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THE JOINT DISTRIBUTION OF THE RIEMANN ZETA - FUNCTION

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In the paper the asymptotic distribution of $\left(|\zeta(s)|, \zeta(s)\right)$, where $\zeta(s)$ is the Riemann zeta - function, in the sense of weak convergence of probability measures is considered. For this, the continuity theorems for probability measures on $\mathbb{R} \times \mathbb{C}$ are used. Some aspects of the dependence of $|\zeta(s)|$ and $\zeta(s)$ are also discussed.

1. INTRODUCTION

Throughout the paper, $\mathbb{N}, \mathbb{Z}, \mathbb{R}$ and $\mathbb{C}$ denote the sets of positive integers, integers, real and complex numbers, respectively. Let $s = \sigma + it$ be a complex variable, and let $\zeta(s)$, as usual, denote the Riemann zeta - function defined, for $\sigma > 1$, by

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s},$$

and by analytic continuation elsewhere. It is well known that the function $\zeta(s)$ has a limit distribution in the sense of the weak convergence of probability measures, see [**4, 5, 6, 9, 13, 14**]. For more precise statements we need some notation. Denote by meas$\{A\}$ the Lebesgue measure of a measurable set $A \subseteq \mathbb{R}$, by $\mathcal{B}(S)$ the class of Borel sets of the space S, and let

$$\nu_T(\ldots) = \frac{1}{T} \operatorname{meas} \left\{t \in [0, T] : \ldots\right\},$$

where in place of dots a condition satisfied by t is to be written. Moreover, let

$$\gamma = \left\{s \in \mathbb{C} : |s| = 1\right\}$$

be the unit circle on $\mathbb{C}$, and

$$\Omega = \prod_p \gamma_p,$$

where $\gamma_p = \gamma$ for all primes p. With the product topology and pointwise multiplication Ω is a compact topological Abelian group. Therefore, on $\left(\Omega, \mathcal{B}(\Omega)\right)$ the probability Haar

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measure m_H exists, and this leads to a probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by $\omega(p)$ the projection of $\omega \in \Omega$ to the coordinate space γ_p. Let, for $\sigma > 1/2$,

$$\zeta(\sigma, \omega) = \prod_p \Big(1 - \frac{\omega(p)}{p^\sigma}\Big)^{-1}.$$

Then $\zeta(\sigma, \omega)$ is a complex-valued random element defined on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Let P_ξ stand for the distribution of the random element ξ, so in the case of $\zeta(\sigma, \omega)$

$$P_\zeta(A) = m_H\big(\omega \in \Omega : \zeta(\sigma, \omega) \in A\big), \qquad A \in \mathcal{B}(\mathbb{C}).$$

THEOREM A. *Let $\sigma > 1/2$ be fixed. Then the probability measure*

$$\nu_T\big(\zeta(\sigma + it) \in A\big), \qquad A \in \mathcal{B}(\mathbb{C}),$$

converges weakly to P_ζ as $T \to \infty$.

A direct proof of Theorem A for Dirichlet L- functions is given in [**2**], it also follows from a limit theorem in the space $M(D)$ of functions meromorphic on $D = \{s \in \mathbb{C} : \sigma > 1/2\}$ equipped with the topology of uniform convergence on compacta, see [**1**] or, more generally, [**10, 11, 12**], since the function $h : M(D) \to \mathbb{C}$ defined by

$$h(f) = f(\sigma), f \in M(D),$$

is continuous.

THEOREM B. *Let $\sigma > 1/2$ be fixed. Then the probability measure*

$$\nu_T\Big(\big|\zeta(\sigma + it)\big| \in A\Big), \qquad A \in \mathcal{B}(\mathbb{R}),$$

converges weakly to $P_{|\zeta|}$ as $T \to \infty$.

The function $h : \mathbb{C} \to \mathbb{R}$ given by $h(s) = |s|$, clearly, is continuous, therefore Theorem B is an immediate consequence of Theorem A.

Now let, for $A \in \mathcal{B}(\mathbb{R})$,

$$L(A) = \begin{cases} \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_A e^{-(\log^2 u)/2} \frac{du}{u}, & A \in (0, \infty), \\ 0, & A \in (-\infty, 0]. \end{cases}$$

L is the lognormal probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

THEOREM C. *The probability measure*

$$\nu_T\bigg(\Big|\zeta\Big(\frac{1}{2} + it\Big)\Big|^{(2^{-1} \log\log T)^{-1/2}} \in A\bigg), \qquad A \in \mathcal{B}(\mathbb{R}),$$

converges weakly to L as $T \to \infty$.

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Theorem C in terms of distribution functions is stated in [9].

Let P be a probability measure on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$. Then its characteristic transform $w(\tau, k), \tau \in \mathbb{R}, k \in \mathbb{Z}$, is defined in [9] by

$$w(\tau, k) = \int_{\mathbb{C}\setminus\{0\}} |z|^{i\tau} e^{ik\arg z} dP. \tag{1}$$

A probability measure on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ is called lognormal if its characteristic transform is

$$\exp\left\{-\frac{\tau^2 + k^2}{2}\right\}, \qquad \tau \in \mathbb{R}, k \in \mathbb{Z}.$$

Let $P_n, n \in \mathbb{N}$, and P be probability measures on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$. We say that P_n converges weakly in the sense of $\mathbb{C}$ to P as $n \to \infty$ if P_n converges weakly to P as $n \to \infty$ and

$$P_n(\{0\}) \underset{n\to\infty}{\longrightarrow} P(\{0\})$$

(see [9]).

Theorem D. *The probability measure*

$$\nu_T\left(\left(\zeta\left(\frac{1}{2} + it\right)\right)^{(2^{-1}\log\log T)^{-1/2}} \in A\right), \qquad A \in \mathcal{B}(\mathbb{C}),$$

converges weakly in the sense of $\mathbb{C}$ *to the lognormal probability measure on* $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ *as* $T \to \infty$.

A theorem, similar to Theorem D, when $\zeta((1/2) + 1/(\log T) + it)$ instead of $\zeta((1/2) + it)$ is considered, can be found in [9]. Theorem D can be obtained by the same way. Also, Theorem D is a consequence of Selberg's result for $\zeta((1/2) + it)$, see, for example, [6]. Note that, for $\zeta(s) \neq 0, a \neq 0, \zeta^a(s)$ is understood as $\exp\{a \log\zeta(s)\}$, where $\arg\zeta(s)$ in $\log\zeta(s)$ is defined by continuous displacement from the point $s = 2$ along the straight lines connecting the points $s = 2, s = 2 + it$ and $s = \sigma + it$. Since

$$\nu_T(\zeta(\sigma + it) = 0) = o(1), \qquad T \to \infty,$$

we set, for simplicity, $\zeta^a(\sigma + it) = 0$ if $\zeta(\sigma + it) = 0$.

Our aim is to consider the joint distribution of $|\zeta(s)|$ and $\zeta(s)$, and to investigate a "measure" of their asymptotic dependence.

Let $\mathbf{X} = \mathbb{R} \times \mathbb{C}$. In Section 2 we shall consider the weak convergence of probability measures on $(\mathbf{X}, \mathcal{B}(\mathbf{X}))$. For points of $\mathbf{X}$, we use the notation $(x, re^{i\varphi})$. Let P be a probability measure on $(\mathbf{X}, \mathcal{B}(\mathbf{X}))$, and

$$P_{\mathbb{R}}(A) = P(A \times \mathbb{C}), \qquad A \in \mathcal{B}(\mathbb{R}).$$

The functions

$$w(\tau) = \int_{\mathbb{R}} e^{i\tau x} dP_{\mathbb{R}}, \qquad \tau \in \mathbb{R}, \tag{2}$$

and

$$w(\tau_1, \tau_2, k) = \int_{\mathbf{X}} e^{i(\tau_1 x + k\varphi)} r^{i\tau_2} dP, \qquad \tau_1, \tau_2 \in \mathbb{R}, k \in \mathbb{Z}, \tag{3}$$

where the integrand is zero if $r = 0$, are called the characteristic transforms of the measure P.

Now we define the weak convergence of probability measures in the sense of the space $\mathbf{X}$. Let $P_n, n \in \mathbb{N}$, and P be probability measures on $(\mathbf{X}, \mathcal{B}(\mathbf{X}))$. We say that P_n converges weakly in the sense of $\mathbf{X}$ to P as $n \to \infty$ if P_n converges weakly to P, and additionally

$$P_n(\mathbb{R} \times \{0\}) \underset{n\to\infty}{\longrightarrow} P(\mathbb{R} \times \{0\}).$$

THEOREM 1. *Let $\sigma > 1/2$ be fixed. Then, as $T \to \infty$, the probability measure*

$$\nu_T\Big(\big(\log|\zeta(\sigma + it)|, \zeta(\sigma + it)\big) \in A\Big), \qquad A \in \mathcal{B}(\mathbf{X}),$$

converges weakly in the sense of $\mathbf{X}$ to the measure P on $(\mathbf{X}, \mathcal{B}(\mathbf{X}))$ defined by its characteristic transforms

$$w(\tau) = \int_{\Omega} |\zeta(\sigma, \omega)|^{i\tau} dm_H, \qquad \tau \in \mathbb{R},$$

$$w(\tau_1, \tau_2, k) = \int_{\Omega} |\zeta(\sigma, \omega)|^{i\tau_1 + i\tau_2} \exp\{ik \arg \zeta(\sigma, \omega)\} dm_H, \qquad \tau_1, \tau_2 \in \mathbb{R}, \quad k \in \mathbb{Z}.$$

THEOREM 2. *As $T \to \infty$, the probability measure*

$$\nu_T\left(\left(\frac{\log|\zeta((1/2) + it)|}{\sqrt{2^{-1}\log\log T}}, \left(\zeta\left(\frac{1}{2} + it\right)\right)^{(2^{-1}\log\log T)^{-1/2}}\right) \in A\right), \qquad A \in \mathcal{B}(\mathbf{X}),$$

converges weakly in the sense of $\mathbf{X}$ to the measure P on $(\mathbf{X}, \mathcal{B}(\mathbf{X}))$ defined by its characteristic transforms

$$w(\tau) = e^{-(\tau^2/2)}, \qquad \tau \in \mathbb{R},$$

$$w(\tau_1, \tau_2, k) = \exp\left\{-\frac{(\tau_1 + \tau_2)^2 + k^2}{2}\right\}, \tau_1, \tau_2 \in \mathbb{R}, \quad k \in \mathbb{Z}.$$

Next we shall discuss the asymptotic dependence of functions. Suppose that ξ_1 and ξ_2 are a real and a complex-valued random variables with distributions P_{ξ_1} and P_{ξ_2}, respectively, defined on some probability space $(\widehat{\Omega}, \mathcal{B}(\widehat{\Omega}), \mathbb{P})$. By the definition, ξ_1 and ξ_2 are independent if, for all $A_1 \in \mathcal{B}(\mathbb{R})$ and $A_2 \in \mathcal{B}(\mathbb{C})$,

$$\mathbb{P}(\xi_1 \in A_1, \xi_2 \in A_2) = P_{\xi_1}(A_1)P_{\xi_2}(A_2). \tag{4}$$

Since the spaces $\mathbb{R}$ and $\mathbb{C}$ are separable, (ξ_1, ξ_2) is a $\mathbf{X}$ - valued random variable. Moreover, $\mathcal{B}(\mathbf{X}) = \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{C})$. Therefore, if ξ_1 and ξ_2 are independent, then by (4)

$$\mathbb{P}\big((\xi_1, \xi_2) \in A\big) = P_{\xi_1}(A_1)P_{\xi_2}(A_2), \tag{5}$$

where

$$A = A_1 \times A_2, A_1 \in \mathcal{B}(\mathbb{R}), A_2 \in \mathcal{B}(\mathbb{C}).$$

Denote by P_{ξ_1,ξ_2} the distribution of the two-dimensional vector (ξ_1, ξ_2). Then, in view of (5), the characteristic transforms of the measure P_{ξ_1,ξ_2} are

$$w(\tau) = \int\limits_{\mathbb{R}} e^{i\tau x} dP_{\xi_1},$$

$$w(\tau_1, \tau_2, k) = \int\limits_{\mathbf{X}} e^{i(\tau_1 x + k\varphi)} r^{i\tau_2} dP_{\xi_1,\xi_2} = \int\limits_{\mathbb{R}} e^{i\tau_1 x} dP_{\xi_1} \int\limits_{\mathbb{C}} r^{i\tau_2} e^{ik\varphi} dP_{\xi_2} = w(\tau_1)w(\tau_2, k).$$

On the other hand, if, for all $\tau_1, \tau_2 \in \mathbb{R}, k \in \mathbb{Z}$,

$$w(\tau_1, \tau_2, k) = w(\tau_1)w(\tau_2, k),$$

then by Theorem 5, see Section 2,

$$P_{\xi_1,\xi_2}(A) = P_{\xi_1}(A_1)P_{\xi_2}(A_2),$$

$A = A_1 \times A_2, A_1 \in \mathcal{B}(\mathbb{R}), A_2 \in \mathcal{B}(\mathbb{C})$. This shows that ξ_1 and ξ_2 are independent. Hence it follows that the quantity

$$W(\xi_1, \xi_2) \stackrel{def}{=} \sup_{\substack{\tau_1, \tau_2 \in \mathbb{R} \\ k \in \mathbb{Z}}} \big| w(\tau_1, \tau_2, k) - w(\tau_1)w(\tau_2, k) \big|$$

is a certain "measure" of the dependence between ξ_1 and ξ_2. As we just have seen, the random variables ξ_1 and ξ_2 are independent if and only if $W(\xi_1, \xi_2) = 0$. Clearly, $0 \leqslant W(\xi_1, \xi_2) \leqslant 2$.

Now we shall apply the last theory to the asymptotic distribution of two functions. Suppose that $f_1(t)$ and $f_2(t)$ are defined on $\mathbb{R}$ with values in $\mathbb{R}$ and $\mathbb{C}$, respectively, and that the probability measures

$$\nu_T(f_1(t) \in A), \qquad A \in \mathcal{B}(\mathbb{R}),$$
$$\nu_T(f_2(t) \in A), \qquad A \in \mathcal{B}(\mathbb{C}),$$

and

$$\nu_T\Big(\big(f_1(t), f_2(t)\big) \in A\Big), \qquad A \in \mathcal{B}(\mathbf{X}),$$

converges weakly to P_{f_1}, converges weakly in the sense of $\mathbb{C}$ to P_{f_2} and converges weakly in the sense of $\mathbf{X}$ to P_{f_1,f_2}, respectively, as $T \to \infty$. Denote by $w_{f_1}(\tau), w_{f_2}(\tau, k)$ and $\big(w_{f_1}(\tau_1), w_{f_1,f_2}(\tau_1, \tau_2, k)\big)$ the characteristic function and characteristic transforms of the measures P_{f_1}, P_{f_2} and P_{f_1,f_2}, respectively, and define

$$W\big(f_1(t), f_2(t)\big) \stackrel{def}{=} \sup_{\substack{\tau_1,\tau_2\in\mathbf{R}\\ k\in\mathbf{Z}}} \big|w_{f_1,f_2}(\tau_1, \tau_2, k) - w_{f_1}(\tau_1) w_{f_2}(\tau_2, k)\big|.$$

Then by the above remarks the quantity $W\big(f_1(t), f_2(t)\big)$ is the "measure" of the asymptotic dependence of the functions $f_1(t)$ and $f_2(t)$.

Let $f(t), t \in \mathbb{R}$, be a complex-valued function. Then, clearly $|f(t)|$ and $f(t)$ are "strongly" asymptotically dependent. In the case of the Riemann zeta - function we have the following results.

THEOREM 3. *We have*

$$W\left(\frac{\log|\zeta((1/2)+it)|}{\sqrt{2^{-1}\log\log T}}, \left(\zeta\Big(\frac{1}{2}+it\Big)\right)^{(2^{-1}\log\log T)^{-1/2}}\right) = 1.$$

In the case $\sigma > 1/2$, the situation is more complicated, and the estimation of

$$W\Big(\log\big|\zeta(\sigma+it)\big|, \zeta(\sigma+it)\Big)$$

remains an open problem.

THEOREM 4. *Let $\sigma > 1/2$. Then, for $\tau \in \mathbb{R}, \tau \neq 0$,*

$$W\Big(\log\big|\zeta(\sigma+it)\big|, \zeta(\sigma+it)\Big) \geqslant 1 - \left|\int_\Omega \big|\zeta(\sigma,\omega)\big|^{i\tau} dm_H\right|^2.$$

It is an interesting problem of the dependence on σ of estimates for

$$W\Big(\log\big|\zeta(\sigma+it)\big|, \zeta(\sigma+it)\Big).$$

2. PROBABILISTIC BACKGROUND

In this section we consider probability measures and their weak convergence on $\big((\mathbf{X}, \mathcal{B}(\mathbf{X})\big)$ where $\mathbf{X} = \mathbb{R} \times \mathbb{C}$.

Clearly, the study of probability measures on $\big(\mathbf{X}, \mathcal{B}(\mathbf{X})\big)$ can be reduced to that of probability measures on $\big(\mathbb{R}^3, \mathcal{B}(\mathbb{R}^3)\big)$. However, in our case it is convenient to use the trigonometric form $re^{i\varphi}$ of complex numbers. For probability measures P on $\big(\mathbb{C}, \mathcal{B}(\mathbb{C})\big)$ this was done in [8], see also [9], by using the characteristic transforms (1). A similar

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method of investigations can be also applied for probability measures on $(\mathbf{X}, \mathcal{B}(\mathbf{X}))$. We define the characteristic transforms $(w(t), w(\tau_1, \tau_2, k))$ of the probability measure P on $(\mathbf{X}, \mathcal{B}(\mathbf{X}))$ by formulae (2) and (3).

The aim of this section is to obtain, by using the characteristic transforms, the uniqueness and continuity theorems for probability measures on $(\mathbf{X}, \mathcal{B}(\mathbf{X}))$.

THEOREM 5. *A probability measure P on $(\mathbf{X}, \mathcal{B}(\mathbf{X}))$ is uniquely determined by its characteristic transforms $(w(\tau), w(\tau_1, \tau_2, k))$.*

THEOREM 6. *Let P_n be a probability measure on $(\mathbf{X}, \mathcal{B}(\mathbf{X}))$, and let $(w_n(\tau), w_n(\tau_1, \tau_2, k))$ be its characteristic transforms, $n \in \mathbb{N}$. Suppose that*

$$\lim_{n\to\infty} w_n(\tau) = w(\tau), \qquad \tau \in \mathbb{R},$$

and

$$\lim_{n\to\infty} w_n(\tau_1, \tau_2, k) = w(\tau_1, \tau_2, k), \qquad \tau_1, \tau_2 \in \mathbb{R}, k \in \mathbb{Z},$$

where the functions $w(\tau)$, $w(0, \tau_2, 0)$ and $w(\tau_1, 0, 0)$ are continuous at the points $\tau = 0, \tau_2 = 0$ and $\tau_1 = 0$, respectively. Then on $(\mathbf{X}, \mathcal{B}(\mathbf{X}))$ there exists a probability measure P such that P_n converges weakly in the sense of $\mathbf{X}$ to P as $n \to \infty$. In this case, $(w(\tau), w(\tau_1, \tau_2, k))$ are the characteristic transforms of the measure P.

THEOREM 7. *Let P_n and $(w_n(\tau), w_n(\tau_1, \tau_2, k))$ be the same as in Theorem 6. Suppose that P_n converges weakly in the sense of $\mathbf{X}$ to some probability measure P on $(\mathbf{X}, \mathcal{B}(\mathbf{X}))$ as $n \to \infty$. Then*

$$\lim_{n\to\infty} w_n(\tau) = w(\tau), \qquad \tau \in \mathbb{R},$$

and

$$\lim_{n\to\infty} w_n(\tau_1, \tau_2, k) = w(\tau_1, \tau_2, k), \qquad \tau_1, \tau_2 \in \mathbb{R}, k \in \mathbb{Z},$$

where $(w(\tau), w(\tau_1, \tau_2, k))$ are the characteristic transforms of the measure P.

To prove Theorems 5–7 we use the following auxiliary space. Let, as above, γ be the unit circle on $\mathbb{C}, \mathbb{T} = \mathbb{R} \times \gamma$ and $\mathbb{Y} = \mathbb{R} \times \mathbb{T}$. We denote the points of the space $\mathbb{Y}$ by (x, y, α) where $x, y \in \mathbb{R}$ and $\alpha \in \gamma$. Define the Fourier transform

$$f(\tau_1, \tau_2, k), \tau_1, \tau_2 \in \mathbb{R}, k \in \mathbb{Z},$$

of the probability measure P on $(\mathbb{Y}, \mathcal{B}(\mathbb{Y}))$ by

$$f(\tau_1, \tau_2, k) = \int_{\mathbf{Y}} e^{i(\tau_1 x + \tau_2 y)} \alpha^k d\mathbf{P}.$$

LEMMA 8. *The probability measure P is uniquely determined by its Fourier transform $f(\tau_1, \tau_2, k)$.*

PROOF: First of all we notice that the space $\mathbb{Y}$ is locally compact. Therefore, the lemma and the next lemmas follow from general theorems for probability measures on locally compact groups, see, for example, [7]. However, we prefer to give, for fulness, a simple direct proof.

Let $f_j(\tau_1, \tau_2, k)$ be the Fourier transform of the probability measure P_j on $(\mathbb{Y}, \mathcal{B}(\mathbb{Y}))$, $j = 1, 2$. We have to prove that

$$f_1(\tau_1, \tau_2, k) = f_2(\tau_1, \tau_2, k), \qquad \tau_1, \tau_2 \in \mathbb{R}, k \in \mathbb{Z},$$

implies the equality

$$P_1(A) = P_2(A)$$

for all $A \in \mathcal{B}(\mathbb{Y})$. It suffices to prove the later equality for the sets

$$A = (a, b] \times (c, d] \times l.$$

where l is an arc of the circle γ, and

$$-\infty < a < b < \infty, -\infty < c < d < \infty.$$

Define a function $\psi : \mathbb{R} \to [0, 1]$ by

$$\psi(u) = \begin{cases} 1 & \text{if} \quad u \leqslant 0, \\ 1 - u & \text{if} \quad 0 \leqslant u \leqslant 1, \\ 0 & \text{if} \quad u \geqslant 1, \end{cases}$$

and let $\psi_n(u) = \psi(nu)$. Moreover, we put

$$\begin{aligned} g_{1,n}(x) &= \psi_n\Big(\rho\big(x, (a, b]\big)\Big), \\ g_{2,n}(y) &= \psi_n\Big(\rho\big(y, (c, d]\big)\Big), \\ g_{3,n}(\alpha) &= \psi_n\big(\rho_1(\alpha, l)\big), \end{aligned}$$

where ρ and ρ_1 are the distance on $\mathbb{R}$ and γ, respectively. Let I_B denote the indicator function of a set B. Then, obviously,

$$\begin{aligned} g_{1,n}(x) &\to I_{(a,b]}(x), \\ g_{2,n}(y) &\to I_{(c,d]}(y), \\ g_{3,n}(\alpha) &\to I_l(\alpha), \end{aligned}$$

as $n \to \infty$. Hence we have

$$P_j(A) = \lim_{n \to \infty} \int_{\mathbb{Y}} g_{1,n}(x) g_{2,n}(y) g_{3,n}(\alpha) dP_j, \qquad j = 1, 2,$$

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and it suffices to prove that, for $n \in \mathbb{N}$,

$$\int_{\mathbb{Y}} g_{1,n}(x)g_{2,n}(y)g_{3,n}(\alpha)dP_1 = \int_{\mathbb{Y}} g_{1,n}(x)g_{2,n}(y)g_{3,n}(\alpha)dP_2. \tag{6}$$

We fix $n \in \mathbb{N}$. Let $0 < \varepsilon < 1$, and let $K_1 > 0$ and $K_2 > 0$ be such that the functions $g_{1,n}(x)$ and $g_{2,n}(y)$ are zeros in the exterior of $[-K_1, K_1]$ and $[-K_2, K_2]$, respectively, and

$$P_j(\mathbb{Y}\backslash A_{K_1,K_2}) < \varepsilon, \qquad j = 1, 2, \tag{7}$$

where

$$A_{K_1,K_2} = \{(x, y, \alpha) \in \mathbb{Y} : |x| \leqslant K_1, |y| \leqslant K_2\}.$$

Since $g_{j,n}(-K_j) = g_{j,n}(K_j)$, the function $g_{j,n}(x)$ by the Weierstrass theorem can be approximated uniformly on $[-K_j, K_j]$ by a finite trigonometric sum

$$\sum_{m_j} a_{j,m_j} e^{(im_j\pi x)/(K_j)}$$

with period $2K_j$, $j = 1, 2$. Similarly, the function $g_{3,n}(\alpha)$ can be approximated by a linear combination of circle functions

$$\sum_{m_3} b_{m_3}\alpha^{m_3}.$$

Therefore, the product $g_{1,n}(x)g_{2,n}(y)g_{3,n}(\alpha)$ can be approximated uniformly on

$$[-K_1, K_1] \times [-K_2, K_2]$$

by a finite sum

$$g(x, y, \alpha) = \sum_{m_1,m_2,m_3} a_{1,m_1}a_{2,m_2}b_3 e^{(im_1\pi x)/(K_1)} e^{(im_2\pi y)/(K_2)} \alpha^{m_3}.$$

We choose the latter sum to satisfy

$$\left|g_{1,n}(x)g_{2,n}(y)g_{3,n}(\alpha) - g(x, y, \alpha)\right| < \varepsilon, \tag{8}$$

for all $(x, y, \alpha) \in A_{K_1,K_2}$.

Since

$$\left|g_{1,n}(x)g_{2,n}(y)g_{3,n}(\alpha)\right| \leqslant 1,$$

by (8)

$$\left|g(x, y, \alpha)\right| < 1 + \varepsilon$$

for all $(x, y, \alpha) \in A_{K_1,K_2}$. Therefore, by periodicity,

$$\left|g_{1,n}(x)g_{2,n}(y)g_{3,n}(\alpha) - g(x, y, \alpha)\right| < 2 + \varepsilon \tag{9}$$

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for $(x, y, \alpha) \in \mathbb{Y}\backslash A_{K_1,K_2}$. Then, in view of (7)–(9),

$$\int\limits_{\mathbb{Y}} \left|g_{1,n}(x)g_{2,n}(y)g_{3,n}(\alpha) - g(x,y,\alpha)\right| dP_j$$
$$= \Big(\int\limits_{A_{K_1,K_2}} + \int\limits_{\mathbb{Y}\backslash A_{K_1,K_2}} \Big) \left|g_{1,n}(x)g_{2,n}(y)g_{3,n}(\alpha) - g(x,y,\alpha)\right| dP_j < \varepsilon + (2+\varepsilon)\varepsilon < 4\varepsilon.$$

From this it follows that

$$(10) \quad \Big|\int\limits_{\mathbb{Y}} g_{1,n}(x)g_{2,n}(y)g_{3,n}(\alpha)dP_1 - \int\limits_{\mathbb{Y}} g_{1,n}(x)g_{2,n}(y)g_{3,n}(\alpha)dP_2\Big|$$
$$< \Big|\int\limits_{\mathbb{Y}} g(x,y,\alpha)dP_1 - \int\limits_{\mathbb{Y}} g(x,y,\alpha)dP_2\Big| + 8\varepsilon.$$

By the hypothesis of the lemma

$$\int\limits_{\mathbb{Y}} g(x,y,\alpha)dP_1 = \int\limits_{\mathbb{Y}} g(x,y,\alpha)dP_2.$$

Since ε is an arbitrary positive number, this shows that (6) is a simple consequence of (10). ∎

PROOF OF THEOREM 5: At first we note that one function $w(\tau_1, \tau_2, k)$ can not determine uniquely the measure P. For example, if P_j has the unit mass at the point $(x_j, 0), j = 1, 2, x_1 \neq x_2$, then $w_1(\tau_1, \tau_2, k) = w_2(\tau_1, \tau_2, k) = 0$ thought $P_1 \neq P_2$. In other words, if $r = 0$, the function $w(\tau_1, \tau_2, k)$ does not separate measures on the component $\mathbb{R}$ of the space $\mathbf{X}$.

Let $\mathbf{X}_0 = \mathbb{R} \times \big(\mathbb{C}\backslash\{0\}\big)$. Then the function $h : \mathbf{X}_0 \to \mathbb{Y}$ given by

$$h(x, re^{i\varphi}) = (x, \log r, e^{i\varphi})$$

is continuous. Therefore,

$$(11) \qquad w(\tau_1, \tau_2, k) = \int\limits_{\mathbb{Y}} e^{i(\tau_1 x + \tau_2 y)}\alpha^k dPh^{-1}, \qquad \tau_1, \tau_2 \in \mathbb{R}, k \in \mathbb{Z},$$

where Ph^{-1} is given by $Ph^{-1}(A) = P(h^{-1}A)$, $A \in \mathcal{B}(\mathbb{Y})$.

Let $\beta = w(0,0,0) = P(\mathbf{X}_0)$. Suppose that $\beta \neq 0$, and define on $\big(\mathbb{Y}, \mathcal{B}(\mathbb{Y})\big)$ the probability measure $\widehat{P}$ by the formula

$$(12) \qquad \widehat{P}(A) = \frac{P(h^{-1}A)}{\beta} = \frac{Ph^{-1}(A)}{\beta}, \qquad A \in (\mathcal{B})(\mathbb{Y}).$$

Substituting this in (11), we obtain that

$$w(\tau_1,\tau_2,k)=\beta\int\limits_{\mathbb{Y}} e^{i(\tau_1 x+\tau_2 y)}\alpha^k d\widehat{P},\quad \tau_1,\tau_2\in\mathbb{R}, k\in\mathbb{Z}. \tag{13}$$

Let

$$\widehat{f}(\tau_1,\tau_2,k)=\int\limits_{\mathbb{Y}} e^{i(\tau_1 x+\tau_2 y)}\alpha^k d\widehat{P},\qquad \tau_1,\tau_2\in\mathbb{R}, k\in\mathbb{Z},$$

denote the Fourier transform of the measure $\widehat{P}$. Then by (13)

$$\widehat{f}(\tau_1,\tau_2,k)=\frac{w(\tau_1,\tau_2,k)}{\beta},\qquad \tau_1,\tau_2\in\mathbb{R}, k\in\mathbb{Z}. \tag{14}$$

By Lemma 8, the measure $\widehat{P}$ is uniquely determined by its Fourier transform $\widehat{f}(\tau_1,\tau_2,k)$, therefore, in view of (14), also by $w(\tau_1,\tau_2,k)$. Consequently, the measure $P(A)$ is uniquely determined by $w(\tau_1,\tau_2,k)$ for $A\in\mathcal{B}(\mathbb{X}), A\in\mathbb{X}_0$. In particular, $P\Big(A\times\big(\mathbb{C}\backslash\{0\}\big)\Big)$ is uniquely determined for all $A\in\mathcal{B}(\mathbb{R})$. Since $P(A\times\mathbb{C})$ is uniquely determined by $w(\tau)$, we derive from this that $P\big(A\times\{0\}\big), A\in\mathcal{B}(\mathbb{R})$, is also uniquely determined by $w(\tau)$ and $w(\tau_1,\tau_2,k)$. This shows that $P(A)$ is uniquely determined by its characteristic transforms also for $A\in\mathcal{B}(\mathbb{X}), A\cap\big(\mathbb{R}\times\{0\}\big)\neq\emptyset$. Thus, in the case $\beta\neq 0$ the theorem is proved. ◻

Now let $\beta=0$, that is, $P(\mathbb{X}_0)=w(0,0,0)=0$. Consequently, $P(A)=0, A\in\mathcal{B}(\mathbb{X}), A\in\mathbb{X}_0$, is uniquely determined. In this case, for every

$$A\in\mathcal{B}(\mathbb{X}), A=A_1\times A_2, A_1\in\mathcal{B}(\mathbb{R}), A_2\in\mathcal{B}(\mathbb{C}), 0\in A_2,$$

we have

$$P(A)=P(A_1\times A_2)=P\Big(A_1\times\big(A_2\setminus\{0\}\big)\Big)+P\big(A_1\times\{0\}\big)=P\big(A_1\times\{0\}\big).$$

However,

$$P\big(A_1\times\{0\}\big)=P(A_1\times\mathbb{C})-P\big(A_1\times\big(\mathbb{C}\backslash\{0\}\big)=P(A_1\times\mathbb{C})=P_{\mathbb{R}}(A_1),$$

and is uniquely determined by $w(\tau)$. The theorem is proved. ◻

We begin the proof of Theorem 6 with a statement on the weak convergence of probability measures on $\big(\mathbb{Y},\mathcal{B}(\mathbb{Y})\big)$.

Lemma 9. *Let P_n be a probability measure on $\big(\mathbb{Y},\mathcal{B}(\mathbb{Y})\big)$, and let $f_n(\tau_1,\tau_2,k)$ be its Fourier transform, $n\in\mathbb{N}$. Suppose that*

$$\lim_{n\to\infty} f_n(\tau_1,\tau_2,k)=f(\tau_1,\tau_2,k),\qquad \tau_1,\tau_2,\in\mathbb{R}, k\in\mathbb{Z},$$

and that the functions $f(0,\tau_2,0)$ and $f(\tau_1,0,0)$ are continuous at the points $\tau_2=0$ and $\tau_1=0$, respectively. Then on $\big(\mathbb{Y},\mathcal{B}(\mathbb{Y})\big)$ there exists a probability measure P such that

P_n converges weakly to P as $\to \infty$. In this case, $f(\tau_1, \tau_2, k)$ is the Fourier transform of the measure P.

PROOF: Let

$$P_{\mathbb{R},n}(A) = P_n(A \times \mathbb{T}), \qquad A \in \mathcal{B}(\mathbb{R}),$$

and

$$P_{\mathbb{T},n}(A) = P_n(\mathbb{R} \times A), \qquad A \in \mathcal{B}(\mathbb{T}).$$

Its is well known that the sequence $\{P_n\}$ is tight (for definition, see [3]) if every sequence of marginal distributions $\{P_{\mathbb{R},n}\}$ and $\{P_{\mathbb{T},n}\}$ is tight. We shall prove the tightness of the sequence $\{P_{\mathbb{T},n}\}$. Clearly,

$$f_n(\tau_2, k) \overset{def}{=} f_n(0, \tau_2, k), \ \tau_2 \in \mathbb{R}, k \in \mathbb{Z},$$

is the Fourier transform of the measure $P_{\mathbb{T},n}$ (for definition, see [9]). By the hypothesis of the lemma

$$\lim_{n \to \infty} f_n(\tau_2, k) = f(0, \tau_2, k) \overset{def}{=} f(\tau_2, k), \qquad \tau_2 \in \mathbb{R}, k \in \mathbb{Z}. \tag{15}$$

By the Fubini theorem, for $u > 0$,

$$\begin{aligned}
\frac{1}{u}\int\limits_{-u}^{u} \big(1 - f_n(\tau_2, 0)\big) d\tau_2 &= \int\limits_{\mathbb{T}} \Big(\frac{1}{u}\int\limits_{-u}^{u}(1 - e^{i\tau_2 y}) d\tau_2\Big) dP_{\mathbb{T},n} \\
&= 2\int\limits_{\mathbb{T}} \Big(1 - \frac{\sin uy}{uy}\Big) dP_{\mathbb{T},n} \leqslant \int\limits_{(y,\alpha)\in,|y|\geqslant\frac{2}{u}} \Big(1 - \frac{1}{uy}\Big) dP_{\mathbb{T},n} \\
&\geqslant P_{\mathbb{T},n}\Big((y, \alpha) \in \mathbb{T} : |y| \geqslant \frac{2}{u}\Big).
\end{aligned} \tag{16}$$

Since $f(\tau_2, 0)$ is continuous at $\tau_2 = 0$, for every $\varepsilon > 0$ there exists $u > 0$ such that

$$\frac{1}{u}\int\limits_{-u}^{u} \big|1 - f(\tau_2, 0)\big| d\tau_2 < \varepsilon.$$

Therefore, by (15) and the Lebesgue theorem on bounded convergence there exists $n_0 \in \mathbb{N}$ such that

$$\frac{1}{u}\int\limits_{-u}^{u} \big|1 - f_n(\tau_2, 0)\big| d\tau_2 < 2\varepsilon$$

for $n \geqslant n_0$. From this and (16) we find that, for $n \geqslant n_0$,

$$P_{\mathbb{T},n}\Big((y, \alpha) \in \mathbb{T} : |y| \geqslant \frac{2}{u}\Big) < 2\varepsilon.$$

Clearly, taking u smaller if this is necessary, we can demonstrate that the later inequality should remain true also for $n < n_0$. This shows that there exists a compact subset $K \subset \mathbb{T}$ such that

$$P_{\mathbb{T},n}(K) > 1 - 2\varepsilon$$

for all $n \in \mathbb{N}$, that is, the sequence $\{P_{\mathbb{T},n}\}$ is tight.

Similarly we obtain that the sequence $\{P_{\mathbb{R},n}\}$ is also tight. Therefore, the sequence of probability measures $\{P_n\}$ is tight. Hence, by the Prokhorov theorem, see, for example, [**2**], it is relatively compact, and we have that every subsequence $\{P_{n_1}\} \subset \{P_n\}$ contains a subsequence $\{P_{n_2}\}$ weakly convergent to some probability measure P on $\big(\mathbb{Y}, \mathcal{B}(\mathbb{Y})\big)$ as $n_2 \to \infty$. Moreover, $f(\tau_1, \tau_2, k)$ is the Fourier transform of the measure P. By Lemma 8 the measure P is the same for all weakly convergent subsequences. Thus, the lemma is a consequence of [**3**, Theorem 2.3]. □

Proof of Theorem 6: Let $\beta_n = w_n(0,0,0)$. By the hypothesis of the theorem

$$\lim_{n\to\infty} \beta_n = w(0,0,0) \overset{def}{=} \beta.$$

If $\beta \neq 0$, there exists $n_0 \in \mathbb{N}$ such that $\beta_n \neq 0$ for $n \geqslant n_0$. For $n \geqslant n_0$, define the measure $\widehat{P}_n$ on $\big(\mathbb{Y}, \mathcal{B}(\mathbb{Y})\big)$ by formula (12), and let $\widehat{f}_n(\tau_1, \tau_2, k)$ be its Fourier transform. Then the hypothesis of the theorem and a formula of the type (14)

$$\widehat{f}_n(\tau_1, \tau_2, k) = \frac{w_n(\tau_1, \tau_2, k)}{\beta_n}$$

imply the existence of the limit

$$\lim_{n\to\infty} \widehat{f}_n(\tau_1, \tau_2, k) = \widehat{f}(\tau_1, \tau_2, k), \qquad \tau_1, \tau_2 \in \mathbb{R}, k \in \mathbb{Z},$$

where the functions $\widehat{f}(\tau_1, 0, 0)$ and $\widehat{f}(0, \tau_2, 0)$ are continuous at $\tau_1 = 0$ and $\tau_2 = 0$, respectively. Therefore, by Lemma 9 on $\big(\mathbb{Y}, \mathcal{B}(\mathbb{Y})\big)$ there exists a probability measure $\widehat{P}$ such that $\widehat{P}_n$ converges weakly to $\widehat{P}$ as $n \to \infty$, and $\widehat{f}(\tau_1, \tau_2, k)$ is the Fourier transform of $\widehat{P}$.

Denote by ∂A the boundary of a set A. The function $h : \mathbf{X}_0 \to \mathbb{Y}$ defined in the proof of Theorem 5 is homeomorphic. Therefore, for $A \in \mathcal{B}(\mathbb{Y})$,

$$\partial(h^{-1}A) = h^{-1}(\partial A). \tag{17}$$

Since $\widehat{P}_n$ converges weakly to $\widehat{P}$ and $\beta_n \to \beta$, we have from the definition of $\widehat{P}$ that on $\big(\mathbf{X}, \mathcal{B}(\mathbf{X})\big)$ there exists a probability measure P such that $\widehat{P}_n(A) \to P(A), n \to \infty$, for the sets $A = h^{-1}B$, where $B \in \mathcal{B}(\mathbb{Y})$ and $P(h^{-1}\partial B) = 0$. However, then in view of (17) $P(\partial h^{-1}B) = P(\partial A) = 0$. Thus, we have that

$$P_n(A) \to P(A), \qquad n \to \infty, \tag{18}$$

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for all continuity sets A of the measure P which do not contain the points $(x, 0)$. In particular case,

$$P_n\Big(A \times (\mathbb{C}\backslash\{0\})\Big) \to P\Big(A \times (\mathbb{C}\backslash\{0\})\Big), \qquad n \to \infty, \tag{19}$$

for all continuity sets A of $P(A \times \mathbb{C}) = P_{\mathbb{R}}(A)$. Moreover, since $w_n(\tau) \to w(\tau), n \to \infty$, and $w(\tau)$ is continuous at $\tau = 0$, it follows that

$$P_n(A \times \mathbb{C}) \to P(A \times \mathbb{C}), \qquad n \to \infty,$$

for all continuity sets A of $P_{\mathbb{R}}$. This together with (19) implies the relation

$$P_n(A \times \{0\}) \to P(A \times \{0\}), \qquad n \to \infty, \tag{20}$$

for all continuity sets A of $P_{\mathbb{R}}$. Suppose that $B \supset \{0\}$ is a continuity set of the measure $P_{\mathbb{C}}, P_{\mathbb{C}}(B) = P(\mathbb{R} \times B)$. Then in view of (18)-(20), for every continuity set A of $P_{\mathbb{R}}$,

$$\begin{aligned} P_n(A \times B) &= P_n\Big(A \times \Big(\big((\mathbb{C} \setminus \{0\}) \setminus B^c\big) \cup \{0\}\Big)\Big) \\ &= P_n\Big(A \times (\mathbb{C} \setminus \{0\})\Big) - P_n(A \times B^c) + P_n\big(A \times \{0\}\big) \to P(A \times B), n \to \infty. \end{aligned}$$

Therefore, we have that P_n converges weakly to P as $n \to \infty$.

Since $\beta_n \to \beta$, we find similarly that $P_n\big(\mathbb{R} \times \{0\}\big) \to P\big(\mathbb{R} \times \{0\}\big), n \to \infty$, and the theorem in the case $\beta \neq 0$ is proved. □

Now suppose that $\beta = 0$. Then $P_n\Big(\mathbb{R} \times \big(\mathbb{C}\backslash\{0\}\big)\Big) \to 0$ as $n \to \infty$. Hence $P_n(A) \to 0, n \to \infty$, for all $A \in \mathbb{R} \times \big(\mathbb{C}\backslash\{0\}\big)$. Since $w_n(\tau) \to w(\tau), n \to \infty$, we have that $P_n(A \times \mathbb{C}) \to P(A \times \mathbb{C}), n \to \infty$, for all continuity sets A of $P_{\mathbb{R}}$. Hence and from relation

$$P_n\Big(A \times \big(\mathbb{C}\backslash\{0\}\big)\Big) \to 0, \qquad n \to \infty,$$

we obtain that $P_n\big(A \times \{0\}\big) \to P(A \times \mathbb{C}) = P\big(A \times \{0\}\big)$. Since $P_n\big(\mathbb{R} \times \{0\}\big) \to 1, n \to \infty$, it follows that P_n converges weakly in the sense of $\mathbf{X}$ as $n \to \infty$ to the measure P the mass of which is concentrated on $\mathbb{R} \times \{0\}$.

Clearly, $\big(w(\tau), w(\tau_1, \tau_2, k)\big)$ are the characteristic transforms of P.

LEMMA 10. *Let $\{P_n\}$ and $\big\{f_n(\tau_1, \tau_2, k)\big\}$ be the same as in Lemma 9. Suppose that P_n converges weakly to some probability measure P on $\big(\mathbb{Y}, \mathcal{B}(\mathbb{Y})\big)$ as $n \to \infty$. Then*

$$\lim_{n\to\infty} f_n(\tau_1, \tau_2, k) = f(\tau_1, \tau_2, k), \qquad \tau_1, \tau_2 \in \mathbb{R}, k \in \mathbb{Z},$$

where $f(\tau_1, \tau_2, k)$ is the Fourier transform of the measure P.

PROOF: The lemma is an immediate consequence of the definition of the Fourier transforms and weak convergence of probability measures. □

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PROOF OF THEOREM 7: The weak convergence of P_n to P implies that of $P_{\mathbb{R},n}$ to $P_{\mathbb{R}}, n \to \infty$. Therefore, we have that

$$\lim_{n\to\infty} w_n(\tau) = w(\tau), \qquad \tau \in \mathbb{R}.$$

We have that

$$\beta \overset{def}{=} w(0,0,0) = \int_{X_0} dP = P\Big(\mathbb{R} \times \big(\mathbb{C}\backslash\{0\}\big)\Big).$$

Since $P_n\big(\mathbb{R} \times \{0\}\big) \to P\big(\mathbb{R} \times \{0\}\big), n \to \infty$, hence we obtain that

$$\beta_n \overset{def}{=} w_n(0,0,0) \to \beta, n \to \infty.$$

If $\beta \neq 0$, then we obtain that $\widehat{P}_n$ converges weakly to $\widehat{P}$ as $n \to \infty$. Now by Lemma 10 it follows that

$$\lim_{n\to\infty} \widehat{f}_n(\tau_1,\tau_2,k) = \widehat{f}(\tau_1,\tau_2,k), \qquad \tau_1,\tau_2, \in \mathbb{R}, k \in \mathbb{Z}.$$

Therefore, from a formula of type (14) we find that

$$\lim_{n\to\infty} w_n(\tau_1,\tau_2,k) = w(\tau_1,\tau_2,k), \qquad \tau_1,\tau_2 \in \mathbb{R}, k \in \mathbb{Z}.$$

If $\beta = 0$, then the limit measure P is concentrated on $\mathbb{R} \times \{0\}$, and its characteristic transform $w(\tau_1,\tau_2,k) \equiv 0$. Then, by the definition of the characteristic transforms

$$\lim_{n\to\infty} w_n(\tau_1,\tau_2,k) \equiv 0.$$

The theorem is proved. ∎

3. PROOF OF THEOREMS 1–4

Theorems 1 and 2 are simple consequences of Theorems A–D and Theorem 6.

PROOF OF THEOREM 1: The characteristic transforms of the measure of the theorem are

$$\left(\frac{1}{T}\int_0^T \big|\zeta(\sigma+it)\big|^{i\tau} d\tau, \frac{1}{T}\int_0^T \big|\zeta(\sigma+it)\big|^{i(\tau_1+\tau_2)} \exp\big\{ik \arg\zeta(\sigma+it)\big\}\, dt\right).$$

By Theorems A and B these characteristic transforms converge to

$$\left(\int_\Omega \big|\zeta(\sigma,\omega)\big|^{i\tau} dm_H, \int_\Omega \big|\zeta(\sigma,\omega)\big|^{i(\tau_1+\tau_2)} \exp\big\{ik \arg\zeta(\sigma,\omega)\big\} dm_H\right)$$

as $T \to \infty$. Therefore, it remains to apply Theorem 6. ∎

PROOF OF THEOREM 2: The characteristic transforms of the measure of the theorem are

$$(21)\quad \left(\frac{1}{T}\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^{(i\tau)/(\sqrt{2^{-1}\log\log T})}dt,\right.$$
$$\left.\frac{1}{T}\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^{(i(\tau_1+\tau_2))/(\sqrt{2^{-1}\log\log T})}\exp\left\{ik\frac{\arg\zeta((1/2)+it)}{\sqrt{2^{-1}\log\log T}}\right\}dt\right).$$

Since the characteristic function of the measure L is $e^{(-\tau^2)/2}$, and the characteristic transforms of the lognormal probability measure on $(\mathbb{C},\mathcal{B}(\mathbb{C}))$ is $e^{(-\tau^2)/2-(k^2)/2}$, by Theorems C and D we obtain that the characteristic transforms (21) converge to

$$\left(e^{-(\tau^2)/2}, e^{-((\tau_1+\tau_2)^2/2)-(k^2/2)}\right)$$

as $T\to\infty$. Hence, by Theorem 6, the theorem follows. ∎

PROOF OF THEOREM 3: By Theorems 2, and C, D

$$W\left(\log\left|\zeta\left(\frac{1}{2}+it\right)\right|/\sqrt{2^{-1}\log\log T}, \left(\zeta\left(\frac{1}{2}+it\right)\right)^{(2^{-1}\log\log T)^{-1/2}}\right)$$
$$=\sup_{\substack{\tau_1,\tau_2\in\mathbb{R}\\ k\in\mathbb{Z}}}\left|e^{-((\tau_1+\tau_2)^2)/2-(k^2/2)}-e^{-\tau_1^2/2}e^{-(\tau_2^2+k^2)/2}\right|.$$

Let

$$f(\tau_1,\tau_2)=e^{-(\tau_1+\tau_2)^2/2}-e^{-\tau_1^2/2}e^{-\tau_2^2/2}.$$

Obviously, $|f(\tau_1,\tau_2)|\leqslant 1$ (as $\tau_1=-\tau_2$ and $\tau_2\to\infty, f(\tau_1,\tau_2)\to 1$).

PROOF OF THEOREM 4: Theorem 1 and Theorems A and B imply

$$W\left(\log|\zeta(\sigma+it)|,\zeta(\sigma+it)\right)$$
$$=\sup_{\substack{\tau_1,\tau_2\in\mathbb{R}\\ k\in\mathbb{Z}}}\left|\int_\Omega |\zeta(\sigma,\omega)|^{i\tau_1+i\tau_2}\exp\{ik\arg\zeta(\sigma,\omega)\}dm_H\right.$$
$$\left.-\int_\Omega|\zeta(\sigma,\omega)|^{i\tau_1}dm_H\int_\Omega|\zeta(\sigma,\omega)|^{i\tau_2}\exp\{ik\arg\zeta(\sigma,\omega)\}dm_H\right|.$$

Taking $\tau_1=-\tau_2\neq 0$ and $k=0$, we find that the expression inside modulo is

$$1-\left|\int_\Omega|\zeta(\sigma,\omega)|^{i\tau_1}dm_H\right|^2.$$

This proves the theorem. ∎

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