2^τ in $\mathcal{M}^\sigma_\tau(k_1, k_2)$ is small we can neglect it and rewrite Eq.(3.12) as

$$\lambda_2 p_2^\tau \mathcal{M}^\sigma_\tau(k_1, k_2) + k_{2\perp}^\tau \mathcal{M}^\sigma_\tau(k_1, k_2) = 0.$$

In the eikonal approximation we have (reinstating the lower quark line) for the contributions from Fig. 3.5(a),(b) and (c)

$$\mathcal{A}^{(8)\sigma}_{abc} = 2p_2^\tau \mathcal{M}^\sigma_\tau(k_1, k_2)$$

† Actually the Ward identity is only exact when the vertical gluon lines are on mass-shell. In fact these lines are off-shell by $\mathbf{k}_1^2$ and $\mathbf{k}_2^2$. However, since these (squared) transverse momenta are small compared with $\lambda_2 s$ or $\rho_1 s$ the identity is obeyed at the order to which we are working.

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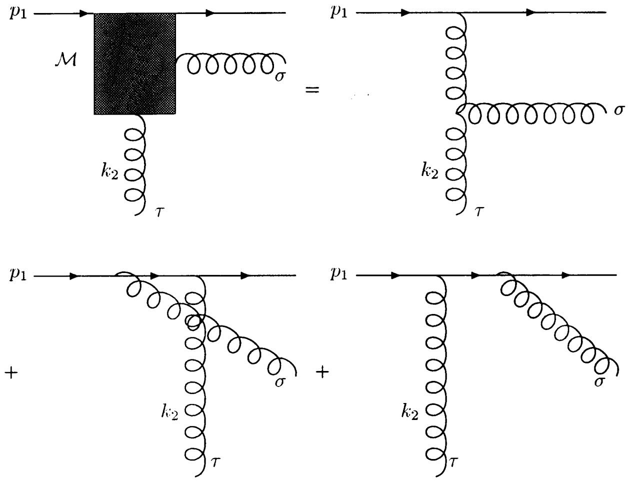

Fig. 3.7

(where we have dropped the colour factor and the coupling constant). This may be rewritten as

$$\mathcal{A}_{abc}^{(8)} = \frac{-2k_{2\perp}^{\tau}}{\lambda_2}\mathcal{M}_{\tau}^{\sigma}(k_1, k_2). \tag{3.13}$$

Since, in the eikonal approximation, $\mathcal{M}_{\tau\sigma}$ has *no transverse components* from Fig. 3.5(b) and (c), it follows that Fig. 3.5(a) dominates.

We can of course play the same game by removing the upper quark line and write the corresponding Green function as $\mathcal{N}_{\tau}^{\sigma}(k_1, k_2)$. The amplitude for the graphs of Fig. 3.5(a),(d) and (e) can now be written

$$\mathcal{A}_{ade}^{(8)\sigma} = \frac{-2k_{1\perp}^{\tau}}{\rho_1}\mathcal{N}_{\tau}^{\sigma}(k_1, k_2) \tag{3.14}$$

and once again it is only Fig. 3.5(a) that contributes at the leading logarithm level.

Now if we replace the eikonal insertion p_2^{ν} on the lower line by $-k_{2\perp}^{\nu}/\lambda_2$ and replace the eikonal insertion p_1^{μ} on the upper line by $-k_{1\perp}^{\mu}/\rho_1$ and consider only the dominant diagram, Fig. 3.5(a), we arrive at an alternative expression for $\mathcal{A}^{(8)\sigma}_{2\to 3}$, i.e.

$$\mathcal{A}^{(8)\sigma}_{2\to 3} = -\frac{4ig^3}{\mathbf{k}_1^2\mathbf{k}_2^2}\frac{k^\mu_{1\perp}k^\nu_{2\perp}}{\rho_1\lambda_2}\delta_{\lambda_1'\lambda_1}\delta_{\lambda_2'\lambda_2}f_{abc}\tau^a\otimes\tau^b$$
$$\left[-g_{\mu\nu}(k_1+k_2)^\sigma + g^\sigma_\nu(2k_2-k_1)_\mu + g^\sigma_\mu(2k_1-k_2)_\nu\right], \quad (3.15)$$

where we have just used the ordinary triple gluon vertex. Writing k_1^μ and k_2^μ in terms of their Sudakov variables and making use of the inequalities $\rho_2 \ll \rho_1$ and $|\lambda_1| \ll |\lambda_2|$, this may be written as

$$\mathcal{A}^{(8)\sigma}_{2\to 3} = \frac{2ig^3}{\mathbf{k}_1^2\mathbf{k}_2^2}\frac{1}{\rho_1\lambda_2}\delta_{\lambda_1'\lambda_1}\delta_{\lambda_2'\lambda_2}f_{abc}\tau^a\otimes\tau^b$$
$$\left\{\left[(\mathbf{k_1}-\mathbf{k_2})^2 - 2\mathbf{k}_1^2\right]\rho_1 p_1^\sigma + \left[(\mathbf{k_1}-\mathbf{k_2})^2 - 2\mathbf{k}_2^2\right]\lambda_2 p_2^\sigma\right.$$
$$-(\mathbf{k_1}-\mathbf{k_2})^2(k_1+k_2)^\sigma_\perp$$
$$\left.+(\mathbf{k}_1^2-\mathbf{k}_2^2)((\rho_1-\rho_2)p_1^\sigma + (\lambda_1-\lambda_2)p_2^\sigma + (k_1-k_2)^\sigma_\perp)\right\}. \quad (3.16)$$

At first sight it does not appear that this works (i.e. we do not appear to be consistent with Eq.(3.11)). However, we note that the terms in the last line of Eq.(3.16) are proportional to $(k_1-k_2)^\sigma$. Since the outgoing gluon is on mass-shell it is transverse, and so terms proportional to $(k_1-k_2)^\sigma$ vanish when contracted with its polarization vector. These terms may therefore be dropped. Finally, using the on-shell condition for the outgoing gluon $(\mathbf{k_1}-\mathbf{k_2})^2 = -\rho_1\lambda_2 s$, we recover precisely Eq.(3.10).[†]

Returning now to the imaginary part of the octet exchange amplitude to order α_s^3, this is given by

$$\Im m\mathcal{A}^{(8)}_2 = \frac{-g_{\sigma\tau}}{2}\int d(P.S.^3)\mathcal{A}^{(8)\sigma}_{2\to 3}(k_1,k_2)\mathcal{A}^{\dagger(8)\tau}_{2\to 3}(k_1-q,k_2-q)$$
$$+ \text{ extra piece}, \quad (3.17)$$

where the prefactor $-g_{\sigma\tau}$ arises from the sum over polarizations of the intermediate gluon and the 'extra piece' will be explained later in this section. We can take the components of q^μ to be transverse (more precisely the longitudinal components are negligible compared with $\rho_1\sqrt{s}$, $\lambda_2\sqrt{s}$).

[†] This is not a gauge choice (we are still working in Feynman gauge), but it is a trick which exploits the gauge invariance to reduce the effective ladder (Fig. 3.6) to the genuine ladder graph Fig. 3.5(a). It will be very useful in the next section.

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We deal first with the colour factor which is

$$-f_{abc}f_{dec}(\tau^a\tau^d)\otimes(\tau^b\tau^e).$$

Anticipating that we shall be adding a contribution from the u-channel which will be equal and opposite to the s-channel contribution, but with τ^b and τ^e interchanged, we antisymmetrize in τ^b and τ^e. In other words we are 'sharing' the octet colour factor between the s-channel and u-channel contributions. We thus obtain

$$-\frac{1}{2}(f_{abc}f_{dec}-f_{aec}f_{cdb})(\tau^a\tau^d)\otimes(\tau^b\tau^e).$$

Making use of the Jacobi identity

$$f_{abc}f_{dec}+f_{aec}f_{bdc}+f_{adc}f_{ebc}=0, \tag{3.18}$$

this becomes

$$-\frac{1}{2}f_{adc}f_{cbe}(\tau^a\tau^d)\otimes(\tau^b\tau^e).$$

The structure constants are antisymmetric in a,d and b,e, so we may replace the products of the colour matrices by commutators and obtain

$$\frac{1}{8}f_{adc}f_{adf}f_{cbe}f_{gbe}\tau^f\otimes\tau^g=\frac{N^2}{8}\tau^a\otimes\tau^a. \tag{3.19}$$

The phase-space integrand can now be written:

$$\begin{aligned}
&-\frac{1}{2}\mathcal{A}^{(8)\sigma}_{2\to3}(k_1,k_2)\mathcal{A}^{\dagger(8)}_{2\to3\sigma}(k_1-q,k_2-q)\\
&=\ -\frac{g^6N^2}{16}\tau^a\otimes\tau^a\delta_{\lambda_1'\lambda_1}\delta_{\lambda_2'\lambda_2}\frac{16p_1^\mu p_2^\nu p_1^{\mu'}p_2^{\nu'}}{\mathbf{k}_1^2\mathbf{k}_2^2(\mathbf{k}_1-\mathbf{q})^2(\mathbf{k}_2-\mathbf{q})^2}\\
&\times\ g_{\sigma\tau}\Gamma^\sigma_{\mu\nu}(k_1,k_2)\Gamma^\tau_{\mu'\nu'}(-(k_1-q),-(k_2-q))
\end{aligned} \tag{3.20}$$

(recall that Hermitian conjugation requires the reversal of the direction of momentum in the right hand effective vertex). After a little algebra the right hand side of Eq.(3.20) becomes

$$\begin{aligned}
&-g^4\frac{N^2s}{4}\mathcal{A}_0^{(8)}\mathbf{q}^2\left[\frac{\mathbf{q}^2}{\mathbf{k}_1^2\mathbf{k}_2^2(\mathbf{k}_1-\mathbf{q})^2(\mathbf{k}_2-\mathbf{q})^2}\right.\\
&\left.-\frac{1}{\mathbf{k}_1^2(\mathbf{k}_1-\mathbf{k}_2)^2(\mathbf{k}_2-\mathbf{q})^2}-\frac{1}{\mathbf{k}_2^2(\mathbf{k}_1-\mathbf{q})^2(\mathbf{k}_1-\mathbf{k}_2)^2}\right]
\end{aligned} \tag{3.21}$$

(the factor $(\mathbf{k}_1-\mathbf{k}_2)^2$ in the denominators of the last two terms comes from replacing $\rho_1\lambda_2 s$ by $-(\mathbf{k}_1-\mathbf{k}_2)^2$). The three-body phase-space integral is

$$
\begin{aligned}
d(P.S.^3) &= \frac{1}{(2\pi)^5}\left(\frac{s}{2}\right)^2 d\rho_1\, d\rho_2\, d\lambda_1\, d\lambda_2\, d^2\mathbf{k_1}\, d^2\mathbf{k_2} \\
&\quad \times\ \delta(-\lambda_1 s - \mathbf{k_1^2})\, \delta(\rho_2 s - \mathbf{k_2^2})\, \delta(-\rho_1\lambda_2 s - (\mathbf{k_1} - \mathbf{k_2})^2)
\end{aligned}
$$

and, after performing the integration over $\rho_2, \lambda_1, \lambda_2$ (absorbing the delta functions), we obtain

$$
\begin{aligned}
\Im\mathrm{m}\mathcal{A}_2^{(8)} &= -\frac{N^2\alpha_s^2}{32\pi^3}\mathcal{A}_0^{(8)}\mathbf{q}^2\int_{\mathbf{k}^2/s}^1 \frac{d\rho_1}{\rho_1} d^2\mathbf{k_1} d^2\mathbf{k_2} \\
&\quad \times \left[\frac{\mathbf{q}^2}{\mathbf{k_1^2}\mathbf{k_2^2}(\mathbf{k_1}-\mathbf{q})^2(\mathbf{k_2}-\mathbf{q})^2} - \frac{1}{\mathbf{k_1^2}(\mathbf{k_1}-\mathbf{k_2})^2(\mathbf{k_2}-\mathbf{q})^2}\right. \\
&\quad \left. - \frac{1}{\mathbf{k_2^2}(\mathbf{k_1}-\mathbf{q})^2(\mathbf{k_1}-\mathbf{k_2})^2}\right] + \text{ extra piece.} \qquad (3.22)
\end{aligned}
$$

Some important cancellations have taken place to obtain the above expression. For example the terms in the product of the two effective vertices which give $\mathbf{k_1^2}, \mathbf{k_2^2}, (\mathbf{k_1}-\mathbf{q})^2$ or $(\mathbf{k_2}-\mathbf{q})^2$ in the numerator have cancelled. Had this not happened there would be integrals over the transverse momenta of the form

$$
\int \frac{d^2\mathbf{k_1} d^2\mathbf{k_2}}{\mathbf{k_1^2}(\mathbf{k_1}-\mathbf{q})^2\mathbf{k_2^2}}, \qquad (3.23)
$$

which is ultra-violet divergent. Of course the upper limit of the transverse momentum integrals is of order $\sqrt{s}$, so such integrals would not really diverge but would introduce a further factor of $\ln s$ (as well as the one we obtain from the integration over ρ_1). This would give an imaginary part proportional to $\ln^2 s$ and a real part proportional to $\ln^3 s$. Calculation of individual diagrams contributing to the order α_s^n correction to the tree amplitude do indeed contain terms proportional to $\alpha_s^n(\ln s)^{2n-1}$ but they cancel between graphs. In the case of QED this cancellation has been verified by explicit calculation up to four loops by McCoy & Wu (1976a–f).

The first term of Eq.(3.22) is encouraging since the integration over the transverse momenta factorizes and together with the logarithm from the integration over ρ_1 we obtain

$$
-\frac{1}{2}\pi\epsilon_G^2(t)\ln(s/\mathbf{k}^2)\mathcal{A}_0^{(8)},
$$

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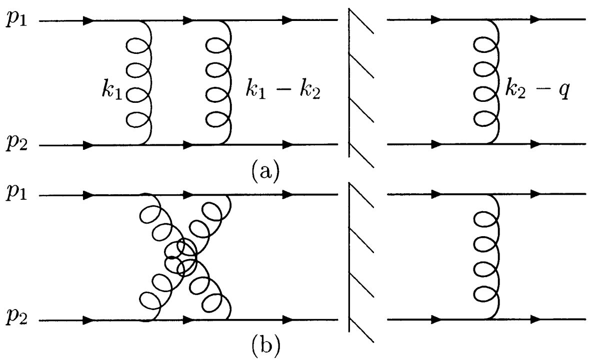

Fig. 3.8. Three gluon exchange graphs.

but the other two terms are not so nice. However, we have forgotten a contribution ('extra piece') coming from the diagrams shown in Fig. 3.8, which also contribute in leading $\ln s$. Note that in these graphs the cut only goes through the quark lines. The contribution which arises when the cut also goes through the middle gluon line of Fig. 3.8(a) has been accounted for already in the interference between Fig. 3.5(c) and (d). There are two relevant contributions – one where there is one gluon exchanged on the right of the cut (shown in Fig. 3.8) and the other where there is one gluon exchanged on the left of the cut. Each of these gives a contribution to the imaginary part of $\mathcal{A}_2^{(8)}$ of

$$-\frac{4g^4}{2}\frac{N}{4}\int d(P.S.^2)\frac{s}{\mathbf{k_2^2}}\epsilon_G(k_2^2)\ln(s/\mathbf{k}^2)\frac{s}{(\mathbf{k_2}-\mathbf{q})^2}\delta_{\lambda_1'\lambda_1}\delta_{\lambda_2'\lambda_2}$$

where we have made use of the result Eq.(3.8) for the amplitude on the left of the cut.

The colour factor, $N/4$, is obtained in the same way as in the preceding section (projecting the colour octet exchange part). Now, from Eq.(3.9)

$$\epsilon_G(k_2^2) = -\frac{N\alpha_s}{4\pi^2}\int d^2\mathbf{k_1}\frac{\mathbf{k_2^2}}{(\mathbf{k_2}-\mathbf{k_1})^2\mathbf{k_1^2}},$$

and integrating over λ_1, ρ_1 using the two-body phase-space ex-

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pression Eq.(3.4) we obtain a contribution of

$$-\frac{N^2\alpha_s^2}{32\pi^3}\mathcal{A}_0^{(8)}\mathbf{q}^2\int\frac{d^2\mathbf{k_1}d^2\mathbf{k_2}}{\mathbf{k}_1^2(\mathbf{k_1}-\mathbf{k_2})^2(\mathbf{k_2}-\mathbf{q})^2}\ln(s/\mathbf{k}^2). \tag{3.24}$$

Together with the contribution from the graphs with one gluon to the left of the cut this exactly cancels the 'unwanted' parts of Eq.(3.22) and we are left with an imaginary part:

$$\Im\mathrm{m}\mathcal{A}_2^{(8)} = -\frac{1}{2}\epsilon_G^2(t)\pi\ln(s/\mathbf{k}^2)\mathcal{A}_0^{(8)}. \tag{3.25}$$

The corresponding real part is

$$\Re\mathrm{e}\mathcal{A}_2^{(8)} = \frac{1}{4}\epsilon_G^2(t)\ln^2(s/\mathbf{k}^2)\mathcal{A}_0^{(8)}. \tag{3.26}$$

We obtain a similar contribution from the crossed diagrams with s replaced by u (and a further sign from the colour factor). Thus up to order α_s^2 we have a colour octet amplitude given in leading $\ln s$ approximation by

$$\mathcal{A}_0^{(8)}\left(1+\epsilon_G(t)\ln(s/\mathbf{k}^2)+\frac{1}{2}\epsilon_G^2(t)\ln^2(s/\mathbf{k}^2)+\cdots\right). \tag{3.27}$$

It is tempting to speculate that these are the first three terms in the expansion of $\mathcal{A}_0^{(8)}s^{\epsilon_G(t)}$. Cheng & Lo (1976) showed that this trend continues up to three loops. In the following section we shall show that it continues to work to all orders of perturbation theory. It is worth emphasizing at this point that the remarkable cancellation between the 'extra piece' from graphs in which three gluons are exchanged between the quarks and the unwanted contribution from the graphs in which three lines are cut depends crucially on the colour factors working out just right. Whereas it works for colour octet exchange, it fails for other channels, particularly for the colour singlet exchange channel which we shall need in order to study the Pomeron.

3.4 The $2 \to (n+2)$ amplitude at the tree level

It was explained earlier in this chapter that the eikonal approximation is independent of the spin of the high energy particle which emits the soft gluon. We may therefore replace the quark lines in Fig. 3.5 by gluons themselves. The eikonal approximation is still valid because of the strong ordering of the momenta. The effective vertex (Eq.(3.11)) is the vertex obtained by adding a gluon with

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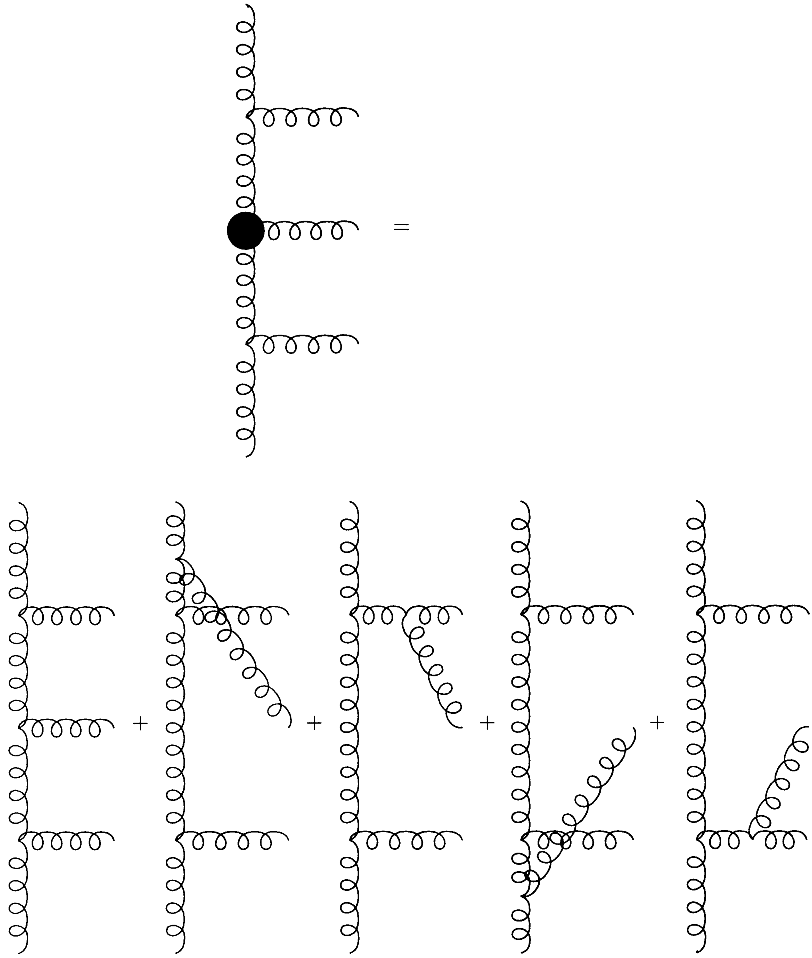

Fig. 3.9

momentum $(k_1 - k_2)^\mu$ to all the gluon lines in a gluon–gluon scattering amplitude with colour octet exchange, as shown in Fig. 3.9.

One might guess that adding more gluons generates more factors of the effective vertices (together with extra propagators for the vertical gluons), giving rise to (the left half of) an n-rung ladder with effective vertices, Γ, at each intersection, so that the amplitude for two quarks to scatter into two quarks and n gluons

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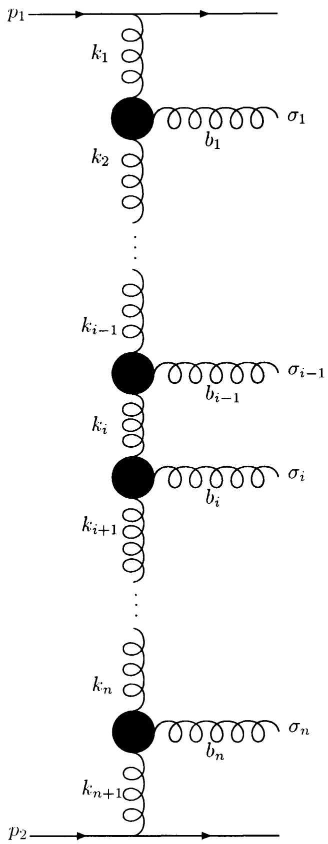

Fig. 3.10. Tree amplitude for two quarks to two quarks plus n gluons.

with octet colour exchange is shown in Fig. 3.10. It turns out that, in the kinematic regime that we are interested in, namely, where the ith emitted gluon has momentum $(k_i - k_{i+1})^\mu$ with Sudakov variables for k_i^μ and k_{i+1}^μ obeying the inequalities

$$\begin{array}{ccccccc} 1 & \gg & \rho_i & \gg & \rho_{i+1} & \gg & \mathbf{k}^2/s \\ 1 & \gg & |\lambda_{i+1}| & \gg & |\lambda_i| & \gg & \mathbf{k}^2/s, \end{array}$$

this guess is correct. Thus in this limit the amplitude for two quarks to scatter into two quarks and n gluons with colour octet

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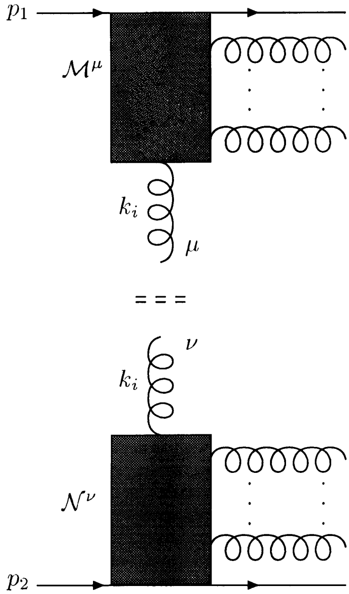

Fig. 3.11

exchange is given by

$$\begin{aligned} \mathcal{A}^{(8)\sigma_1\cdots\sigma_n}_{2\to(n+2)} &= i2s\,(g)^{n+2}\,\delta_{\lambda_1'\lambda_1}\delta_{\lambda_2'\lambda_2}G_n^{(8)} \\ &\times \frac{i}{\mathbf{k}_1^2}\prod_{i=1}^{n}\left\{\frac{2p_1^{\mu_i}p_2^{\nu_{i+1}}}{s}\Gamma^{\sigma_i}_{\mu_i\nu_{i+1}}(k_i,k_{i+1})\frac{i}{\mathbf{k}_{i+1}^2}\right\}, \end{aligned} \quad (3.28)$$

where the colour factor $G_n^{(8)}$ (for gluons with colours b_1 to b_n) is

$$G_n^{(8)}(b_1,\cdots b_n) = \prod_{i=1}^{n} f_{a_ia_{i+1}b_i}\;\tau^{a_1}\otimes\tau^{a_{n+1}}. \quad (3.29)$$

A rather elegant derivation of Eq.(3.28) is given by Gribov, Levin & Ryskin (1983). We reproduce their derivation here. The reader who is prepared to accept Eq.(3.28) on trust may skip to the next section.

Consider the amplitude for two quarks to scatter into two quarks plus n gluons. As described in the last section if we cut the ith vertical gluon, whose momentum is k_i, the amplitude separates into an upper part $\mathcal{M}_\mu(p_1, k_1, \cdots k_i)$ and a lower part

$\mathcal{N}_\nu(p_2, k_i, \cdots k_n)$ (see Fig. 3.11). Since all but the cut gluon line are on shell, these Green functions obey the Ward identities

$$k_i^\mu \mathcal{M}_\mu(p_1, k_1, \cdots k_i) = 0 \tag{3.30}$$

$$k_i^\nu \mathcal{N}_\nu(p_2, k_i \cdots k_n) = 0. \tag{3.31}$$

The largest momentum in the amplitude $\mathcal{M}_\mu$ is p_1 and so the largest part of $\mathcal{M}_\mu$ will be proportional to p_1^μ. Likewise, the largest part of $\mathcal{N}_\nu$ will be proportional to p_2^ν. Therefore in leading logarithm approximation we may rewrite Eqs.(3.30) and (3.31) as

$$k_{i\perp}^\mu \mathcal{M}_\mu(p_1, k_1, \cdots k_i) = -\lambda_i p_2^\mu \mathcal{M}_\mu(p_1, k_1, \cdots k_i)$$

$$k_{i\perp}^\nu \mathcal{N}_\nu(p_2, k_i, \cdots k_n) = -\rho_i p_1^\nu \mathcal{N}_\nu(p_2, k_i, \cdots k_n).$$

This means that we may replace the numerator of the cut gluon propagator by

$$\frac{2k_{i\perp}^\mu k_{i\perp}^\nu}{\rho_i \lambda_i s}. \tag{3.32}$$

We can cut any of the intermediate vertical gluon lines and perform the same manipulations. Therefore, we end up with an amplitude which can be obtained from (the left half of) a genuine uncrossed ladder in which the numerator of the vertical gluon lines is replaced by the expression (3.32). We associate a factor of $\sqrt{(2/s)}k_{i\perp}^\mu/\lambda_i$ with the vertex at the top of the ith vertical gluon and a factor of $\sqrt{(2/s)}k_{i\perp}^\nu/\rho_i$ with the vertex at the bottom of the ith vertical gluon. The amplitude thus becomes

$$\begin{aligned}\mathcal{A}_{2\to(n+2)}^{(8)\sigma_1\cdots\sigma_n} &= 2isg^2\frac{i}{\mathbf{k}_1^2}\delta_{\lambda_1'\lambda_1}\delta_{\lambda_2'\lambda_2}G_n^{(8)}\prod_{i=1}^n \frac{ig}{\mathbf{k}_{i+1}^2}\frac{2k_{i\perp}^{\mu_i}k_{i+1\perp}^{\nu_i}}{\lambda_{i+1}\rho_i s} \\ &\quad\times \Big[g_{\mu_i\nu_i}(-k_i - k_{i+1})^{\sigma_i} + g_{\mu_i}^{\sigma_i}(2k_i - k_{i+1})_{\nu_i} \\ &\qquad + g_{\nu_i}^{\sigma_i}(2k_{i+1} - k_i)_{\mu_i}\Big].\end{aligned} \tag{3.33}$$

We showed in the last section that

$$\begin{aligned}&\frac{2k_{i\perp}^{\mu_i}k_{i+1\perp}^{\nu_i}}{\lambda_{i+1}\rho_i s}\Big[g_{\mu_i\nu_i}(-k_i - k_{i+1})^{\sigma_i} + g_{\mu_i}^{\sigma_i}(2k_i - k_{i+1})_{\nu_i} \\ &\quad + g_{\nu_i}^{\sigma_i}(2k_{i+1} - k_i)_{\mu_i}\Big] = \frac{2p_2^{\mu_i}p_1^{\nu_i}}{s}\Gamma_{\mu_i\nu_i}^{\sigma_i}(k_i, k_{i+1}),\end{aligned} \tag{3.34}$$

plus terms proportional to $(k_i - k_{i+1})^{\sigma_i}$, which vanish because the outgoing gluon is on shell and therefore transverse. The result, Eq.(3.28), then follows.

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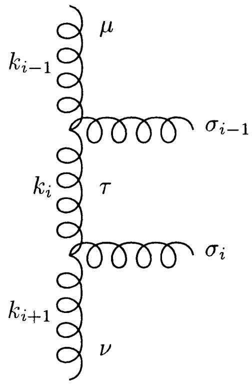

Fig. 3.12

We now need to show that, *using this gauge technique*, the diagrams which are not of the form of uncrossed ladders give contributions which are suppressed by at least one power of ρ_i/ρ_{i+1} and therefore will only contribute to sub-leading logarithm terms when the (phase-space) integrals over all ρ_i s are performed.

First of all let us look at the ith section of (the left half of) the uncrossed ladder (Fig. 3.12). The contribution from this section is proportional to the two effective vertices

$$\Gamma^{\sigma_{i-1}}_{\mu\tau}\Gamma^{\tau\sigma_i}_{\nu}.$$

The leading contribution proportional to $p_1^{\sigma_{i-1}}p_2^{\sigma_i}$ is

$$\sim \frac{2p_{2\mu}p_{1\nu}}{s}\rho_{i-1}\lambda_{i+1}p_1^{\sigma_{i-1}}p_2^{\sigma_i} \tag{3.35}$$

and the contribution proportional to $k^{\sigma_{i-1}}_{i-1\perp}k^{\sigma_i}_{i\perp}$ is

$$\sim \frac{2p_{2\mu}p_{1\nu}}{s}k^{\sigma_{i-1}}_{i-1\perp}k^{\sigma_i}_{i+1\perp}. \tag{3.36}$$

Since cross-rung graphs involve sections of the ladder where the momenta of incoming and outgoing gluons at the ith vertex are not simply k_i, k_{i+1} (see Fig. 3.13) we need to generalize the formula (3.34) for the case where the upper line entering the vertex has momentum k_i^μ and the lower line has momentum k_j^μ. This leads to

$$\frac{2k^\mu_{i\perp}k^\nu_{j\perp}}{\rho_i\lambda_j s}\left[-g_{\mu\nu}(k_i+k_j)^\sigma + g^\sigma_\mu(2k_i-k_j)_\nu + g^\sigma_\nu(2k_j-k_i)_\mu\right],$$

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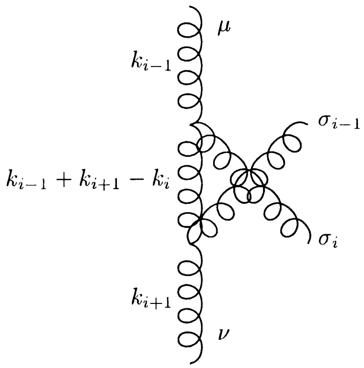

Fig. 3.13. Section of a crossed ladder diagram

which is

$$\sim \frac{2}{\rho_i \lambda_j s} \mathbf{k}^2 \left[\rho_i p_1^\sigma + \lambda_j p_2^\sigma + (k_i + k_j)_\perp^\sigma\right].$$

Since $\rho_{j-1}\lambda_j$ is of order $\mathbf{k}^2/s$ from the on-shell condition of the jth outgoing gluon, we have a contribution of order

$$2\frac{\rho_{j-1}s}{\rho_i} \left[\rho_i p_1^\sigma + \lambda_j p_2^\sigma + (k_i + k_j)_\perp^\sigma\right].$$

Now imagine a section of a crossed-rung ladder (shown in Fig. 3.13) where the middle vertical line has momentum $(k_{i-1} + k_{i+1} - k_i)$, giving rise to a denominator from its propagator which is approximately equal to $\rho_{i-1}\lambda_{i+1}s$. The two vertices have a component proportional to $p_1^{\sigma_{i-1}} p_2^{\sigma_i}$ which is of order

$$\frac{\rho_i}{\rho_{i-1}} \frac{\rho_i}{\rho_{i-1}} \frac{2p_2^\mu p_1^\nu}{s} \rho_{i-1}\lambda_{i+1} p_1^{\sigma_{i-1}} p_2^{\sigma_i}. \tag{3.37}$$

The factors of ρ_i in the numerator of Eq.(3.37) occur because $\lambda_j \approx \lambda_{i+1}$. This is true at both vertices because $|\lambda_{i+1}| \gg |\lambda_i|$ (or $|\lambda_{i-1}|$). Using the on-shell conditions we may therefore replace ρ_{j-1} by ρ_i to arrive at Eq.(3.37).

Since $\rho_i \ll \rho_{i-1}$, expression (3.37) is much smaller than the uncrossed ladder product of two effective vertices, expression (3.35). In addition to this suppression the denominator from the propagator of the intermediate line is much larger than $\mathbf{k}_\mathbf{i}^2$, which is what we obtain from the section of the ladder shown in Fig. 3.12. Thus there is a double suppression of the crossed ladder diagram. If we cross more rungs we get an even greater suppression.

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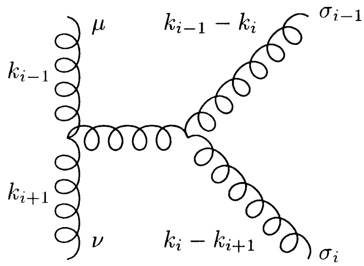

Fig. 3.14

Let us now consider a section of a graph in which two of the outgoing gluons meet at a point. Such a section of a graph involving the triple gluon vertex is shown in Fig. 3.14. Again contracting the left hand triple gluon vertex with $k^{\mu}_{i-1\perp}k^{\nu}_{i+1\perp}$, we obtain a term proportional to $k^{\sigma_{i-1}}_{i-1\perp}k^{\sigma_i}_{i\perp}$ which is of order

$$\frac{2p_1^{\mu}p_2^{\nu}}{s}\mathbf{k}^2\rho_{i-1}\lambda_{i+1}sk^{\sigma_{i-1}}_{i-1\perp}k^{\sigma_i}_{i\perp}$$

and again using the fact that $\rho_i\lambda_{i+1}$ is of order $\mathbf{k}^2/s$ this is of order

$$\frac{2p_1^{\mu}p_2^{\nu}}{s}\frac{\rho_i}{\rho_{i-1}}k^{\sigma_{i-1}}_{i-1\perp}k^{\sigma_i}_{i\perp},$$

which is suppressed relative to the equivalent term from the uncrossed ladder (expression (3.36)) by a factor of ρ_i/ρ_{i-1}. In addition to this the internal gluon propagator has a denominator which is again much larger than $\mathbf{k}^2$, so we get a double suppression.

From the four-point gluon vertex we get a section of a graph shown in Fig. 3.15. Once again the contribution from the vertex has a term proportional to $k^{\sigma_{i-1}}_{i-1\perp}k^{\sigma_i}_{i\perp}$ which is of order

$$\frac{1}{\rho_{i-1}\lambda_{i+1}s}k^{\sigma_{i-1}}_{i-1\perp}k^{\sigma_i}_{i\perp}$$

and we are missing a propagator factor of $\mathbf{k}_{\mathbf{i}}^2$ present in the section of the graph shown in Fig. 3.12. Thus this graph also gives a contribution which is suppressed relative to the uncrossed ladder contribution by a factor of order ρ_i/ρ_{i-1}.

Comparison of other components of the tensor structure (e.g. terms proportional to $p_1^{\sigma_{i-1}}p_2^{\sigma_i}$) yield similar suppression factors.

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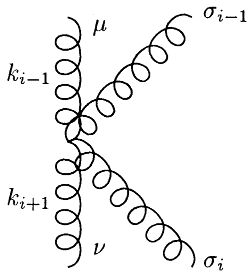

Fig. 3.15

This, then, completes the proof that the amplitude for two quarks to scatter into two quarks and n gluons via colour octet exchange is given by Eq.(3.28) in the kinematic region that leads to leading logarithms.

3.5 Absence of fermion loops

We have so far only considered outgoing gluons in addition to the two quarks present in the initial state. In principle we must also consider the production of extra fermion–antifermion pairs, since such amplitudes must be included in the dispersion relation for the imaginary part of the elastic scattering amplitude. However, these also turn out to be suppressed and do not contribute in leading logarithm approximation. The essential reason for this is that a fermion exchanged in the t-channel gives an s-dependence which is lower than that of an exchanged vector particle due to the fact that the fermion has spin $\frac{1}{2}$.

Looking at this in more detail, in Fig. 3.16(a) we display a section of a ladder in which two of the gluons are replaced by a fermion–antifermion pair. Once again we may use the gauge technique to replace the factor of p_1^μ from the upper gluon by $\sqrt{(2/s)}k_{i-1\perp}^{\mu}/\rho_{i-1}$ and the factor of p_2^ν from the lower gluon line by $\sqrt{(2/s)}k_{i+1\perp}^{\nu}/\lambda_{i+1}$. Having done this the contribution from the section shown in Fig. 3.16(a) contains terms proportional to

$$\frac{1}{\rho_{i-1}\lambda_{i+1}s}\bar{u}(k_{i-1}-k_i)\gamma\cdot k_{i-1\perp}\gamma\cdot k_i\gamma\cdot k_{i+1\perp}u(k_{i+1}-k_i).$$

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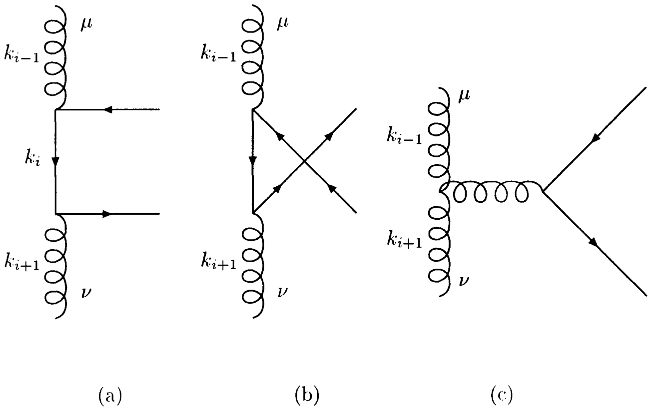

Fig. 3.16. Section of a ladder with a fermion loop.

This is of order

$$\frac{\mathbf{k}^2}{\rho_{i-1}\lambda_{i+1}s}\{k_i \cdot k_{i-1}, k_i \cdot k_{i+1}, k_i^2\}.$$

Now all the scalar products inside the braces are of order $\mathbf{k}^2$ ($\rho_{i-1}\lambda_i$ and $\rho_i\lambda_{i+1}$ are both of order $\mathbf{k}^2/s$), and the factor outside the braces is of order ρ_i/ρ_{i-1}. Thus we obtain a contribution which is suppressed by ρ_i/ρ_{i-1} compared with a typical term from the gluon ladder.

Examination of the graphs shown in Fig. 3.16(b) and (c) also give a similar suppression factor, although in these cases it is due to the presence of a hard fermion or gluon propagator.

Thus we see that it is sufficient to neglect fermion–antifermion pair production in the final state in order to obtain the imaginary part of the elastic amplitude to leading logarithm order.

3.6 Ladders within ladders

We now have an expression for the tree level amplitude for two quarks to scatter to two quarks and n gluons, which when multiplied by the conjugate amplitude and integrated over phase space contributes to the imaginary part of the 'reggeized gluon' amplitude. We now consider loop corrections. A strong hint on how to handle these is given by the fact that it was necessary to consider

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the graphs of Fig. 3.8 at the two-loop level in order to obtain a result that looks like the first three terms of the expansion of the required reggeized form. The subgraphs on the left of the cut in Fig. 3.8 may be viewed as the beginning of an expansion of a ladder itself.

The upshot of all this is that the imaginary part of the octet exchange amplitude in leading $\ln s$ is

> a superposition of n-rung ladders with effective vertices at each rung, whose vertical lines are a superposition of n-rung ladders with effective vertices at each rung, whose vertical lines are a superposition of n-rung ladders with effective vertices at each rung, whose vertical lines are a superposition of n-rung ladders with effective vertices at each rung whose vertical lines are a superposition of n-rung ladders with effective vertices at each rung ...

(n runs from 0 to ∞). The effect of these ladders is to 'reggeize' the gluon, i.e. if we consider the ith section of the ladder (see Fig. 3.12) the square of the centre-of-mass energy coming into this section is

$$\hat{s}_i = (k_{i-1} - k_{i+1})^2 \approx -\rho_{i-1}\lambda_{i+1}s = \frac{\rho_{i-1}}{\rho_i}(\mathbf{k_i} - \mathbf{k_{i+1}})^2 \tag{3.38}$$

(where in the last step we have used the on-shell condition for the ith outgoing gluon).

The reggeization simply means that the propagator of the ith vertical gluon (in Feynman gauge) is replaced by

$$\tilde{D}_{\mu\nu}(\hat{s}_i, k_i^2) = \frac{ig_{\mu\nu}}{\mathbf{k_i^2}}\left(\frac{\hat{s}_i}{\mathbf{k}^2}\right)^{\epsilon_G(k_i^2)}. \tag{3.39}$$

Since all the transverse momenta are of the same order we may replace $(\mathbf{k_i} - \mathbf{k_{i+1}})^2$ in Eq.(3.38) by a typical squared transverse momentum, $\mathbf{k}^2$, and rewrite this as

$$\tilde{D}_{\mu\nu}(\hat{s}_i, k_i^2) = \frac{ig_{\mu\nu}}{\mathbf{k_i^2}}\left(\frac{\rho_{i-1}}{\rho_i}\right)^{\epsilon_G(k_i^2)}. \tag{3.40}$$

We shall establish the validity of this proposition by a 'bootstrap' method. Encouraged by the results of the first few orders in perturbation theory, we shall assume that Eq.(3.40) is true. This will enable us to establish an integral equation for the (Mellin transform of) the amplitude for colour octet exchange. The integral equation has a solution in which the Mellin transform has a

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pole at $\omega = \epsilon_G(t)$, (implying an $s^{\alpha_G(t)}$ behaviour) and this justifies the assumption of reggeization used to establish the integral equation in the first place. It demonstrates the self-consistency of the proposition and, together with the results of the first few orders in perturbation theory, provides an inductive derivation of reggeization valid to all orders in perturbation theory.

The horizontal gluon rungs are attached to the vertical lines via effective vertices $\Gamma^{\sigma}_{\mu\nu}(k_i, k_{i+1})$ and so the amplitude for two quarks to scatter into two quarks plus n gluons via colour octet exchange becomes

$$\begin{aligned} \mathcal{A}^{(8)\sigma_1\cdots\sigma_n}_{2\to(n+2)} &= i2sg^{n+2}\delta_{\lambda_1'\lambda_1}\delta_{\lambda_2'\lambda_2}G^{(8)}_n \frac{i}{\mathbf{k}_1^2}\left(\frac{1}{\rho_1}\right)^{\epsilon_G(k_1^2)} \\ &\times \prod_{i=1}^{n} \frac{2p_1^{\mu_i}p_2^{\nu_{i+1}}}{s}\Gamma^{\sigma_i}_{\mu_i\nu_{i+1}}(k_i, k_{i+1})\frac{i}{\mathbf{k}_{i+1}^2}\left(\frac{\rho_i}{\rho_{i+1}}\right)^{\epsilon_G(k_{i+1}^2)} . \quad (3.41) \end{aligned}$$

In actual fact this is the **multi-Regge exchange** amplitude for the $2 \to 2+n$ amplitude via the exchange of $n+1$ reggeized particles with Regge trajectory $\alpha_G(k_i^2)$. This can be established using techniques of Regge theory, exploiting unitarity in all possible final state sub-channels. This long calculation was performed by Bartels (1975) and is outlined by Lipatov (1989) and we refer the reader to the literature for further details. We shall now proceed to demonstrate the self-consistency of the reggeization *ansatz.*

3.7 The integral equation

The imaginary part of the octet exchange amplitude is given by (see Fig. 3.17 in which a dash on the vertical gluon lines indicates that they are reggeized gluons)

$$\begin{aligned} \Im\mathrm{m}\mathcal{A}^{(8)}(s,t) &= \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n \int d(P.S.^{(n+2)}) \Big(\mathcal{A}^{(8)\sigma_1\cdots\sigma_n}_{2\to(n+2)}(k_1, \cdots k_n) \\ &\times \mathcal{A}^{(8)\dagger}_{2\to(n+2)\sigma_1\cdots\sigma_n}(k_1 - q, \cdots k_n - q) \Big) , \quad (3.42) \end{aligned}$$

and $d(P.S.^{(n+2)})$ is the $(n+2)$-body phase space given in Eq.(2.27) in the preceding chapter.

The colour factor is readily calculated using repetitions of the Jacobi identity (Eq.(3.18)) as was done to obtain the colour factor

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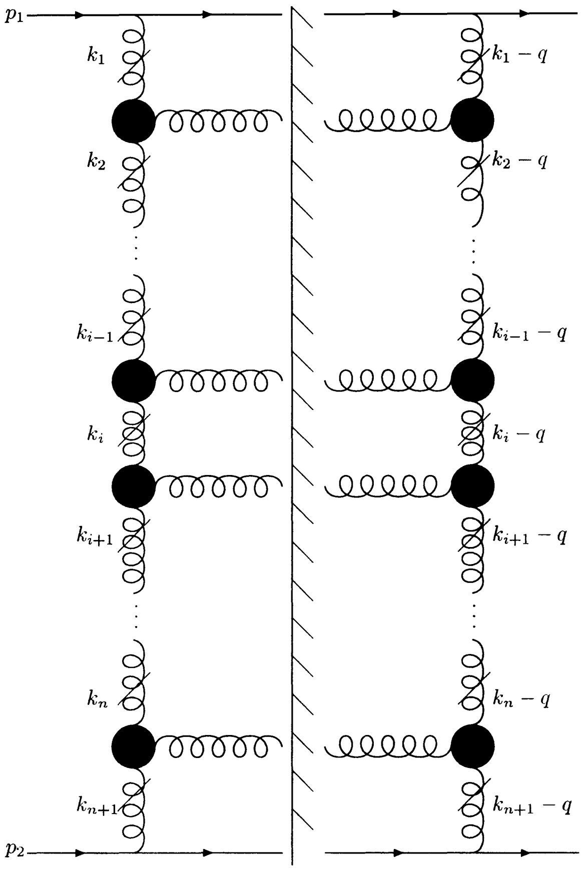

Fig. 3.17. n-rung ladder contribution to imaginary part of amplitude. The dashes on the vertical gluon lines indicate that they are reggeized gluons.

at the two-loop level (Eq.(3.19)). The result is

$$G_n^{(8)}(b_1, \cdots b_n) \times G_n^{(8)}(b_1, \cdots b_n) = \left(\frac{N}{2}\right)^n \frac{N}{4} \tau^a \otimes \tau^a.$$

Performing the contractions of the effective vertices we obtain

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$$
\begin{aligned}
\Im m\mathcal{A}^{(8)}(s,t) &= \sum_{n=0}^{\infty}\int d(P.S.^{(n+2)}) \\
&\times \frac{N}{4}\mathcal{A}_0^{(8)}(s,t)(-N)^n \frac{g^2 s\mathbf{q}^2}{\mathbf{k_1^2}(\mathbf{k_1}-\mathbf{q})^2}\left(\frac{1}{\rho_1}\right)^{\epsilon_G(k_1^2)+\epsilon_G((k_1-q)^2)} \\
&\times \prod_{i=1}^{n}\left[\frac{g^2}{\mathbf{k_{i+1}^2}(\mathbf{k_{i+1}}-\mathbf{q})^2}\left(\mathbf{q}^2-\frac{\mathbf{k_i^2}(\mathbf{k_{i+1}}-\mathbf{q})^2+(\mathbf{k_i}-\mathbf{q})^2\mathbf{k_{i+1}^2}}{(\mathbf{k_i}-\mathbf{k_{i+1}})^2}\right)\right. \\
&\qquad \left.\times\left(\frac{\rho_i}{\rho_{i+1}}\right)^{\epsilon_G(k_{i+1})^2+\epsilon_G((k_{i+1}-q)^2)}\right] \qquad (3.43)
\end{aligned}
$$

(for $n = 0$ the product in Eq.(3.43) is replaced by 1). The reader can check that, apart from the reggeization factors, the $n = 0$ and $n = 1$ terms correspond to (the s-channel contributions to) $\Im m\mathcal{A}_1^{(8)}$ (Eq.(3.5)) and $\Im m\mathcal{A}_2^{(8)}$ (Eq.(3.22)) respectively.

We note that the integrations over the ρ_i are nested and the best way to unravel them is to take the Mellin transform and make use of the convolution formula, Eq.(A.2.7). To this end we define a quantity $\mathcal{F}^{(8)}(\omega,\mathbf{k},\mathbf{q})$ by

$$
\int\left(\frac{\Im m\mathcal{A}^{(8)}(s,t)}{\mathcal{A}_0^{(8)}(s,t)}\right)\left(\frac{s}{\mathbf{k}^2}\right)^{-\omega-1} d\left(\frac{s}{\mathbf{k}^2}\right) = \int\frac{d^2\mathbf{k}}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2}\mathcal{F}^{(8)}(\omega,\mathbf{k},\mathbf{q}). \qquad (3.44)
$$

The Mellin transform and integration over the ρ_i then leaves us with

$$
\begin{aligned}
\mathcal{F}^{(8)}(\omega,\mathbf{k},\mathbf{q}) &= \sum_{n=0}^{\infty}\frac{\pi}{2}\left(\frac{\alpha_s N}{4\pi^2}\right)^{n+1}(-1)^n \\
&\times \frac{\mathbf{q}^2}{(\omega-\epsilon_G(-\mathbf{k}^2)-\epsilon_G(-(\mathbf{k}-\mathbf{q})^2))}\int d^2\mathbf{k_{n+1}} \\
&\times \prod_{i=1}^{n}\left[\int\frac{d^2\mathbf{k_i}}{\mathbf{k_i^2}(\mathbf{k_i}-\mathbf{q})^2}\frac{1}{(\omega-\epsilon_G(-\mathbf{k_i^2})-\epsilon_G(-(\mathbf{k_i}-\mathbf{q})^2))}\right. \\
&\times \left.\left(\mathbf{q}^2-\frac{\mathbf{k_i^2}(\mathbf{k_{i+1}}-\mathbf{q})^2+\mathbf{k_{i+1}^2}(\mathbf{k_i}-\mathbf{q})^2}{(\mathbf{k_i}-\mathbf{k_{i+1}})^2}\right)\right]\delta^2(\mathbf{k}-\mathbf{k_{n+1}}). \qquad (3.45)
\end{aligned}
$$

This sum of all ladders is most easily treated by obtaining an integral equation for $\mathcal{F}^{(8)}(\omega,\mathbf{k},\mathbf{q})$. This integral equation, shown

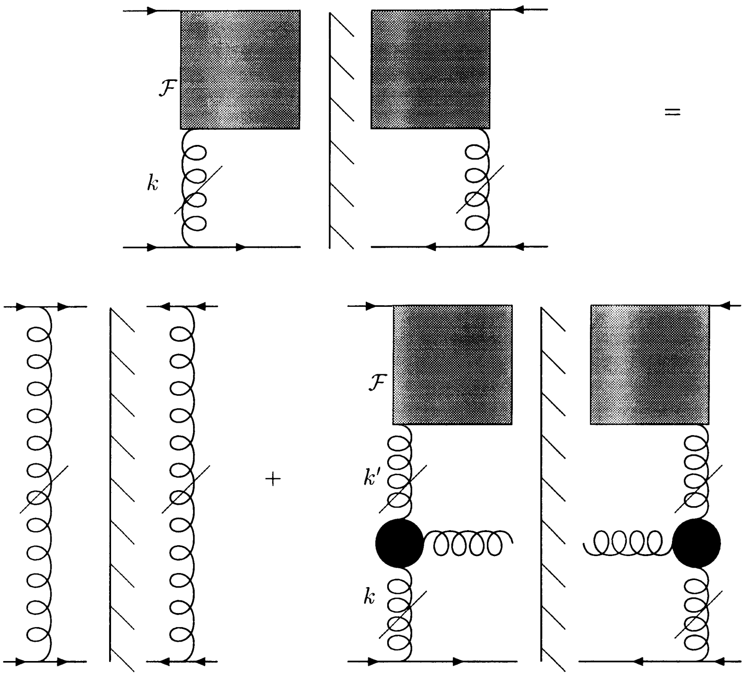

Fig. 3.18. Integral equation for imaginary part of the octet exchange amplitude.

diagrammatically in Fig. 3.18 (where again a dash on a gluon line indicates that it is a reggeized gluon), is

$$\begin{aligned}\mathcal{F}^{(8)}(\omega, \mathbf{k}, \mathbf{q}) &= \frac{\pi}{2}\frac{\alpha_s N}{4\pi^2}\frac{\mathbf{q}^2}{(\omega - \epsilon_G(-\mathbf{k}^2) - \epsilon_G(-(\mathbf{k}-\mathbf{q})^2))} \\ &- \frac{\alpha_s N}{4\pi^2}\int d^2\mathbf{k}'\frac{\mathcal{F}^{(8)}(\omega, \mathbf{k}', \mathbf{q})}{(\omega - \epsilon_G(-\mathbf{k}^2) - \epsilon_G(-(\mathbf{k}-\mathbf{q})^2))} \\ &\times \frac{1}{\mathbf{k}'^2(\mathbf{k}'-\mathbf{q})^2}\left(\mathbf{q}^2 - \frac{\mathbf{k}^2(\mathbf{k}'-\mathbf{q})^2 + \mathbf{k}'^2(\mathbf{k}-\mathbf{q})^2}{(\mathbf{k}-\mathbf{k}')^2}\right). \qquad (3.46)\end{aligned}$$

The first term represents the exchange of two reggeized gluons with no rungs on the ladder. The second term represents the effect of adding a rung which couples with effective vertices to the vertical lines, which are themselves reggeized, and serves to build up the sum of all ladders (as discussed in Section 2.5).

This rather forbidding looking equation in actual fact has a rather simple solution in which $\mathcal{F}^{(8)}(\omega, \mathbf{k}, \mathbf{q})$ is independent of $\mathbf{k}$. To see this we multiply by $(\omega - \epsilon_G(-\mathbf{k}^2) - \epsilon_G(-(\mathbf{k}-\mathbf{q})^2))$ to obtain

$$\begin{aligned}(\omega - \epsilon_G(-\mathbf{k}^2) - \epsilon_G(-(\mathbf{k}-\mathbf{q})^2))\mathcal{F}^{(8)}(\omega, \mathbf{k}, \mathbf{q}) \\ = \frac{\pi}{2}\frac{\alpha_s N \mathbf{q}^2}{4\pi^2} - \frac{\alpha_s N}{4\pi^2}\int d^2\mathbf{k}' \mathcal{F}^{(8)}(\omega, \mathbf{k}', \mathbf{q}) \\ \times \left(\frac{\mathbf{q}^2}{\mathbf{k}'^2(\mathbf{k}'-\mathbf{q})^2} - \frac{\mathbf{k}^2}{\mathbf{k}'^2(\mathbf{k}-\mathbf{k}')^2} - \frac{(\mathbf{k}-\mathbf{q})^2}{(\mathbf{k}'-\mathbf{q})^2(\mathbf{k}-\mathbf{k}')^2}\right).\end{aligned} \tag{3.47}$$

Now we note that

$$\epsilon_G(-\mathbf{k}^2) = -\frac{\alpha_s N}{4\pi^2}\int d^2\mathbf{k}' \frac{\mathbf{k}^2}{\mathbf{k}'^2(\mathbf{k}-\mathbf{k}')^2} \tag{3.48}$$

$$\epsilon_G(-(\mathbf{k}-\mathbf{q})^2) = -\frac{\alpha_s N}{4\pi^2}\int d^2\mathbf{k}' \frac{(\mathbf{k}-\mathbf{q})^2}{(\mathbf{k}'-\mathbf{q})^2(\mathbf{k}-\mathbf{k}')^2}. \tag{3.49}$$

Thus if $\mathcal{F}^{(8)}(\omega, \mathbf{k}, \mathbf{q})$ is independent of $\mathbf{k}$ we have

$$\begin{aligned}(\omega - \epsilon_G(-\mathbf{k}^2) - \epsilon_G(-(\mathbf{k}-\mathbf{q})^2))\mathcal{F}^{(8)}(\omega, .., \mathbf{q}) &= \frac{\pi}{2}\frac{\alpha_s N \mathbf{q}^2}{4\pi^2} \\ +(\epsilon_G(-\mathbf{q}^2) - \epsilon_G(-\mathbf{k}^2) - \epsilon_G(-(\mathbf{k}-\mathbf{q})^2))\mathcal{F}^{(8)}(\omega, .., \mathbf{q}).\end{aligned} \tag{3.50}$$

The terms with factors of $\epsilon_G(-\mathbf{k}^2)$ and $\epsilon_G(-(\mathbf{k}-\mathbf{q})^2)$ cancel out exactly. It is worth emphasizing that this remarkable cancellation only works in the octet exchange channel. It depends crucially on the fact that the colour factor from the addition of an extra rung is $N/2$. It is the generalization of the seemingly miraculous cancellation of those terms corresponding to Figs. 3.5 and 3.8 which spoiled the exponentiation up to order α_s^2.

The solution to Eq.(3.50) is simply

$$\mathcal{F}^{(8)}(\omega, .., \mathbf{q}) = \frac{\pi}{2}\frac{\alpha_s N \mathbf{q}^2}{4\pi^2}\frac{1}{(\omega - \epsilon_G(-\mathbf{q}^2))}, \tag{3.51}$$

so that the imaginary part of the amplitude (inserting into Eq.(3.44) and recalling that $t = -\mathbf{q}^2$) is

$$\Im m\mathcal{A}^{(8)}(s,t) = -\frac{\pi}{2}\epsilon_G(t)\left(\frac{s}{\mathbf{k}^2}\right)^{\epsilon_G(t)} \mathcal{A}_0^{(8)}. \tag{3.52}$$

The analytic function of which this is the imaginary part is

$$= \frac{1}{2}\left(\frac{-s}{\mathbf{k}^2}\right)^{\epsilon_G(t)} \mathcal{A}_0^{(8)}.$$

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Since $\mathcal{A}_0^{(8)}$ is proportional to s we have an s-dependence of

$$-(-s)^{1+\epsilon_G(t)}.$$

Adding the contribution from the u-channel graphs and using $u \approx -s$ we obtain a total expression for the octet exchange amplitude from summing the leading $\ln s$ to all orders in perturbation theory given by

$$\mathcal{A}^{(8)} = 8\pi\alpha_s \frac{\mathbf{k}^2}{t} \tau^a \otimes \tau^a \delta_{\lambda_1\lambda_1'} \delta_{\lambda_2\lambda_2'} \left(\frac{s}{\mathbf{k}^2}\right)^{\alpha_G(t)} \frac{1 - e^{i\pi\alpha_G(t)}}{2}, \quad (3.53)$$

where

$$\alpha_G(t) = 1 + \epsilon_G(t).$$

This is a Regge trajectory of odd signature and we have justified the *ansatz* made in Eq.(3.39) for the 'reggeized' gluon propagator.

Although $\alpha_G(t)$ is infra-red divergent, if we regularize using dimensional regularization, i.e. if we perform the integration over transverse momenta in $2 + \epsilon$ dimensions, then we have†

$$\alpha_G(-\mathbf{q}^2) = 1 - \frac{N\alpha_s \mathbf{q}^2}{(2\pi)^{2+\epsilon}} \int \frac{d^{2+\epsilon}\mathbf{k}}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2} = 1 - \frac{N\alpha_s}{4\pi} \frac{2(\mathbf{q}^2)^{\epsilon/2}}{\epsilon},$$

such that $\alpha_G(0) = 1$ and we find that the massless, spin-1 gluon does indeed lie on the trajectory. This has been shown by Fadin, Kuraev & Lipatov (1976), Frankfurt & Sherman (1976), Tyburski (1976), and Cheng & Lo (1976) to be true also in the case where the gauge group is broken spontaneously so that the 'gluon' acquires a mass, M, and it turns out that $\alpha_G(M^2) = 1$. In this case graphs involving Higgs bosons (which do not occur in the treatment described in this chapter) play a crucial role.

We have now done most of the hard work. In the next chapter we shall be using these reggeized gluons to construct the perturbative Pomeron.

3.8 Summary

• The first few terms in the perturbative expansion for the amplitude involving spin-1, colour octet exchange suggest that the gluon reggeizes, i.e. its propagator is given by Eq.(3.1) with $\epsilon_G(t)$ given by Eq.(3.9). After regularization of the infra-red divergence

† We have absorbed $\ln 4\pi$ and the Euler constant γ_E into $1/\epsilon$.

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we find $\alpha_G(0)\,(= 1 + \epsilon_G(0)) = 1$ so that the gluon lies on this trajectory.

- The two-quark to two-quark plus n-gluon amplitude at the tree level, in the kinematic regime which leads to leading $\ln s$ in the octet exchange amplitude, is given by the left half of uncrossed ladder diagrams with effective vertices, $\Gamma^{\sigma}_{\mu\nu}$, given by Eq.(3.11) coupling the rungs of the ladder and the vertical gluon lines.
- Loop corrections in leading $\ln s$ approximation are introduced by replacing the propagators for the vertical gluon lines of the ladder by reggeized gluons.
- An integral equation for the Mellin transform of the imaginary part of the octet exchange amplitude can be obtained using a dispersion relation involving these ladders.
- The integral equation has a solution which consists of a simple pole at $\omega = \epsilon_G(t)$, thereby justifying the proposition that the gluon reggeizes.