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PAIRWISE BALANCED DESIGNS WITH BLOCK SIZES THREE AND FOUR

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ABSTRACT. Given integers v, a and b, when does a pairwise balanced design on v elements with a triples and b quadruples exist? Necessary conditions are developed, and shown to be sufficient for all $v \geq 96$. An extensive set of constructions for pairwise balanced designs is used to obtain the result.

1. **Preliminaries.** Let X be a finite set, $|X| = v$. For a set $K \subseteq \{2,3,4,\ldots,v\}$, let $\binom{X}{K}$ denote the set of all subsets of X whose cardinalities appear in K. For $\mathcal{B} \subseteq \binom{X}{K}$, $(X, \mathcal{B})$ is a $(v; K)$-*pairwise balanced design* (or $(v; K)$-PBD) if every 2-subset of X appears in precisely one member of $\mathcal{B}$. Members of $\mathcal{B}$ are called *blocks,* and K is the set of *block sizes.*

Let $W \subseteq X$, $|W| = w$. If $\mathcal{B} \subseteq \binom{X}{K}$ has the property that $(X, \mathcal{B} \cup \{W\})$ is a pairwise balanced design then $(X, W, \mathcal{B})$ is called a $(v, w; K)$-*incomplete* pairwise balanced design, or $(v, w; K)$-IPBD. The set W is called a *hole.*

In this paper, we study PBDs and IPBDs with block sizes 3 and 4, which we call *triples* and *quadruples,* respectively. It has long been known that a $(v, \{3,4\})$-PBD exists if and only if $v \equiv 0, 1 \pmod 3$, $v \neq 6$ (see [3], for example). We address a more complicated problem, the determination of the possible numbers of blocks of each size in such a PBD of order v. Define

$$\mathrm{Spec}_4(v) = \{ s : \exists (v, \{3,4\})\text{-PBD having } s \text{ quadruples}\},$$

and

$$\mathrm{Spec}_4(v, w) = \{ s : \exists (v, w, \{3,4\})\text{-IPBD having } s \text{ quadruples}\}.$$

Our goal is to determine $\mathrm{Spec}_4(v)$, leaving only a handful of exceptions for small values of v. In the process, we employ some results on $\mathrm{Spec}_4(v, w)$. We shall see that there are substantial connections to fundamental problems in design theory.

Determining the possible numbers of pairs and triples in a PBD with blocks of sizes two and three is straightforward using the solution for the maximum packing problem for triples (see [28]). Similarly, determining the possible numbers of pairs and quadruples in a PBD with block sizes two and four also is easy given the solution for the packing problem for quadruples (see [5]). Hence the determination for triples and quadruples is the next step, and as we shall see it is substantially more complicated.

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We first determine necessary conditions on $\mathrm{Spec}_4(v)$, and subsequently adapt a large battery of recursive constructions to establish sufficiency for $v \geq 96$. We introduce definitions as needed, but refer the reader to [3,33] for standard background in combinatorial design theory.

At the outset, let us remark that pairwise balanced designs have wide applications in the construction of combinatorial designs [3,33], and have proved to be very useful in the statistical design of experiments [18].

2. **Necessary conditions.** In this section, we employ some elementary observations to establish necessary conditions on $\mathrm{Spec}_4(v)$. For each integer $t \geq 0$, let $\mathcal{A}(3t+2) = \emptyset$; otherwise define $\mathcal{A}(v)$ according to the following table:

v	$\mathcal{A}(v)$
$12t$	$\{3t, \ldots, t(12t-3)\}$
$12t+1$	$\{0, \ldots, t(12t+1)\} \setminus \{1,2,3,4, t(12t+1)-3, t(12t+1)-2, t(12t+1)-1\}$
$12t+3$	$\{0, \ldots, t(12t+3)\} \setminus \{1,2,3,4\}$
$12t+4$	$\{3t+1, \ldots, 12t^2+7t+1\} \setminus \{12t^2-7t-2, 12t^2-7t-1, 12t^2-7t\}$
$12t+6$	$\{3t+2, \ldots, 12t^2+9t+1\}$
$12t+7$	$\{0, \ldots, 12t^2+13t\} \setminus \{1,2,3,4\}$
$12t+9$	$\{0, \ldots, 12t^2+15t+4\} \setminus \{1,2,3,4\}$
$12t+10$	$\{3t+3, \ldots, 12t^2+19t+4\}$

For convenience, we let m_v denote the smallest number in $\mathcal{A}_4(v)$, and we let M_v denote the largest number.

LEMMA 2.1. *For all $v \geq 0$, $\mathrm{Spec}_4(v) \subseteq \mathcal{A}(v)$.*

PROOF. For $v \equiv 2 \pmod 3$, there is no $(v, \{3,4\})$-PBD. For the remaining cases, consider an element x of the PBD, and let d_i $(i = 3,4)$ be the number of blocks of size i containing x. Now $2d_3 + 3d_4 = v-1$, and hence $2d_3 \equiv v-1 \pmod 3$, and $d_4 \equiv v-1 \pmod 2$. Hence for $v \equiv 0 \pmod 3$, we have at least $\lceil v/3 \rceil$ triples, and hence at most $\lfloor v(v-3)/12 \rfloor$ quadruples. Similarly when v is even, we have at least $\lceil v/4 \rceil$ quadruples.

Observe further that the number of triples is always congruent to $v(v-1)/6 \pmod 2$; hence when $v \equiv 7, 10 \pmod{12}$, the number of triples is odd. Since $d_3 \equiv 0 \pmod 6$ in such a PBD, the smallest number of triples is 7; this gives an upper bound of $(v(v-1)-42)/12$ quadruples in this cases. These arguments establish the lower and upper bounds; now we turn to the other missing values.

When v is odd, consider the configuration of quadruples. Every element is in an even number of quadruples; it is easy to verify that this requires either zero or at least five quadruples. Hence $\{1,2,3,4\} \cap \mathrm{Spec}_4(v) = \emptyset$ for v odd.

When $v \equiv 1, 4 \pmod{12}$, every element is in a number of triples which is $0 \pmod 3$, and the number of triples is even. Hence if there are any triples at all, there must be at least eight of them (and they partition the unique 6-regular graph on eight vertices into triangles if the number of triples equals eight). Hence in these cases we have that $v(v-1)/12 - s \notin \mathrm{Spec}_4(v)$ for $s \in \{1,2,3\}$. ∎

The maximum values in $\mathcal{A}(v)$ arise as follows. When $v \equiv 1,4 \pmod{12}$, the PBD contains only quadruples. The maximum for $v \equiv 0,3 \pmod{12}$ is obtained by omitting one point from the maximum solution for $v+1$ points. When $v \equiv 7,10 \pmod{12}$, the maximum is obtained by taking a PBD with one 7-block and all other blocks of size 4, and then replacing the 7-block with the 7 triples of a PBD on the same points. The maximum for $v \equiv 6,9 \pmod{12}$ is *not* obtained by omitting a point from the maximum solution on $v+1$ points. Rather it has $(v-9)/3$ disjoint triples, and four triples intersecting in a single point.

Our main result in this paper is the following

MAIN THEOREM. $\mathrm{Spec}_4(v) = \mathcal{A}(v)$ *for* $v \geq 96$.

The proof of sufficiency involves a large number of recursive constructions, that we introduce in section 3. Then in section 4, we determine various values in $\mathrm{Spec}_4(v,w)$ for small v and w. In section 5, we apply the recursive techniques to the small values to prove the Main Theorem. Finally, in section 6, we outline some applications of the results.

3. **Recursive constructions.** In addition to PBDs and IPBDs defined earlier, we require a few further basic definitions in design theory. We call $(X, \mathcal{G}, \mathcal{B})$ a K-GDD with group type $g_1^{t_1} \cdots g_k^{t_k}$ if $\mathcal{B} \subseteq \binom{X}{K}$, $(X, \mathcal{B} \cup \mathcal{G})$ is a PBD, and $\mathcal{G}$ is a partition of X into sets (called *groups*); for $1 \leq i \leq k$, $\mathcal{G}$ contains t_i groups with k_i elements. The groups form essentially a spanning set of holes.

The basic construction that we use in general forms $\{3,4\}$-GDDs, and then "fills in groups" with IPBDs and PBDs as follows:

LEMMA 3.1 (FILLING IN GROUPS). *Let* $(X, \mathcal{G}, \mathcal{B})$ *be a* $\{3,4\}$-GDD *with* $|X| = v$ *and groups* $G_1, \ldots, G_m$. *Let* b_4 *be the number of quadruples in* $\mathcal{B}$. *Let* w *be a nonnegative integer. Let* $f_i \in \mathrm{Spec}_4(|G_i| + w, w)$, *and* $h_i \in \mathrm{Spec}_4(|G_i| + w)$. *Then for* s *such that* $1 \leq s \leq m$,

$$b_4 + \sum_{i=1}^{m} f_i \in \mathrm{Spec}_4(v,w),$$

$$b_4 + \sum_{\substack{i=1 \\ i \neq s}}^{m} f_i + h_s \in \mathrm{Spec}_4(v), \text{ and}$$

$$b_4 + \sum_{\substack{i=1 \\ i \neq s}}^{m} f_i \in \mathrm{Spec}_4(v, |G_s| + w).$$

PROOF. Add w new elements W to X. Now on $G_i \cup W$, place an IPBD leaving a hole on the set W; do this for all $i \neq s$. Then we may either leave the final hole, place an IPBD on it, or a PBD on it, to obtain the three outcomes above. ∎

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A (v, w, K)-IPBD is equivalent to a K-GDD of type $w^1 1^{v-w}$, and hence we can apply Lemma 3.1 (with $w = 0$) to IPBDs as well. This is referred to as "filling the hole".

For large v, we can use Wilson's fundamental construction for GDDs, which we state next:

LEMMA 3.2 (FUNDAMENTAL CONSTRUCTION [35]). *Suppose that* $(X, \mathcal{G}, \mathcal{B})$ *is a* K-GDD, *and that* $w: X \longrightarrow Z^+ \cup \{0\}$ *is any function (which we call the weight function). Let* $\mathcal{G}$ *consist of groups* $G_1, \ldots, G_t$. *If for every* $\{x_1, \ldots, x_m\} \in \mathcal{B}$, *there is a* K-GDD *with* m *groups, in which the ith group has size* $w(x_i)$, *then there is a* K-GDD *with* t *groups, so that the size of the ith group is* $\sum_{x \in G_i} w(x)$. ∎

The application of the fundamental construction requires that we develop a substantial collection of GDDs with block sizes 3 and 4; filling in groups then requires that we develop some IPBDs and PBDs with block sizes 3 and 4. Hence we recall a large number of constructions that can be used to produce such GDDs and PBDs.

LEMMA 3.3 [8]. *Let* g, t, u *be nonnegative integers satisfying* $g \geq 1$, $t \geq 3$, $u \leq g(t-1)$, $\frac{1}{2}g^2\binom{t}{2} + gtu \equiv 0 \pmod 3$, $pg(t-1) + u \equiv 0 \pmod 2$, *and if* $u \neq 0$ *then* $gt \equiv 0 \pmod 2$. *Then there exists a* $\{3\}$-GDD *of group-type* $g^t u^1$. ∎

Lemma 3.3 includes as special cases three important results that we employ in a substantial way. When $g = u = 1$, Lemma 3.3 is equivalent to the existence of Steiner triple systems, determined in 1847 by Kirkman [15]. When $g = 1$, Lemma 3.3 gives the Doyen-Wilson theorem [11] that a $(v, w, \{3\})$-IPBD exists whenever $v, w \equiv 1, 3 \pmod 6$ and $v \geq 2w + 1$. It also yields a theorem of Rosa and Hoffman [27]: a $\{3\}$-GDD of group-type $4^t u^1$ exists for all even $u \leq 4t - 4$ for which $t \equiv 0$ or $1 - u \pmod 3$, $t \geq 3$.

LEMMA 3.4 [23]. *Let* $v, w \equiv 1 \pmod 3$, $v(v-1) \equiv w(w-1) \pmod{12}$, *and* $v \geq 3w + 1$. *Then there exists a* $(v, w; \{4\})$-IPBD. ∎

The spectrum of $(v, \{4\})$-PBDs was first determined by Hanani [12]. Lemma 3.4 has some useful corollaries. The *truncation* of a PBD is another PBD obtained by removing some elements, and all occurrences of those elements in blocks (and naturally removing all "blocks" of size 0 and 1 that result). Truncations of the IPBDs in Lemma 3.4 are particularly valuable. Removing a single element from the hole of size w gives a $\{4\}$-GDD with group-type $3^{(v-w)/3}(w-1)^1$. More generally, truncating the hole to $w - x$ elements yields a $(v - x, w - x, \{3, 4\})$-IPBD with $x(v - w)/3$ triples. One can also truncate by removing $x = 1, 3$ or 4 points from a block with the elements not in the hole to produce a $(v - x, w, \{3, 4\})$-IPBD with $x(v-4)/3$ triples for $x = 3, 4$, and $(v-4)/3 + 1$ triples for $x = 1$. (The case $x = 4$ only applies here when $v \neq 3w - 1$, since a quadruple disjoint from the hole is needed.) Naturally, one can truncate one point from the hole, and then two or three from a resulting triple as well. In general, we do not comment on the PBDs and GDDs from such obvious truncations; however, they prove very useful in constructing needed ingredients.

Next we consider a special type of GDD. A $\{k\}$-GDD of group-type m^k is often called a *transversal design* and denoted TD(k, m). An *incomplete* transversal design

ITD(k,m,n) is a set of k disjoint groups $G_1,\ldots,G_k$ of size m, a set H intersecting each G_i in n points, and a collection of blocks of size k, so that every 2-subset in H or in one of the groups does not appear in a block, and every other 2-subset appears precisely once. There is a TD(k,m) if and only if there is an ITD$(k,m,1)$ (simply choose H to be a block, and omit that block).

LEMMA 3.5 [13]. *For $m \geq 3n$, $n \geq 1$ there is an* ITD$(4,m,n)$ *except when $m = 6$ and $n = 1$.* ∎

The additional hole in the ITD can be filled by using a $\{3,4\}$-GDD of group-type n^k to form a $\{3,4\}$-GDD of group-type m^k.

Next we exploit a process that essentially reverses the truncation operation. Suppose that a PBD, IPBD, GDD or TD contains a set of blocks that contain every element precisely once; this is termed a *parallel class* of blocks. Let $P_1,\ldots,P_t$ be a parallel class of blocks. Then if there exist $(|P_i| + w, w, \{3,4\})$-IPBDs for each i, one can "fill in the parallel class" — this is analogous to filling in groups as in Lemma 3.1. Hence we are interested in designs with many parallel classes, so that we can extend *many* parallel classes in this way. A design is *resolvable* if its block set can be partitioned into parallel classes.

LEMMA 3.6 [3,34]. *There exists a resolvable* TD$(4,m)$ *except for $m \in \{2,3,6\}$ and possibly for $m = 10$.*

To use Lemma 3.6, for any parallel class we can add three fixed elements, and put a $(7,3,\{3\})$-IPBD on each block and the three elements, leaving the hole on the new elements. If s parallel classes are extended in this way, we add $3s$ elements that produce a hole of size $3s$ (that can then be filled). This essentially gives a GDD of group-type $m^4 3s^1$.

LEMMA 3.7 [19]. *For nonnegative integers t,x,y satisfying $x + 2y = 6t - 1$, there is a resolvable* $(6t,\{2,3\})$-PBD *with x parallel classes of 2-blocks and y parallel classes of triples, except when $x = 1$ and $t \in \{1,2\}$.* ∎

When $x = 1$, such resolvable PBDs are called *nearly Kirkman triple systems*. To use such PBDs to construct $\{3,4\}$-PBDs, we extend each parallel class of 2-blocks to form triples, and then extend some of the parallel classes of triples to form quadruples. In the process of proving Lemma 3.7, Rees also proves a similar result on resolvable GDDs that we can exploit:

LEMMA 3.8 [19]. *For even n and all $n \leq r \leq 2n$, there exists a resolvable $\{2,3\}$-*GDD *of group-type n^3 having $2r - 2n$ parallel classes of 2-blocks and $2n - r$ parallel classes of triples, except when $n = r = 2$ or $n = r = 6$.* ∎

Lemma 3.8 is used similarly to Lemma 3.7, but enables us to fill in groups at the end. In order to produce many quadruples using the extension of parallel classes, we desire primarily parallel classes of quadruples, or of triples (that can then be extended). A particularly useful result in this vein was proved by Rees and Stinson [24], with some further cases settled by Assaf and Hartman [1]:

LEMMA 3.9. *Let g, t satisfy $t \geq 3$, $gt \equiv 0 \pmod 3$ and $g(t-1) \equiv 0 \pmod 2$. Then there is a resolvable* $\{3\}$-GDD *of group-type g^t except for 2^3, 2^6, 6^3, and with the possible exceptions of $t = 6$ and $g \equiv 2, 10$ (mod 1)2.* ∎

Thus far, we have considered only resolvable designs in which each parallel class is *uniform*, in that every block in the parallel class has the same size. Rees has made substantial advances on $\{2,3\}$-PBDs in which the parallel classes are nonuniform:

LEMMA 3.10 [20]. *There exists a resolvable* $\{2,3\}$-PBD *with an even number p of elements and r parallel classes if and only if $\frac{1}{2}p \leq r \leq p-1$ and $p(r-p+1) \equiv 0$* (mod 3), *with the exceptions* $(p,r) = (6,3), (12,6)$. ∎

LEMMA 3.11 [21,22]. *There exists a resolvable* $\{2,3\}$-PBD *with an odd number p of elements and r parallel classes provided $p(r-p+1) \equiv 0 \pmod 3$ and one of the following holds:*

(i) $\frac{1}{2}p \leq r \leq p-4$,
(ii) $p \equiv 3 \pmod 6$ *and* $r = \frac{1}{2}(p-1)$, *or*
(iii) $(p,r) = (9,6)$. ∎

A resolvable PBD produced by Lemma 3.10 or 3.11 has $p(p-1-r)/3$ triples and $p(2r-p+1)/2$ pairs.

Next we require further $\{4\}$-GDDs for use in the Fundamental Construction.

LEMMA 3.12 [7]. *Let g, t be integers satisfying $t \geq 4$, $g(t-1) \equiv 0 \pmod 3$ and $g^2t(t-1) \equiv 0 \pmod 4$. Then there exists a* $\{4\}$-GDD *of group-type g^t except when* $(g,t) \in \{(2,4),(6,4)\}$. ∎

At this point, $\{4\}$-GDDs are available from Lemma 3.4 (by truncation), Lemma 3.5, and Lemma 3.12. We require a few further small GDDs:

LEMMA 3.13 [25,26]. *There exist* $\{4\}$ − GDD*s of group-type* $3^4 6^2$, $3^1 6^4$, $3^6 9^2$.

Combining Lemmas 3.4, 3.12 and 3.13, we observe that $\{4\}$-GDDs with group sizes 3 and 6 exist on v elements except when $v = 18$; when at least one group of size 6 is required, we have

LEMMA 3.14. *There is a* $\{4\}$-GDD *with groups of sizes 3 and 6, having at least one group of size 6, for all $v \equiv 0 \pmod 3$, $v > 18$.*

PROOF. Using the GDD of type $6^2 3^4$, we have such a GDD for all $v \equiv 0, 3 \pmod{12}$, $v \geq 75$. Lemma 3.4 provides such GDDs for $v \equiv 6, 9 \pmod{12}$, and Lemma 3.12 gives such GDDs for $v \in \{36, 48, 60, 72\}$. Hence we need only treat the cases $v = 39$, 51 and 63. For $v = 39$, take elements $\{1, \ldots, 39\}$, and take the blocks obtained from the starter blocks $\{1,3,11,18\}, \{1,4,15,24\}, \{1,2,6,37\}$ under the action of the permutation $(1\,2 \cdots 36)(37\,38\,39)$. This is a GDD of type $3^1 6^6$. For $v = 51$, take elements $\{1, \cdots, 51\}$, and take the blocks obtained from starter blocks $\{1,2,6,49\}$, $\{1,3,16,22\}$, $\{1,4,13,27\}$ and $\{1,8,18,38\}$ under the action of the permutation $(1\,2\,3 \ldots 48)(49\,50\,51)$. This is a GDD of type $3^1 6^8$.

For $v = 63$, take elements $\{1, \ldots, 63\}$, and take the blocks to be those obtained from starter blocks $\{1,2,6,61\}$, $\{1,3,9,22\}$, $\{1,4,13,36\}$, $\{1,8,25,39\}$ and $\{1,12,27,45\}$ under the action of the permutation $(1\,2\,3 \cdots 60)(61\,62\,63)$. ∎

When $v \equiv 0,1,3,4,7,10 \pmod{12}$, we have seen the PBD with the maximum number of quadruples; all are obtained from Lemma 3.4. However, for $v \equiv 6,9 \pmod{12}$, this maximum is *not* obtained in this way. Instead we employ a result of Mills on "coverings"; his result implies the following:

LEMMA 3.15 [17]. *For $v \equiv 6,9 \pmod{12}$, there is a $\{3,4\}$-PBD with precisely $(v+3)/3$ triples.* ∎

Actually, Mills proved that the minimum covering of pairs on a set of size $v \equiv 7,10 \pmod{12}$ by quadruples has an excess that is a single pair covered four times rather than once. Truncating Mills's covering by removing either of the elements in this excess pair produces Lemma 3.15.

In the constructions that follow, we assume that whenever possible, the basic designs given by Lemmas 3.3–3.15 are employed as ingredients to fill in groups, fill holes, extend parallel classes, and truncate. We typically state only the basic design that is constructed, and assume that the operations mentioned are performed in a suitable way to obtain the specified number of quadruples.

4. **Small ingredients.** In this section, we develop quite a large collection of small $\{3,4\}$-PBDs, IPBDs and GDDs for use in the recursive constructions of section 3. Since we require IPBDs to fill in groups effectively, we remark first on some trivial connections between $\mathrm{Spec}_4(v,w)$ and $\mathrm{Spec}_4(v)$. First observe that $\mathrm{Spec}_4(v) = \mathrm{Spec}_4(v,0) = \mathrm{Spec}_4(v,1)$. Now, $\mathrm{Spec}_4(v,3) = \mathrm{Spec}_4(v) \setminus \{v(v-1)/12\}$. Furthermore,

$$\mathrm{Spec}_4(v,4) = \{s-1 : s \in \mathrm{Spec}_4(v),\ s \neq 0\}.$$

Finally, if $s \in \mathrm{Spec}_4(v,w)$ and $t \in \mathrm{Spec}_4(w)$, then filling the hole gives $s+t \in \mathrm{Spec}_4(v)$.

Since $\mathcal{A}(v)$ has approximately $v^2/12$ elements, we are naturally unable to present explicit constructions for each case. We organize the presentation by defining the *period* of v to be $\lfloor v/12 \rfloor$. In the zeroth and first periods, we give explicit constructions for each design. In the second and third periods, we simply summarize the consequences of Lemmas 3.3–3.15 supplemented by filling in groups and holes, extending parallel classes, and truncating. Additional designs in these periods are presented explicitly in a supplementary report. The solution for the fourth and higher periods is then pursued in section 5.

4.1. Zeroth and first periods. The systematic investigation of small PBDs was first undertaken by Kelly and Nwankpa [14]; they classified all PBDs on at most fourteen elements. The classification of PBDs was extended to $v = 15$ by Brouwer [6]. Beyond that point, no complete classification is available. Nevertheless, we can exploit the available catalogues to determine $\mathrm{Spec}_4(v,w)$ for $v \leq 15$.

In the zeroth period, there are *unique* $(v, \{3,4\})$-PBDs except for $v = 6$ where no such PBD exists:

v	# triples	# quadruples
0	0	0
1	0	0
3	1	0
4	0	1
7	7	0
9	12	0
10	9	3

Truncating the $(9, \{3\})$-PBD gives a $\{3\}$-GDD of type 2^4 of which we make extensive use.

In the first period, a variety of PBDs begins to appear. For $v = 12$, we have $\mathrm{Spec}_4(12) = \{3,9\}$. Using parallel classes in these PBDs, we obtain a $\{3\}$-GDD of type 4^3, a $\{4\}$-GDD of type 3^4, and a $\{3,4\}$-GDD of type 3^4 with 3 quadruples and 12 triples.

For $v = 13$, we have $\mathrm{Spec}_4(13) = \{0,6,7,13\}$. Brouwer [6] established that $\mathrm{Spec}_4(15) = \{0,5,6,7,10,14,15\}$. In the process, he established the following:

LEMMA 4.1. *There exist* $\{3,4\}$-GDD*s of type* 3^5 *having 0, 5, 6, 10 and 15 quadruples.* ■

The Doyen-Wilson Theorem (see Lemma 3.3) establishes that for $w \equiv 1,3 \pmod 6$, $\mathrm{Spec}_4(2w+1, w) = \{0\}$; hence $\mathrm{Spec}_4(15,7) = \{0\}$. In addition, $\mathrm{Spec}_4(15,6) = \{6\}$.

For $v \geq 16$, we can no longer rely on exhaustive catalogues.

LEMMA 4.2. $\mathrm{Spec}_4(16) = \{4,5,6,7,9,10,11,12,15,20\}$. *Moreover, there exist* $\{3,4\}$-GDD*s of type* 4^4 *having 0, 8 and 16 quadruples.*

PROOF. The GDDs are constructed as follows. Use Lemma 3.8 with $n = 4$, $r = 4$, 5 and 6 to produce a resolvable $\{2,3\}$-GDD of type 4^3 with 0, 2 or 4 parallel classes of 2-blocks (and hence 4, 3 or 2 parallel classes of triples). Extend four parallel classes to produce the required GDD. Taking groups as blocks in these GDDs gives $\{4,12,20\} \subseteq \mathrm{Spec}_4(16)$.

Filling in groups in Lemma 4.1 gives $\{5,10,11,15\} \subseteq \mathrm{Spec}_4(16)$. For $6 \in \mathrm{Spec}_4(16)$, take the following PBD:

dehi	dfjk	dglm	efno	egpa	fgbc	dnb	doa	dpc	ejb	ekm	elc	fhp
fil	fma	ghj	gin	gko	hkl	hmn	hob	hac	ijc	ika	imb	iop
jlo	jmp	jna	knc	kpb	lnp	lab	moc					

In this notation, we use letters to represent the elements of the design, and use abc to denote a block $\{a,b,c\}$.

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$7 \in \mathrm{Spec}_4(16)$

abch	cdei	aefj	bdfk	adlm	beno	cfgp	agn	aik	aop	bgl	bij	bmp
cjn	ckl	cmo	dgo	dhn	djp	egk	ehm	elp	fhl	fio	fmn	ghj
gim	hip	hko	iln	jkm	jlo	knp						

$9 \in \mathrm{Spec}_4(16)$

abcd	aefg	ahij	behk	cfil	dgjm	bfjn	cgho	deip	akp	aln	amo	bgp
bio	blm	cej	ckn	cmp	dfk	dhl	dno	elo	emn	fhm	fop	gin
gkl	hnp	ikm	jko	jlp								

We next show that the remaining values in $\mathcal{A}(16)$ do not appear in $\mathrm{Spec}_4(16)$. Now if $s > 12$, any such PBD with s quadruples must have an element that meets only quadruples. Truncating to remove this element gives $s - 5 \in \mathrm{Spec}_4(15)$. This rules out $s \in \{13, 14, 16\}$ from $\mathrm{Spec}_4(16)$.

The final case to consider is $s = 8$. Elementary counting shows that there is a unique possible configuration of eight quadruples up to isomorphism, namely 012a, 345b, 036c, 147d, 057e, 246f, 156g and 237h. A exhaustive search by computer showed that among the remaining pairs, the closest one can come to a partition into triples is to obtain 22 triples and one hexagon. Hence no solution exists here. ∎

It is easy to verify that $10 \in \mathrm{Spec}_4(16, 6)$ using a resolvable $(10, \{2, 3\})$-PBD with six parallel classes from Lemma 3.10.

For $v = 18$, we have the following:

LEMMA 4.3. *$\{5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16\} \subseteq \mathrm{Spec}_4(18)$, and $\{20, 21, 22\} \cap \mathrm{Spec}_4(18) = \emptyset$.*

PROOF. First we treat the affirmative cases. The PBD with 5 quadruples is given by Lemma 6.14 of [27], and that with 6 quadruples by Lemma 6.15 of [27]. The PBD with 11 quadruples is obtained from a resolvable $(11, \{2, 3\})$-PBD with 7 parallel classes from Lemma 3.11. The PBD with 15 quadruples is obtained by extending three parallel classes of a resolvable $(15, \{3\})$-PBD. The PBD with 16 blocks is obtained by truncating a point from a PBD given by Stanton [29].

We next exhibit a number of designs explicitly. In each case, we chose a set of quadruples meeting the necessary conditions on degrees; then we used a hill-climbing algorithm similar to that of Stinson [32] to partition the remaining pairs into triples.

$7 \in \mathrm{Spec}_4(18)$

abcd	aefg	ahij	dgjr	dklm	gkno	jkpq	akr	alq	amo	anp	ber	bfi
bgl	bhk	bjm	bnq	bop	cem	cfk	cgq	chl	cin	cjo	cpr	dep
dfn	dho	diq	ehq	eik	ejn	elo	fhr	fjl	fmp	foq	ghp	gim

hmn ilp ior lnr mqr

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$8 \in \mathrm{Spec}_4(18)$

abcd	aefg	ahij	cfik	dgjl	dimn	cgop	fjqr	akm	alr	anp	aoq	bej
bfn	bgm	bho	bir	bkq	blp	cer	chl	cjm	cnq	dek	dfo	dhq
dpr	ehn	eip	elq	emo	fhp	flm	ghk	giq	gnr	hmr	ilo	jkp
jno	kln	kor	mpq									

$9 \in \mathrm{Spec}_4(18)$

abek	bcfl	cdgm	dehn	efio	fgap	ghbq	hicr	iadj	acq	aho	alm	anr
bdp	bim	bjr	bno	cej	cko	cnp	dfk	dlr	doq	egr	elq	emp
fhj	fmn	fqr	gik	gjn	glo	hkm	hlp	iln	ipq	jkl	jmq	jop
knq	kpr	mor										

$10 \in \mathrm{Spec}_4(18)$

abcd	aefg	ahij	behk	cfik	dgjk	bfjl	cemn	dhop	giqr	akl	amo	anq
apr	bgp	bim	bnr	boq	cgo	chq	cjr	clp	der	dfq	din	dlm
eip	ejo	elq	fhn	fmp	for	ghm	gln	hlr	ilo	jmq	jnp	kmr
kno	kpq											

$12 \in \mathrm{Spec}_4(18)$

abcd	efgh	ijkl	mnop	aeim	bfjn	cgko	dhlq	afkr	bglm	chin	dejo	agp
aho	ajq	aln	bek	bhr	bip	boq	cel	cfm	cjp	cqr	dfp	dgi
dkn	dmr	enr	epq	fiq	flo	gjr	gnq	hjm	hkp	ior	kmq	lpr

$13 \in \mathrm{Spec}_4(18)$

abcd	aefg	ahij	dlmn	glop	jlqr	dgjk	bfio	einq	bhnp	mpre	orch	qcfm
akl	amo	anr	apq	bel	bgq	bjm	bkr	cek	cgn	cil	cjp	deh
dfr	dip	doq	ejo	fhl	fjn	fkp	ghm	gir	hkq	ikm	kno	

$14 \in \mathrm{Spec}_4(18)$

abhn	aclp	adij	aegq	afmo	rbgl	rcio	rdfp	rejn	rhmq	bcde	fghi	jklm
nopq	akr	bfk	biq	bjo	bmp	cfn	cgm	chj	ckq	dgo	dhk	dlq
dmn	efl	ehp	eim	eko	fjq	gjp	gkn	hlo	ikp	iln		

The nonexistence results follow from the nonexistence of PBDs on 17 elements with 31 or fewer blocks [30]; for truncating a $(18, \{3, 4\})$-PBD gives a PBD with maximum block size four on 17 points, having the same number of blocks. ∎

We also have $\{3,4,11,12\} \subseteq \text{Spec}_4(18,6)$. These four values are obtained as follows. The first is a $\{3\}$-GDD of type $4^3 6^1$. The second is obtained by filling groups in a $\{3\}$-GDD of type $3^4 5^1$. The third is obtained by taking a resolvable $\{2,3\}$-GDD with type 4^3, and 2 parallel classes of triples (from Lemma 3.8 with $n = 4$ and $r = 6$), and extending all six parallel classes. The fourth is obtained by removing one point from a TD(4,4), then filling in groups using a $(7,\{3\})$-PBD and leaving a 6-hole.

We have further that $11 \in \text{Spec}_4(18,7)$; this is the same construction as that for 11 quadruples in Lemma 4.3.

LEMMA 4.4. *$\mathcal{A}(19) \setminus \{17,23,24,25\} \subseteq \text{Spec}_4(19)$. $\{23,24,25\} \cap \text{Spec}_4(19) = \emptyset$. Moreover, $\{0,7,8,16\} \subseteq \text{Spec}_4(19,7)$.*

PROOF. For $7 \in \text{Spec}_4(19,7)$, take a resolvable $\{2,3\}$-GDD of type 4^3 from Lemma 3.8 (with $n = 4$, $r = 7$) and extend all seven parallel classes. For $8 \in \text{Spec}_4(19,7)$, take a resolvable $(12,\{2,3\})$-PBD with five parallel classes of 2-blocks and three parallel classes of triples (Lemma 3.7 with $x = 5$, $y = 3$); extend 7 parallel classes. For $16 \in \text{Spec}_4(19,7)$, fill in groups in a $\{4\}$-GDD of type 4^4 using $(7,3,\{3\})$-IPBDs.

Now $\text{Spec}_4(19,7) \subseteq \text{Spec}_4(19)$. The case $18 \in \text{Spec}_4(19)$ is handled by truncating two points in a group from a $\{4\}$-GDD of type 2^{10}, and filling in the remaining groups. The case $19 \in \text{Spec}_4(19)$ is handled by removing two points on a triple from the PBD on 21 elements given in Lemma 3.15. The case $20 \in \text{Spec}_4(19)$ was found by hand using an *ad hoc* construction, and has blocks as follows: abcd, aepq, ahmr, alos, bgms, bkoq, blnr, cekr, cips, cjmq, dglq, dhks, djpr, efgh, ejns, fqrs, gior, hinq, ijkl, mnop, afi, agj, akn, bei, bfp, bhj, cfl, cgn, cho, deo, dfn, dim, elm, fjo, fkm, gkp, hlp.

For $21 \in \text{Spec}_4(19)$, remove three points from the hole of a $(22,7,\{4\})$-IPBD. Stanton [29] gives a solution with 22 quadruples.

The case $12 \in \text{Spec}_4(19)$ is handled by taking elements $Z_6 \times \{1,2,3\} \cup \{\infty\}$, and starter blocks $0_1 1_1 2_2 2_3$, $0_1 2_1 0_2 5_3$, $0_1 3_2 5_2$, $0_1 0_3 4_3$, $0_2 1_2 4_3$, $0_2 1_3 2_3$, and $\{\infty\} 0_i 3_i$ for $i \in \{1,2,3\}$.

$5 \in \text{Spec}_4(19)$

abcd	aefg	behi	cfhj	dgij	ahr	ais	ajp	ako	aln	amq	bfq	bgl
bjo	bkp	bms	bnr	cer	cgq	cio	ckn	cls	cmp	dep	dfn	dhk
dlm	doq	drs	ejs	ekq	elo	emn	fim	fkr	flp	fos	ghm	gks
gno	gpr	hlq	hns	hop	ikl	inp	iqr	jkm	jlr	jnq	mor	pqs

$6 \in \text{Spec}_4(19)$

abcd	efgh	aeij	bfkl	cgik	dhjl	afr	agn	ahp	ako	alq	ams	bes
bgo	bhi	bjr	bmq	bnp	cer	cfo	chm	cjp	cln	cqs	deq	dfs
dgp	dio	dkm	dnr	ekp	elo	emn	fin	fjm	fpq	gjq	gls	gmr
hkq	hns	hor	ilm	ips	iqr	jkn	jos	krs	lpr	mop	noq	

$9 \in \mathrm{Spec}_4(19)$

abcd	aefg	nhij	nklm	ehko	cfip	gjmq	blqr	opdr	ahs	aik	ajl	amp
anr	aoq	ben	bfm	bgk	bhp	bio	bjs	cem	cgh	cjr	cks	clo
cnq	dej	dfn	dgs	dhl	dim	dkq	eil	epq	ers	fhq	fjo	fkr
fls	gir	glp	gno	hmr	iqs	jkp	mos	nps				

$10 \in \mathrm{Spec}_4(19)$

abcd	aefg	ahij	aklm	beno	hkpq	cfrs	ilns	dgop	jmqr	anq	aor	aps
bfk	bgj	bhs	bir	blq	bmp	ceq	cgl	cho	cim	cjp	ckn	dem
dfi	dhn	djk	dlr	dqs	ehl	eip	ejs	ekr	fhm	fjn	flp	foq
ghr	giq	gks	gmn	iko	jlo	mos	npr					

$11 \in \mathrm{Spec}_4(19)$

bcfh	degi	nprs	almr	ajqs	hiqr	gjkr	fkls	emos	abno	acdp	aeh	afg
aik	bdr	bel	bgs	bij	bkp	bmq	cer	cgo	cis	cjm	ckq	cln
dfm	dhs	djl	dko	dnq	efq	ejp	ekn	fip	fjn	for	ghn	glq
gmp	hjo	hkm	hlp	ilo	imn	opq						

$13 \in \mathrm{Spec}_4(19)$

bcde	bfkp	bhna	bqrs	cjoa	dhmq	efgh	einr	ekqa	glsa	hijk	klmn	nopq
adr	afi	amp	bgm	bio	bjl	cfn	cgp	chs	cim	ckr	clq	dfo
dgn	dil	djp	dks	ejm	elp	eos	fjq	flr	fms	giq	gjr	gko
hlo	hpr	ips	jns	mor								

$14 \in \mathrm{Spec}_4(19)$

bcda	efga	hija	klma	nopa	qrsa	behk	bfio	cenq	cjmr	dfjq	bglq	cflp
ejls	bjp	bms	bnr	cgh	cik	cos	dem	dgk	dhn	dis	dlo	dpr
eip	eor	fhs	fkr	fmn	gir	gjo	gmp	gns	hlr	hmo	hpq	iln
imq	jkn	koq	kps									

$15 \in \mathrm{Spec}_4(19)$

bcdk	efgl	hijm	behn	cfio	dgjp	bfjq	cghr	deis	bgia	ceja	dfha	kpra
lnsa	moqa	blm	bor	bps	clp	cms	cnq	dlq	dmr	dno	ekm	eop
eqr	fks	fmp	fnr	gko	gmn	gqs	hkl	hos	hpq	ikq	ilr	inp
jkn	jlo	jrs										

For the nonexistence results, see Stanton [29]. ■

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LEMMA 4.5. $\{0,8\} \subseteq \text{Spec}_4(21,9)$, $\{0,8,28\} \subseteq \text{Spec}_4(21,7)$, $30 \in \text{Spec}_4(21,6)$, *and*

$$\{0,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,28,30,31\} \subseteq \text{Spec}_4(21).$$

PROOF. For $8 \in \text{Spec}_4(21,9)$, extend nine parallel classes of a resolvable $(12,\{2,3\})$-PBD with 2 parallel classes of triples and 7 parallel classes of 2-blocks (from Lemma 3.7). For $8 \in \text{Spec}_4(21,7)$, truncate one group of an ITD(4,6,2) to the two points of the hole, and extend the groups that result by one point. For $28 \in \text{Spec}_4(21,7)$, puncture one point not in the hole of a $(22,7,\{4\})$-PBD. For $30 \in \text{Spec}_4(21,6)$, remove instead a point of the 7-hole.

This settles 8 and 28 in $\text{Spec}_4(21)$. For $12 \in \text{Spec}_4(21)$, extend two parallel classes and the groups of a nearly Kirkman triple system of order 18 (from Lemma 3.7). For $21 \in \text{Spec}_4(21)$, remove the four points of a block in a $(25,\{4\})-$PBD. For $30 \in \text{Spec}_4(21)$, add one infinite point to the groups in a $\{4\}$-GDD of type 2^{10}. For $31 \in \text{Spec}_4(21)$, use Lemma 3.15.

For $7 \in \text{Spec}_4(21)$, on elements $Z_7 \times \{1,2,3\}$, take starter blocks $0_1 3_1 1_2 2_2$, $0_1 0_2 0_3$, $0_1 1_1 4_2$, $0_1 2_1 6_2$, $0_1 1_3 3_3$, $0_1 2_3 5_3$, $0_2 2_2 1_3$, $0_2 4_2 2_3$ and $0_2 3_3 4_3$ (modulo $(7,-)$).

For $10 \in \text{Spec}_4(21)$, on elements $(Z_{10} \times \{1,2\}) \cup \{\infty\}$, take starter blocks $0_1 4_1 0_2 3_2$, $0_1 1_1 3_1$, $0_1 1_2 5_2$, $0_1 2_2 4_2$, $0_1 7_2 8_2$, $\infty 0_1 5_1$ and $\infty 0_2 5_2$.

For $14 \in \text{Spec}_4(21)$, on elements $Z_7 \times \{1,2,3\}$, take starter blocks $0_1 3_1 1_2 2_2$, $0_1 2_1 4_3 5_3$, $0_1 0_2 0_3$, $0_1 1_1 4_2$, $0_1 1_3 6_3$, $0_2 2_2 5_3$, $0_2 3_2 4_3$ and $0_2 2_3 6_3$.

For $15 \in \text{Spec}_4(21)$, on elements $(Z_{10} \times \{1,2\}) \cup \{\infty\}$, take starter blocks $0_1 1_2 2_2 4_2$, $0_1 5_1 0_2 5_2$, $0_1 3_2 7_2$, $0_1 1_1 4_1$, $0_1 2_1 8_2$ and $\infty 1_1 0_2$.

For $18 \in \text{Spec}_4(21)$, on elements $Z_{18} \cup \{A,B,C\}$, take starter blocks $\{0,1,3,8\}$ and $\{0,6,12\}$. This leaves differences 4 and 9 unused in Z_{18}. The corresponding cubic graph has a 1-factorization [31] F_1, F_2, F_3. Form triples consisting of pairs in F_1 together with A, pairs in F_2 together with B and pairs in F_3 together with C. Finally, add the triple ABC.

For $20 \in \text{Spec}_4(21)$, on elements $(Z_{10} \times \{1,2\}) \cup \{\infty\}$ take starter blocks $0_1 1_1 4_1 5_2$, $0_1 2_2 3_2 6_2$, $0_1 2_1 0_2$, $0_1 7_2 9_2$, $\infty 0_1 5_1$ and $\infty 0_2 5_2$.

$5 \in \text{Spec}_4(21)$

abcd	aefg	behi	cfhj	dgij	ahq	aio	aju	akr	alm	ans	apt	bfk
bgq	bjm	blu	bnt	bos	bpr	cer	cgt	cik	clq	cmn	cou	cps
den	dfl	dhm	dkp	doq	drs	dtu	ejt	ekl	emu	eop	eqs	fir
fmq	fno	fpu	fst	gho	gkn	gls	gmp	gru	hkt	hlr	hnp	hsu
ilp	ims	inu	iqt	jks	jln	jor	jpq	kmo	kqu	lot	mrt	nqr

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$6 \in \mathrm{Spec}_4(21)$

abcd efgh aeij bfkl cgik dhjl afn agp aho aks alq amu art
bem bgq bhr bin bjt bos bpu cer cfo chp cjs clm cnu
cqt deo dfp dgu diq dkr dms dnt ekt eln epq esu fis
fjq fmr ftu gjo gls gmt gnr hiu hkm hnq hst ilr imp
iot jkp jmn jru kno kqu lou lpt moq nps opr qrs

$9 \in \mathrm{Spec}_4(21)$

abcd aefg nhij nklm ehko cfip gjmq blqr opdr ahr ais ajt akq
alu amo anp bej bfu bgh bim bkp bno bst cem cgo chl
cjk cnr cqt csu deu dfn dgk dhs diq djl dmt eir els
ent epq fhq fjo fks flt fmr giu glp gns grt hmu hpt
ikt ilo jpu jrs kru mps nqu oqs otu

$11 \in \mathrm{Spec}_4(21)$

bcfh degi nprs almr ajqs hiqr gjkr fkls emos abno acdp aeh afg
aiu akt bdk beq bgs bim bju blp brt cej cgn cis ckm
clq cot cru dfu dhs djt dln dmq dor efr ekp elu ent
fin fjo fmt fpq ghm glo gpu gqt hjn hku hlt hop ijl
iko ipt jmp knq mnu oqu stu

$13 \in \mathrm{Spec}_4(21)$

bcde bfkp bhna bqrs cjoa dhmq efgh einr ekqa glsa hijk klmn nopq
adf aim apt aru bgm biu bjt blo cfs cgq chr cil ckt
cmp cnu dgn dis djl dkr dot dpu ejp elu emt eos fio
fjn flq fmu frt gip gjr gko gtu hlt hou hps iqt jms
jqu ksu lpr mor nst

$16 \in \mathrm{Spec}_4(21)$

qrst rcei csfj deuk eftl fuim ghjn hiko ijlp jkmq klnb lmoc mnpd
noae opbf pacg abh adf aiq aju aks alr amt bcu bdr bej
bgq bit bms cdh ckt cnq dgi djt dlq dos egs ehm epq
fgk fhq fnr glu gmr got hls hpt hru ins jor kpr ntu
oqu psu

$17 \in \mathrm{Spec}_4(21)$

rsth scei ctfj degk efhl fgim ghjn hiko ijlp jkmq klnr lmob mnpc
noqd opae pqbf qacg abi ads afk ahm ajr alt anu bch bdj
ben bgs bkt bru cdl cku cor dfr dhp diu dmt eju emr
eqt fns fou glu got gpr hqu int iqr jos kps lqs msu
ptu

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$19 \in \mathrm{Spec}_4(21)$

abdi	bcej	cdfk	degl	efhm	fgin	ghjo	hikp	ijlq	jkmr	klns	lmoa	mnpb
noqc	oprd	pqse	qraf	rsbg	sach	aeu	agp	ajn	akt	bfo	bht	bkq
blu	cgm	ciu	clp	crt	dhn	dju	dmq	dst	eir	eko	ent	fjs
flt	fpu	gku	gqt	hlr	hqu	imt	ios	jpt	msu	nru	otu	■

LEMMA 4.6. $4 \in \mathrm{Spec}_4(22,10)$. $13 \in \mathrm{Spec}_4(22,9)$. $\{7,15,35\} \subseteq \mathrm{Spec}_4(22,7)$. $4 \in \mathrm{Spec}_4(22,6)$. *Finally,* $\mathrm{Spec}_4(22)$ *contains*

$$\{6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,25,28,35\}.$$

PROOF. For $4 \in \mathrm{Spec}_4(22,10)$, extend the groups of a $\{3\}$-GDD of type $9^1 3^4$. For $13 \in \mathrm{Spec}_4(22,9)$, use a resolvable $(13,\{2,3\})$-PBD with nine parallel classes (from Lemma 3.11). For $7 \in \mathrm{Spec}_4(22,7)$, truncate a group of a TD(4,7) to one point. For $15 \in \mathrm{Spec}_4(22,7)$, remove three points not in the hole from a group of an ITD(4,6,2) and add one point at infinity to the groups. For $35 \in \mathrm{Spec}_4(22,7)$, take a $(22,7,\{4\})$-PBD. For $4 \in \mathrm{Spec}_4(22,6)$, take a $\{3\}$-GDD of type $6^1 4^4$.

In consequence, we obtain 7, 13, 15 and 35 in $\mathrm{Spec}_4(22)$. For $19 \in \mathrm{Spec}_4(22)$, extend three parallel classes and the groups of a nearly Kirkman triple system of order 18. For $28 \in \mathrm{Spec}_4(22)$, remove three points from a block of a $(25,\{4\})$-PBD. Now $6 \in \mathrm{Spec}_4(22)$ is given by Lemma 6.23 in [27]. We obtain 14, 21 and 25 from extending parallel classes in the solutions for 7, 14 and 18 in Lemma 4.5 for 21 points. For $11 \in \mathrm{Spec}_4(22)$, on elements $Z_{11} \times \{1,2\}$ take starter blocks $0_1 4_1 5_1 0_2, 0_2 6_1 9_1, 0_2 8_1 10_1, 0_2 1_2 2_1, 0_2 4_2 7_1$ and $0_2 2_2 5_2$. For $22 \in \mathrm{Spec}_4(22)$, take instead starter blocks $0_1 4_1 0_2 1_2$, $0_1 2_1 3_1 5_2, 0_2 2_1 7_1, 0_2 4_2 5_1$ and $0_2 2_2 5_2$.

For $16 \in \mathrm{Spec}_4(22)$, add a point to the parallel class {afg, bim, clq, dor, ekp, hjn, stu} of the the solution for $9 \in \mathrm{Spec}_4(21)$. For $18 \in \mathrm{Spec}_4(22)$, add a point to the parallel class {ahr, bfu, cjk, dmt, epq, gns, ilo} of the solution for $11 \in \mathrm{Spec}_4(21)$.

$8 \in \mathrm{Spec}_4(22)$

abcd	aefg	ahij	dgjr	dklm	gkno	jkpq	stuv	akt	alp	amn	aos	aqu
arv	beu	bft	bgs	bhr	bip	bjm	bkv	blo	bnq	ceo	cfi	cgh
cjl	cks	cmr	cnv	cpu	cqt	deh	dfu	div	dnt	doq	dps	eik
ejs	eln	emq	epv	ert	fho	fjv	fkr	flq	fms	fnp	giq	glv
gmu	gpt	hku	hlt	hmp	hns	hqv	ils	imt	inr	iou	jnu	jot
lru	mov	opr	qrs									

$9 \in \mathrm{Spec}_4(22)$

abcd	aefg	ahij	cfik	dgjl	dimn	cgop	fjqr	stuv	akv	alm	ans	aot
apq	aru	bes	bfn	bgk	bhq	bio	bju	blv	bmt	bpr	ceh	cjv
clu	cmr	cnt	cqs	deq	dft	dhs	dkp	dou	drv	eir	ejm	eku
elt	enp	eov	fhu	flp	fmv	fos	ghm	giu	gnv	gqt	grs	hko
hln	hpv	hrt	ils	ipt	iqv	jkt	jno	jps	klq	kms	knr	lor
moq	mpu	nqu										

$10 \in \mathrm{Spec}_4(22)$

abek	bcfl	cdgm	dehn	efio	fgap	ghbq	hicr	iadj	stuv	acv	aht	alo
amu	anr	aqs	bds	bim	bjt	bnu	bop	brv	cet	cjk	cno	cps
cqu	dfu	dkt	dlv	doq	dpr	egs	ejm	elq	epv	eru	fhm	fjs
fkr	fnv	fqt	giu	gjn	gkv	glr	got	hjv	hko	hls	hpu	ikp
ilt	ins	iqv	jlp	jou	jqr	klu	kms	knq	lmn	mov	mpq	mrt
npt	ors											

$12 \in \mathrm{Spec}_4(22)$

sbcd	efgh	ijkl	mnop	teim	bfjn	cgko	vhlq	afkr	bglm	chin	uejo	abq
acj	adg	aep	aho	aiu	alt	ams	anv	bek	bht	bir	bov	bpu
cev	cfl	cmr	cpq	ctu	der	dfq	dhm	div	djp	dkn	dlu	dot
eln	eqs	fip	fmu	fos	ftv	gis	gjt	gnu	gpv	gqr	hjr	hkp
hsu	ioq	jmq	jsv	kmv	kqu	kst	lor	lps	nqt	nrs	prt	ruv

$17 \in \mathrm{Spec}_4(22)$

abcd	aefg	ahij	aklm	beno	hkpq	cfrs	ilns	dgop	jmqr	anrv	aou	aps
aqt	bfk	bgjv	bhr	biu	blt	bmp	bqs	ceh	cgq	cipv	cjt	ckn
clu	cmo	deu	dfhv	dim	djk	dlr	dnq	dst	eiq	ejl	ekr	emsv
ept	fit	fjo	flp	fmn	fqu	ghl	gir	gks	gmt	gnu	hmu	hnt
hos	iko	jnp	jsu	ktuv	loqv	ort	pru					

$20 \in \mathrm{Spec}_4(22)$

bcde	bfkp	bhna	bqrs	cjoa	dhmq	efgh	einr	ekqa	glsa	hijk	klmn	nopq
adf	aip	amrv	atu	bgt	bilv	bju	bmo	cfm	cgr	chuv	ciq	cks
clp	cnt	dgp	diu	djl	dkt	dnsv	dor	ejpv	elt	emu	eos	fis
fjn	flr	fou	fqtv	gim	gjq	gkov	gnu	hlo	hpr	hst	iot	jms
jrt	kru	lqu	mpt	psu								

$23 \in \text{Spec}_4(22)$

qrst	rcei	csfj	deuk	eftl	fuim	ghjn	hiko	ijlp	jkmq	klnb	lmoc	mnpd
noae	opbf	pacg	abh	adfv	aiq	aju	akr	als	amt	bcu	bdr	bej
bgq	bitv	bms	cdh	ckt	cnqv	dgi	djt	dlq	dos	egs	ehmv	epq
fgk	fhq	fnr	gluv	gmr	got	hlr	hpt	hsu	ins	jorv	kpsv	ntu
oqu	pru											∎

4.2. Second and Third Periods We simply state the results obtained from the recursive constructions of section 3 using the small designs from section 4.1. The verification is purely mechanical; in fact, we have employed a sequence of computer programs embodying the constructions of Lemmas 3.1–3.15 in order to verify that the results stated are correct. We have also constructed a number of specific designs in these periods. These are given explicitly in the supplementary report. In Tables 4.1 and 4.2, we list the *possible* exceptions, i.e. the values in $\mathcal{A}(v)$ not shown to be in $\text{Spec}_4(v)$ by example or by recursive construction. In Table 4.2, we list a value s when $M_v - s$ is a possible exception, since in the third period all remaining exceptions are near the maximum number of quadruples. We do not comment on $\text{Spec}_4(v, w)$ in these periods; although we obtain many results from the recursive constructions for $w \geq 6$, few are needed for the proof of the Main Theorem to follow.

TABLE 4.1 Second Period.

v	Possible Values in $\mathcal{A}(v) \setminus \text{Spec}_4(v)$
24	19 22 23 24 26 27 28 29 31 32 33 34 36 37 38 39 40 41
25	27 28 30 31 33 34 35 37 39 40 41 42 44 45 46
27	22 23 28 29 31 33 34 35 38 39 40 41 43 44 46 47 49 50 51 53
28	31 32 34 35 38 40 42 43 44 47 48 50 52 53 55 56 57 58 59
30	25 26 28 29 31 32 34 35 36 37 38 40 41 42 43 44 46 50 53 55 56 57 58 59 60 61 62 65 66
31	27 28 29 34 35 38 39 40 43 45 46 49 50 51 52 53 54 63 65 66 67 68 69 70 71 72
33	28 29 30 34 36 38 41 44 45 50 51 55 56 57 61 62 63 67 68 69 70 71 72 73 75 76 81
34	28 35 48 54 58 60 61 64 66 67 70 72 73 74 76 78 79 80 81 82 83 84 86 87

5. **Proof of the Main Theorem.** In order to prove the Main Theorem, we treat congruence classes of $v \pmod{1}2$. In all eight classes, we begin with two applications of the Fundamental Construction; hence we examine these first before considering particular classes.

LEMMA 5.1. *For* $t \equiv 0, 1 \pmod 4$, $0 \leq s \leq 12t^2 - 12t$, $s \equiv 0 \pmod 8$, *there is a* $\{3,4\}$-GDD *of group-type* 12^t *with precisely* s *quadruples. For* $t \equiv 2, 3 \pmod 4$,

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TABLE 4.2 Third Period.

v	Values $M_v - s$ for possible $s \in \mathcal{A}(v) \setminus \mathrm{Spec}_4(v)$
36	1 3 4 5 7 10 11 13 15 16 19 21 27 46 47 51 53 55 61 63 65 71
37	4 5 6 7 8 10 11 12 13 14 15 16 17 19 20 23 26 27
39	1 2 5 8 9 15
40	5 8 9 15
42	1 2 5 6 7 8 9 13
43	2 3 4 5 6 7 8 9 10 11 12
45	1 5 8 9 13 14 15
46	2 3 4 5 6 7 8 9 10 11 12 14

$t \geq 7$, $0 \leq s \leq 12t^2 - 12t - 24$ and $s \equiv 0 \pmod 8$, there is a $\{3,4\}$-GDD of group-type $12^{t-2}24^t$ with precisely s quadruples.

PROOF. Use Lemma 3.4 to produce $\{4\}$-GDDs with group-types 3^t when $t \equiv 0, 1 \pmod 4$, and $3^{t-2}6^1$ when $t \equiv 2, 3 \pmod 4$. Apply Lemma 3.2 with weight 4 to each element, using the $\{3,4\}$-GDDs of type 4^4 having 0, 8 or 16 quadruples (see Lemma 4.2). ∎

The omission of $t = 6$ in Lemma 5.1 can be remedied to a certain extent by a different application of the Fundamental Construction:

LEMMA 5.2. *There is a $\{3,4\}$-GDD of group-type 12^6 having precisely s quadruples for all $s \equiv 0 \pmod 5$, $0 \leq s \leq 360$.*

PROOF. Truncate a $(25, \{5\})$-PBD to obtain a $\{5\}$-GDD of type 4^6. Now apply Lemma 3.2 giving every element weight 3, and using the $\{3,4\}$-GDDs of type 3^5 having 0, 5, 10 or 15 quadruples (see Lemma 4.1). ∎

In specific cases, we also employ variants of Lemmas 5.1 and 5.2 that assist in particular classes. However our general strategy is to fill in groups in the GDDs produced by Lemma 5.1, using the GDDs of Lemma 5.2 to handle the exception in 5.1.

5.1. $v \equiv 1 \pmod{12}$

We write $v = 12t + 1$, and first apply the general construction.

LEMMA 5.3. *For $12t + 1 \geq 49$, $t \neq 6$, if $s \in \mathcal{A}(12t+1)$ then $s \in \mathrm{Spec}_4(12t+1)$ for $s \notin \{5, 9, 10, 11, 17\}$ and $M_v - s \notin \{4, 5, 9, 10, 11, 17\}$.*

PROOF. Using $\mathrm{Spec}_4(13)$ and results on $\mathrm{Spec}_4(25)$, fill in groups in the 12^t or $12^{t-2}24^1$ $\{3,4\}$-GDDs. When $t \equiv 0, 1 \pmod 4$, we choose t numbers from $\{0, 6, 7, 13\}$ and $(3t^2 - t)/4$ numbers from $\{0, 8, 16\}$ to form the number of quadruples. When $t \equiv 2, 3 \pmod 4$, we choose $t - 2$ numbers from $\{0, 6, 7, 13\}$, one from $\{0, 50\}$, and $(3t^2 - t - 6)/4$ from $\{0, 8, 16\}$. This produces all numbers of quadruples in $\mathcal{A}(12t+1)$ with the exceptions stated. ∎

We next consider order 73:

LEMMA 5.4. *For* $s \in \mathcal{A}(73)$, $s \notin \{8, 9, M_{73}-9, M_{73}-8, M_{73}-4\}$, *there is a* $(73, \{3,4\})$-PBD *with precisely* s *quadruples.*

PROOF. Fill in groups in the GDD of type 12^6 using $\mathrm{Spec}_4(13)$. ∎

Now we clear up the remaining exceptions. In view of Lemma 3.3, if $v, w \equiv 1, 3 \pmod 6$, $v \geq 2w+1$ and $s \in \mathrm{Spec}_4(w)$, we have $s \in \mathrm{Spec}_4(v)$. Hence we have the following:

LEMMA 5.5. *For* $5 \leq s \leq 17$, *if* $v \equiv 1, 3 \pmod 6$, $v \geq 49$, *then* $s \in \mathrm{Spec}_4(v)$.

PROOF. Apply the observations using Lemma 3.3 to $\{6, 7, 13\} \subseteq \mathrm{Spec}_4(13)$, $\{5, 10, 14, 15\} \subseteq \mathrm{Spec}_4(15)$, $\{8, 9, 11, 12, 16\} \subseteq \mathrm{Spec}_4(19)$, and $17 \in \mathrm{Spec}_4(25)$. This leaves only $17 \in \mathrm{Spec}_4(49)$ to construct; this is straightforward using a $\{3\}$-GDD of type 15^3 and filling groups with $(19,4)$-IPBDs. ∎

We use Lemma 3.4 in a similar way to fill in the "top end":

LEMMA 5.6. *For* $v \equiv 1, 4 \pmod{12}$, $M_v - s \in \mathrm{Spec}_4(v)$ *for*

(i) $s = 4$ *and* $v \geq 121$,
(ii) $s \in \{5, 8, 9, 10, 11, 14, 15, 16\}$ *and* $v \geq 49$,
(iii) $s \in \{6, 7, 13\}$ *and* $v \geq 40$,
(iv) $s = 12$ *and* $v \geq 76$, *and*
(v) $s = 17$ *and* $v \geq 85$.

PROOF. Use Lemma 3.4 to produce a $(v, w, \{4\})$-IPBD with $w = 40$ (case (i)), $w = 16$ (case (ii)), $w = 13$ (case (iii)), $w = 25$ (case (iv)) or $w = 28$ (case (v)). In each case, $M_w - s \in \mathrm{Spec}_4(w)$, and this PBD is used to fill the hole in the IPBD. ∎

The case of $M_v - 4$ can often be handled by the following construction:

LEMMA 5.7. *Let* $v \equiv 1, 4, 13, 16, 40 \pmod{48}$, $v \geq 40$. *Then* $M_v - 4 \in \mathrm{Spec}_4(v)$.

PROOF. For $v \equiv 4, 16 \pmod{48}$, there exists an ITD$(4, v/4, 2)$ using Lemma 3.5. Form a $\{3,4\}$-GDD of type $(v/4)^4$ with precisely 8 triples by filling the hole in the ITD with a $\{3\}$-GDD of type 2^4. Now fill groups using a $((v/4), \{4\})$-PBD. For $v \equiv 1, 13 \pmod{48}$, use an ITD$(4, ((v-1)/4), 2)$ in the same way, filling groups with $(((v-1)/4)+1, 1, \{4\})$-IPBDs. For $v \equiv 40 \pmod{48}$, we use an ITD$(4, (v-4)/4, 4)$ and fill in groups with $(((v-4)/4)+4, 4, \{4\})$-IPBDs; at the end we place a quadruple on the resulting hole of order four. ∎

As a consequence of the previous three lemmas, when $v \geq 49$ we are left with the possible exception of $M_v - 4$ for $v \in \{73, 85\}$, and $M_v - 17$ for $v \in \{49, 61\}$. These last two cases can be treated by modifying the construction in Lemma 5.7 to use one IPBD$(13, 1, \{3\})$ (for $v = 49$) and one IPBD$(16, 1, \{3,4\})$ (for $v = 61$), both of which have 26 triples (and hence have 13 quadruples fewer than the maximum).

To summarize, we have

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LEMMA 5.8. *For* $v \equiv 1 \pmod{12}$ *and* $v \geq 49$, $\mathcal{A}(v) = \mathrm{Spec}_4(v)$ *except possibly for* $M_v - 4$, $v \in \{73, 85\}$. ∎

5.2. $v \equiv 3 \pmod{12}$

We write $v = 12t + 3$. From filling in groups in the GDDs of Lemma 5.1 using $\mathrm{Spec}_4(15,3)$ and $\mathrm{Spec}_4(27,3)$, along with Lemma 5.5, we have

LEMMA 5.9. *For* $12t + 3 \geq 51$, $t \neq 6$, $\mathrm{Spec}_4(12t+3) = \mathcal{A}(12t+3)$. ∎

Now filling in groups of the GDD of type 12^6 using $\mathrm{Spec}_4(15,3)$, we obtain all values in $\mathcal{A}(75)$ except 8 and 9. Hence by using Lemma 5.5 as well, we obtain

LEMMA 5.10. $\mathrm{Spec}_4(75) = \mathcal{A}(75)$. ∎

Hence we have shown

LEMMA 5.11. *For* $v \equiv 3 \pmod{12}$ *and* $v \geq 51$, $\mathcal{A}(v) = \mathrm{Spec}_4(v)$. ∎

5.3. $v \equiv 4 \pmod{12}$

We write $v = 12t + 4$. Applying the basic construction using $\mathrm{Spec}_4(16,4)$ and $\mathrm{Spec}_4(28,4)$, we obtain

LEMMA 5.12. *For* $12t + 4 \geq 52$, $t \neq 6$, *if* $s \in \mathcal{A}(12t+4)$ *and* $s \notin \{M_v - 12, M_v - 7, M_v - 6, M_v - 4\}$ *then* $s \in \mathrm{Spec}_4(12t+4)$. ∎

For $v = 76$, we use $\mathrm{Spec}_4(16,4)$ to fill groups in the GDD of type 12^6 to obtain

LEMMA 5.13. $\mathcal{A}(76) \setminus \{M_{76} - 12, M_{76} - 7, M_{76} - 6, M_{76} - 4\} \subseteq \mathrm{Spec}_4(76)$. ∎

Now we treat the remaining cases. Applying Lemmas 5.6 and 5.7 leaves only the cases $M_v - 4$ for $v = 76$, and $M_v - 12$ for $v \in \{52, 64\}$. For the first case, apply the Fundamental Construction giving every point weight 3 to an ITD(4, 6, 2) and fill the resulting 6-hole with an ITD(4, 6, 2). The result is an ITD(4, 18, 2) with a sub-TD(4, 3). Now fill the hole with a $\{3\}$-GDD of type 2^4, and "unplug" the TD(4, 3). Add four points at infinity. On each group together with these four points, place a $(22, 7, \{4\})$-IPBD so that the hole coincides with the four additional points and the three points of the TD(4, 3) in this group. Finally, on the twelve points of the TD(4, 3) and the four additional points, place a $(16; \{4\})$-PBD.

For $M_{52} - 12 \in \mathrm{Spec}_4(52)$, fill groups of a TD(4, 13) using $\{7, 13\} \subseteq \mathrm{Spec}_4(13)$. For $M_{64} - 12 \in \mathrm{Spec}_4(64)$, fill the hole of an ITD(4, 16, 2) with a $\{3\}$-GDD of type 2^4, and fill groups using $\{12, 20\} \subseteq \mathrm{Spec}_4(16)$.

To summarize,

LEMMA 5.14. *For* $v \equiv 4 \pmod{12}$ *and* $v \geq 52$, $\mathcal{A}(v) = \mathrm{Spec}_4(v)$. ∎

5.4. $v \equiv 0 \pmod{12}$

We write $v = 12t$. This case poses a special problem, because $\mathrm{Spec}_4(12)$ only contains two different values. Hence filling in groups as usual in the GDDs of Lemma 5.1 gives only:

LEMMA 5.15. *For* $t \equiv 2,3 \pmod 4$, $t \geq 7$, *if* $s \in \mathcal{A}(12t)$ *and* $s \neq M_v - s\prime$ *for* $s\prime \in \{1,2,3,4,5,9,10,11\}$ *then* $s \in \mathrm{Spec}_4(12t)$. ■

LEMMA 5.16. *For* $t \equiv 0,1 \pmod 4$, *if* $s \in \mathcal{A}(12t)$ *and* $s - m_v$ *is even, then if* $\{s - m_v, M_v - s\} \cap \{2,4,10\} = \emptyset$, $s \in \mathrm{Spec}_4(12t)$. ■

Lemma 5.16 is quite weak, in that it produces only values of the same parity. Hence instead of using Lemma 5.1, we apply the Fundamental Construction using weight 4 to GDDs of type 3^a6^b, $b \geq 1$, to obtain $\{3,4\}$-GDDs with type 12^a24^b; these exist provided $t \geq 7$ (Lemma 3.14). In this way, we refine Lemma 5.16 to obtain:

LEMMA 5.17. *For* $t \equiv 0,1 \pmod 4$, $t \geq 8$, *if* $s \in \mathcal{A}(12t)$ *and* $s \neq M_v - s\prime$ *for* $s\prime \in \{1,2,3,4,5,9,10,11\}$, *then* $s \in \mathrm{Spec}_4(12t)$. ■

The case $v = 72$ is also complicated by the sparsity of $\mathrm{Spec}_4(12)$. Here we take a $(25,\{5\})$-PBD; truncating a point leaves a $\{4,5\}$-GDD of group-type 5^44^1 having 5 quadruples and 20 blocks of size 5. Use the $\{3,4\}$-GDDs of type 3^5 and 3^4 in the Fundamental Construction to form a $\{3,4\}$-GDD of group-type 15^412^1 in which the number of quadruples is s; one can choose any s that is the sum of 20 numbers in $\{0,5,6,10,15\}$ and five numbers in $\{3,9\}$. Now use $\mathrm{Spec}_4(12)$ and $\mathrm{Spec}_4(15)$ to fill in groups. In consequence, we obtain

LEMMA 5.18. *If* $s \in \mathcal{A}(72)$, $s \notin \{19,20,21,22,26,27\}$ *then* $s \in \mathrm{Spec}_4(72)$. ■

The cases in Lemma 5.18 are $m_v + x$ for $x \in \{1,2,3,4,8,9\}$. Using Lemma 3.3, there is a $\{3\}$-GDD of group-type 24^14^{12}. Since $\{7,8,9,10,14,15\} \subseteq \mathrm{Spec}_4(24)$, the exceptions left in the Lemma are all handled by filling groups in the GDD.

It remains to consider the exceptions in Lemmas 5.15 and 5.17. To do this, we establish the following:

LEMMA 5.19. *For* $v \equiv 0,3 \pmod{12}$ *and* $1 \leq s \leq 11$, $M_v - s \in \mathrm{Spec}_4(v)$ *for*

(*i*) $s = 6$ *and* $v \geq 39$,
(*ii*) $s \in \{1,5,8,9,10\}$ *and* $v \geq 48$,
(*iii*) $s = 7$ *and* $v \geq 75$,
(*iv*) $s = 2$ *and* $v \geq 84$,
(*v*) $s \in \{3,4,11\}$ *and* $v \geq 120$.

PROOF. Truncate a $(v+1, w+1, \{4\})$-IPBD to form a $\{3,4\}$-GDD of group-type $3^{(v-w)/3}w^1$, where $w = 12$, 15, 24, 27, or 39 in the five cases respectively. Fill the hole with a $(w,\{3,4\})$-PBD having $M_w - s$ quadruples. ■

Now we treat the remaining exceptions:

$\{M_{84} - 11, M_{84} - 4, M_{84} - 3\} \subseteq \mathrm{Spec}_4(84)$:

Use a TD(4,21) and observe that $4M_{21} + 21^2 = M_{84} - 2$. Since $\{M_{21} - 3, M_{21} - 1\} \subseteq \mathrm{Spec}_4(21)$, we obtain the desired results by filling in groups of the TD.

$M_{96} - 3 \in \mathrm{Spec}_4(96)$; $M_{108} - 3 \in \mathrm{Spec}_4(108)$:

For the first, take a resolvable TD(4, 7). Extend the groups by adding one point, and extend two parallel classes of blocks by adding a point to each. The result is a $\{4,5,8\}$-GDD with group-type $4^7 3^1$. Apply the Fundamental Construction with weight 3 to form a $\{4\}$-GDD of group-type $12^7 9^1$ using $\{4\}$-GDDs of type 3^4, 3^5 and 3^8. Now fill in groups using a $(12,3,\{3,4\})$-IPBD and four $(15,3,\{3,4\})$-IPBDs having the maximum number of quadruples, and three $(15,3,\{3,4\})$-IPBDs having one fewer than the maximum.

For the second, use instead a resolvable TD(4, 8) and extend three parallel classes and no groups.

$M_{96}-4 \in \mathrm{Spec}_4(96)$; $M_{108}-4 \in \mathrm{Spec}_4(108)$:

Truncate the solutions given by Lemma 5.7 for 97 and 109 elements.

$M_{96}-11 \in \mathrm{Spec}_4(96)$:

Use an ITD(4, 24, 2) along with $M_{24}-7 \in \mathrm{Spec}_4(24)$.

$M_{108}-11 \in \mathrm{Spec}_4(108)$:

Use an ITD(4, 26, 2) and fill in groups using $(30,4,\{3,4\})$-IPBDs and a $(30,\{3,4\})$-PBD. Each IPBD has the maximum number of quadruples, and we use $M_{30}-3 \in \mathrm{Spec}_4(30)$.

We have also verified by a set of tedious computations (by computer) that $\mathrm{Spec}_4(48) = \mathcal{A}(48)$ and $\mathrm{Spec}_4(60) = \mathcal{A}(60)$, using the constructions of section 3 and this section together with the ingredients of section 4.

We have shown

LEMMA 5.20. *For* $v \equiv 0 \pmod{12}$ *and* $v \geq 48$, $\mathcal{A}(v) = \mathrm{Spec}_4(v)$. ∎

5.5. $v \equiv 6 \pmod{12}$

Write $v = 12t + 6$. We apply the Fundamental Construction as in Lemma 5.1, but to a general class of GDDs:

LEMMA 5.21. *If there exists a* $\{4\}$*-GDD of group-type* $3^a 6^b$ *with* $a \geq 1$ *and* $a+2b = t$, *then if* $s \in \mathcal{A}(12t+6)$ *and* $s \leq M_{12t+6} - 9a + 3$, *then* $s \in \mathrm{Spec}_4(12t+6)$. *In particular, this holds for* $t \equiv 0, 1 \pmod 4$ *and* $a = t$, *and* $t \equiv 2, 3 \pmod 4$, $t \neq 6$, *and* $a = t - 2$.

PROOF. Form a $\{3,4\}$-GDD of group-type $12^a 24^b$ as in Lemma 5.1. Fill in groups using b $(30,6,\{3,4\})$-IPBDs, $a-1$ $(18,6,\{3,4\})$-IPBDs, and one $(18,\{3,4\})$-PBD. For the IPBD of order 30, Lemma 3.3 provides an IPBD with 6 quadruples, and Lemma 3.4 provides one with 64 quadruples; simply delete a point of the hole of a $(31,7,\{4\})$-IPBD. The particular cases mentioned are from Lemma 3.4. ∎

We employ a further general construction:

LEMMA 5.22. *Let* $z \in \{0,4,6,8,16\}$, $s_1 \in \mathrm{Spec}_4(3t+3,3)$, *and* $\{s_2,s_3,s_4\} \subseteq \mathrm{Spec}_4(3t+4,3)$. *Then for* $t \geq 4$,

$$(3t)(3t+1) - z + s_1 + s_2 + s_3 + s_4 \in \mathrm{Spec}_4(12t+6).$$

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PROOF. Set $y = 2, 3$ or 4 when $z = 4$, $z = 6$, or $z \in \{0, 8, 16\}$ respectively. Puncture an ITD$(4, 3t+1, y)$ by removing an element outside the hole, and fill the hole with a $\{3,4\}$-GDD of type y^4 having $2z$ triples. This gives a $\{3,4\}$-GDD of group-type $(3t)^1(3t+1)^3$. Fill in the groups. ∎

In Lemma 5.22, we come "close" to the maximum number of quadruples, but do not attain it. In particular, the maximum number obtainable in this way in general is $M_{12t+6} - 10$ for $t \equiv 0,3 \pmod 4$, and $M_{12t+6} - 9$ for $t \equiv 1,2 \pmod 4$. We therefore comment next on the cases close to the maximum:

LEMMA 5.23. *Let $v \equiv 6,9 \pmod{12}$, and $s\prime = M_v - s$. Then $s\prime \in \mathrm{Spec}_4(v)$ in each of the following cases:*

(i) $s = 4$ *and* $v \geq 30$,
(ii) $6 \leq s \leq 17$ *and* $v \geq 57$,
(iii) $s \in \{1, 3, 18, 19, 20, 21, 22, 23, 24, 25, 26, 31\}$ *and* $v \geq 66$,
(iv) $s = 28$ *and* $v \geq 93$,
(v) $s \in \{2, 5, 29, 30\}$ *and* $v \geq 102$,
(vi) $s = 27$ *and* $v \geq 129$,

PROOF. Using Lemma 3.4, form a $(v+1, w+1, \{4\})$-IPBD with $w = 9, 18, 21, 30, 33$ and 42 in the six cases above. Truncate to form a $\{4\}$-GDD of type $3^{(v-w)/3}w^1$. Place a $(w, \{3,4\})$-IPBD on the hole having $M_w - s$ quadruples. ∎

Now we turn to specific cases. For $v = 54$, Lemmas 5.21 and 5.22 give all values up to $M_{54} - 13$, except $M_{54} - 15$. We obtain M_{54} from Lemma 3.15, and $M_{54} - 4$ from Lemma 5.23. $M_{54} - 3$ is obtained by puncturing a $(55, 7, \{4\})$-PBD outside the hole, and filling the hole. For $M_v - s$, $s \in \{8, 9, 11, 12, 15\}$, modify Lemma 5.22 to use one $(16, \{4\})$-PBD in place of one of the IPBDs. For $M_{54} - 10$, use Lemma 3.7 with $6t = 36$, $x = 1$ and $y = 17$, and extend parallel classes to form a $(54, 18, \{3,4\})$-IPBD with 18 triples; then fill the hole using $15 \in \mathrm{Spec}_4(18)$. This leaves as possible exceptions $M_{54} - s$ for $s \in \{1, 2, 5, 6, 7\}$.

For $v = 66$, use Lemmas 5.21 and 5.23 to obtain all but $M_{66} - s$ for $s \in \{2, 5\}$.

For $v = 78$, we cannot apply Lemma 5.1. Instead, we use a construction similar to Lemma 5.2. Take a $\{5\}$-GDD of type 5^5 (i.e., the affine plane of order 5), and apply the Fundamental Construction giving each element weight 3. This gives a $\{3,4\}$-GDD of type 15^5 with s quadruples for any $0 \leq s \leq 375$, $s \equiv 0 \pmod 5$. Use $(18, 3, \{3,4\})$-IPBDs to fill in groups. This handles all values up to $M_{78} - 32$. Now applying Lemmas 5.22 and 5.23 leaves only $M_{78} - s$ for $s \in \{2, 5\}$.

For larger v, we proceed as follows. For all values except those in an interval of length at most $9t - 3$ at the top end, we use Lemma 5.1. For the remaining values near the maximum, we employ Lemma 5.22 recursively. The recursion uses both the determinations for $v \equiv 0,3 \pmod{12}$ already completed, and it uses determinations for smaller orders in the classes $v \equiv 6,9 \pmod{12}$. The last case, $v \equiv 9 \pmod{12}$, is examined in a later section. Using Lemmas 3.4, 5.8 and 5.14, one can always choose the $(3t+4, 3, \{3,4\})$-IPBD to have the maximum number of quadruples allowed by the necessary conditions, with the possible exceptions of $v \in \{25, 28, 37, 73, 76, 85\}$. In these cases, one can use instead the solutions for $M_v - 5$ for $v \in \{73, 76, 85\}$, $M_v - 7$ for $v = 25$ and $M_v - 9$ for $v \in \{28, 37\}$; this reduces the maximum in Lemma 5.22 by three, nine and fifteen quadruples, respectively.

Hence if we have $\mathcal{A}(3t+3) = \mathrm{Spec}_4(3t+3)$, the recursion provides all values up to $M_{12t+6} - 25$ for $t \in \{8, 11\}$, $M_{12t+6} - 19$ for $t = 7$, $M_{12t+6} - 13$ for $t \in \{23, 24, 27\}$, $M_{12t+6} - 10$ for other $t \equiv 0, 3 \pmod 4$, $t \geq 7$, and $M_{12t+6} - 9$ for $t \equiv 1, 2 \pmod 4$, $t \geq 6$. Lemma 5.23 can then be used to provide the missing values.

It remains to consider the "small" cases, in which $\mathrm{Spec}_4(3t+3, 3) = \mathcal{A}(3t+3)$ has not been established. In general, we observe that if $\mathrm{Spec}_4(3t+3)$ contains all values up to $M_{3t+3} - 18$ and $t \notin \{7, 8, 11\}$, Lemma 5.22 gives all values up to $M_{12t+6} - 31$ at least. One can verify that this holds for all $t \geq 12$ using the results of section 4.2, Lemmas 5.11 and 5.20, and the induction. Then Lemma 5.23 completes the determination.

At this point, we must consider the cases $7 \leq t \leq 11$. For $t = 7$, we obtain all values up to $M_{90} - 42$ from Lemma 5.21. Using Lemma 5.22 with $42 \in \mathrm{Spec}_4(24)$ and the determination of $\mathrm{Spec}_4(25)$ in section 4.2, we obtain all values up to $M_{90} - 30$, and $M_{90} - 28$. Using $35 \in \mathrm{Spec}_4(24)$ instead, we also obtain $M_{90} - 27$ and $M_{90} - 29$.

For $t = 8$, use the $\{4\}$-GDD of type $3^4 6^2$ (Lemma 3.13) in Lemma 5.21 to obtain all values up to $M_{102} - 36$. Filling in groups of a TD(4, 27) using $\mathrm{Spec}_4(27)$ then gives all values up to $M_{102} - 32$ (at least), and $M_{102} - 27$.

For $t \in \{9, 10\}$, using the determination of $\mathrm{Spec}_4(31, 3)$ and $\mathrm{Spec}_4(34, 3)$ in Lemma 5.22, along with Lemma 5.23, handles all values.

For $t = 11$, Lemma 5.21 handles all values up to $M_{138} - 78$. Filling the hole of a $(138, 42, \{3,4\})$-IPBD with the maximum number of quadruples using $\mathrm{Spec}_4(42)$ handles all values up to $M_{138} - 14$.

In each case, provided that the case $v \equiv 9 \pmod{12}$ is handled, we have established that the possible exceptions in the ingredients of Lemma 5.22 only cause possible exceptions that are eliminated by Lemma 5.23. Hence although the determination is not completed for the small cases, the possible exceptions do not propagate. Once we have completed the case $v \equiv 9 \pmod{12}$ (in the next section), we have finished the case $v \equiv 6 \pmod{12}$.

We treat a few of the exceptions remaining for $v \leq 90$:

LEMMA 5.24. *For* $t \equiv 0, 3 \pmod 4$, $t \geq 4$, $M_{12t+6} - 5 \in \mathrm{Spec}_4(12t+6)$. *Moreover,* $\{M_{54} - 7, M_{54} - 6\} \subseteq \mathrm{Spec}_4(54)$.

PROOF. Form an ITD$(4, 3t+1, 2)$. Add two points "at infinity". On each group of the ITD plus the two extra points, place a $(3t+3, 4, \{3,4\})$-IPBD, so that the 4-hole in the IPBD coincides with the two extra points and the two points in the hole of the ITD. The result is a $(12t+6, 10, \{3,4\})$-IPBD; fill the final hole. Now if each IPBD is taken to have the maximum number of quadruples possible, the PBD produced has $4t+13$ triples, and hence has $M_{12t+6}-5$ quadruples. In the case $t=4$, we obtain the two further values by using $(15,4,\{3,4\})$-PBD with one fewer quadruple than the maximum. ■

We have shown

LEMMA 5.25. *For* $v \equiv 6 \pmod{12}$ *and* $v \geq 54$, $\mathcal{A}(v) = \mathrm{Spec}_4(v)$ *except possibly for* $M_v - 1$ *with* $v = 54$, $M_v - 2$ *with* $v \in \{54,66,78,90\}$ *and* $M_v - 5$ *with* $v \in \{66,78\}$. ■

5.6. $v \equiv 9 \pmod{12}$

Write $v = 12t+9$. We first adapt Lemma 5.1 to the case at hand.

LEMMA 5.26. *If there exists a* $\{4\}$*-GDD of type* 3^a6^b *with* $a \geq 1$ *and* $a+2b=t$, *and if* $s \in \mathcal{A}(12t+9)$ *and* $s \leq M_{12t+9}-19a+9$ *or* $M_{12t+9}-s \in \{19a-16, 19a-18, 19a-19\}$, *Then* $s \in \mathrm{Spec}_4(12t+9)$.

PROOF. Apply the Fundamental Construction with weight four to the GDD and fill in groups using b $(33,9,\{3,4\})$-IPBDs, $a-1$ $(21,9,\{3,4\}$-IPBDs and one $(21,\{3,4\})$-PBD. ■

In Lemma 5.26, we take in general the $(33,9,\{3,4\})$-IPBDs to have no quadruples (from Lemma 3.3), or the maximum number (from Lemma 3.4).

To supplement this, we require a construction for large values in $\mathcal{A}(12t+9)$:

LEMMA 5.27. *For* $i=1,2,3,4$, *let* $s_i \in \mathrm{Spec}_4(3t+3)$. *Let* $z \in \{0,4,6,8,16\}$. *Then for* $t \geq 4$,

$$(3t+2)^2 - z + s_1 + s_2 + s_3 + s_4 \in \mathrm{Spec}_4(12t+9).$$

PROOF. Set $y = 2$, 3 or 4 when $z = 4$, $z = 6$ or $z \in \{0,8,16\}$, respectively. Fill the hole in an ITD$(4,3t+2,y)$ with a $\{3,4\}$-GDD of type y^4 having $2z$ triples. Now fill in groups of the resulting GDD using $(3t+3,1,\{3,4\})$-IPBDs. ■

In Lemma 5.27, when $t \equiv 0,3 \pmod 4$, the maximum value produced is M_{12t+9}; when $t \equiv 1,2 \pmod 4$, the maximum value is $M_{12t+9}-2$.

When $t \not\equiv 6$, we apply Lemmas 5.26 and 5.27, using Lemma 5.23 to take care of certain exceptions. As in the case $v \equiv 6 \pmod{12}$, we employ an induction from smaller values; however, in this case, the induction is dramatically simplified by the fact that Lemma 5.27 comes quite close to the maximum. When $t \equiv 1,2 \pmod 4$, there are no exceptions left; more precisely, Lemma 5.23 handles all exceptions left by applying Lemma 5.27 inductively, since Lemma 5.23 handles $M_v - s$ for $s \leq 26$ except $s \in \{2,5\}$ for $v \in \{69,81\}$. Lemma 5.27 handles these cases directly (recall that the maximum in Lemma 5.27 is $M_v - 2$ in these congruence classes).

When $t \equiv 0,3 \pmod 4$, no exceptions result whenever $M_{3t+3} - 1 \in \mathrm{Spec}_4(3t+3)$; otherwise we must treat the possible exceptions $M_v - 5$ and $M_v - 2$ for values not handled by Lemma 5.23. This leaves the cases $M_v - 2$ and $M_v - 5$ only for $v = 93$.

It remains to treat the case $v = 81$:

LEMMA 5.28. $\text{Spec}_4(81) = \mathcal{A}(81)$.

PROOF. Apply the Fundamental Construction with weight 4 to a TD(4, 5) to obtain a $\{3,4\}$-GDD of type 20^4 having s quadruples for any $0 \leq s \leq 400$, $s \equiv 0 \pmod 8$. Fill in groups with a $(21, 1, \{3,4\})$-IPBD. This construction establishes that $M_{81} - x \in \text{Spec}_4(81)$ except for $x \in \{0, 1\}$. These two values are provided by Lemma 3.15 and 5.23. ∎

We should remark that the induction required is immediate once we have $\text{Spec}_4(3t+3) = \mathcal{A}(3t+3)$ in the above; below that point, one must verify that the possible exceptions in $\text{Spec}_4(3t+3)$ do not propagate to make new possible exceptions in $\text{Spec}_4(12t+9)$, other than those explicitly mentioned above. This is a straightforward computation.

Now we treat the remaining exceptions. For $v = 93$, take a resolvable TD(4, 7); add a point to the groups and a point to one parallel class of blocks to obtain a $\{4,5,8\}$-GDD of type $2^1 4^7$. Apply the Fundamental Construction with weight 3, using $\{4\}$-GDDs of types 3^4, 3^5 and 3^8. Then fill groups with a $(9, 3, \{3\})$-IPBD and seven $(15, 3, \{3,4\})$-IPBDs. This produces $M_{93} - 5$.

Hence we have

LEMMA 5.29. *For* $v \equiv 9 \pmod{12}$ *and* $v \geq 57$, $\mathcal{A}(v) = \text{Spec}_4(v)$ *except possibly for* $M_v - 2$, $v = 93$. ∎

5.7. $v \equiv 7 \pmod{12}$

Write $v = 12t + 7$.

LEMMA 5.30. *If there exists a* $\{4\}$-GDD *of type* $3^a 6^b$ *with* $a \geq 1$ *and* $a + 2b = t$, *and if* $s \in \mathcal{A}(12t+7)$ *and* $s \leq M_{12t+7} - 9a + 6$, *then* $s \in \text{Spec}_4(12t+7)$.

PROOF. Apply the Fundamental Construction to obtain a $\{3,4\}$-GDD of type $12^a 24^b$. Fill in groups using b $(31, 7, \{3,4\})$-IPBDs, $a - 1$ $(19, 7, \{3,4\})$-IPBDs, and one $(19, \{3,4\})$-PBD. ∎

The $(31, 7, \{3,4\})$-IPBD is taken to have no quadruples (Lemma 3.3) or all quadruples (Lemma 3.4).

Next we treat the bulk of the cases at the top end.

LEMMA 5.31. *For* $t \geq 4$, $z \in \{0, 4, 6, 8, 16\}$ *and* $s_i \in \text{Spec}_4(3t+4, 3)$ $(i = 1, 2, 3, 4)$,

$$(3t+1)^2 - z + s_1 + s_2 + s_3 + s_4 \in \text{Spec}_4(12t+7).$$

PROOF. Choose y as in Lemma 5.27. Fill groups in an ITD$(4, 3t+1, y)$ using $(3t+4, 3, \{3,4\})$-IPBDs. ∎

The maximum realizable in Lemma 5.31 is $M_{12t+7} - 9$ for $t \equiv 1, 2 \pmod 4$, and $M_{12t+7} - 11$ for $t \equiv 0, 3 \pmod 4$. Hence we need some values near the maximum:

LEMMA 5.32. *For* $v \equiv 7, 10 \pmod{12}$, $M_v - s \in \mathrm{Spec}_4(v)$ *when*
(i) $s = 1$ *and* $v \geq 31$,
(ii) $s \in \{3,4,5,6,7,9,10,11,12,13,14,15,16,17,18,19,20,25\}$ *and* $v \geq 58$, *and*
(iii) $s \in \{21,22,23,24,26,27,28,29\}$ *and* $v \geq 67$.
(iv) $s = 2$ *and* $v \geq 103$.

PROOF. Set $w = 10$, 19, 22 or 34 according to the case considered. Fill the hole of a $(v, w, \{4\})$-IPBD from Lemma 3.4 using $M_w - s \in \mathrm{Spec}_4(w)$. ∎

There remain four issues: the case $v = 79$, the induction using Lemmas 5.31 and 5.32, the remaining exceptions for $v = 55$, and the missing value $M_v - 8$. We treat each in turn.

LEMMA 5.33. $\mathcal{A}(79)\setminus\{M_{79} - 8, M_{79} - 2\} \subseteq \mathrm{Spec}_4(79)$.

PROOF. Apply the Fundamental Construction with weight 3 to a $\{5\}$-GDD of type 5^5, and fill groups with five $(19, 4, \{3,4\})$-IPBDs. This handles all values up to $M_{79} - 29$. Use Lemma 5.32 to complete the proof. ∎

Now for the induction, we require a solution for all $v \equiv 1 \pmod 3$, and hence we require in particular the solution for $v \equiv 10 \pmod{12}$ yet to come. We can state the following:

LEMMA 5.34. *For* $t \equiv 0, 3 \pmod 4$, $t \geq 4$, $\mathcal{A}(12t+7)\setminus\{M_{12t+7} - 8\} \subseteq \mathrm{Spec}_4(12t+7)$ *except possibly for* $M_v - 5$ *for* $v = 55$, *and* $M_v - 2$ *for* $v \in \{55, 91\}$.

PROOF. When $\mathrm{Spec}_4(3t+4) = \mathcal{A}(3t+4)$, the verification is routine. Hence we need only consider small values of t; with the results on small cases, one can check that all required values are constructed by Lemmas 5.30, 5.31 and 5.32. ∎

The cases $t \equiv 1, 2 \pmod 4$ are treated inductively using solutions for $3t + 4 \equiv 7, 10 \pmod{12}$. It is easy to establish the following:

LEMMA 5.35. *For* $t \equiv 1, 2 \pmod 4$, *if* $\mathcal{A}(3t+4)\setminus\{M_{3t+4} - 8, M_{3t+4} - 5, M_{3t+4} - 2\} \subseteq \mathrm{Spec}_4(3t + 4)$, *then* $\mathcal{A}(12t + 7)\setminus\{M_{12t+7} - 8\} \subseteq \mathrm{Spec}_4(12t + 7)$, *except possibly for* $M_{12t+7} - 2$ *for* $t \in \{5, 6\}$. ∎

The induction now proceeds in a manner analogous to the cases $v \equiv 6, 9 \pmod{12}$. It is easy to verify from the results in section 4 and using Lemma 5.32 that for $v \geq 55$, all values up to $M_v - 29$ are handled inductively, and Lemma 5.32 then provides a number of further values near the maximum. Hence we need only treat the cases missed by Lemma 5.32. The particular case $M_v - 8$ is not addressed by Lemma 5.32, and hence we also need to consider this special value.

In the induction, there remain a number of cases to be checked when $\mathrm{Spec}_4(3t+4)$ has further possible exceptions. We remark that $t = 6$ is handled by Lemma 5.33. For $t = 5$, we have $M_{19} - 3 \in \mathrm{Spec}_4(19)$, and hence Lemma 5.32 provides all of the additional cases that result. For $t \geq 9$, no further exceptions result, since we always have M_{3t+4} and $M_{3t+4} - 1$ in $\mathrm{Spec}_4(3t + 4)$ (where $t \equiv 1, 2 \pmod 4$).

To treat $M_v - 8$, we prove the following:

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LEMMA 5.36. *Suppose there exists a* GDD *on t elements with one group of size 1 or 6, all other groups of sizes* $\equiv 0, 3 \pmod 4$, *and all blocks of sizes* $\equiv 0, 1 \pmod 4$. *If the* GDD *contains a group of size 4, then* $M_{3t+4} - 8 \in \mathrm{Spec}_4(3t+4)$.

PROOF. Apply the Fundamental Construction with weight 3 to the GDD, using GDDs of types 3^x for $x \equiv 0, 1 \pmod 4$, and $(3g+4, 4, \{3,4\})$-IPBDs for each group of size g. It is evident that all but the resulting group of size 3 or 18 can be replaced entirely by quadruples. Choose a $(16, 4, \{3,4\})$-IPBD with 16 triples to fill the group of size 4 and the other IPBDs to have only quadruples. ■

We apply Lemma 5.36 to a number of cases. For $v = 55$, use a TD(4, 4) and extend the groups to form a $\{4,5\}$-GDD of type $4^4 1^1$. The construction of Lemma 5.36 can be generalized to give $M_{55} - s \in \mathrm{Spec}_4(55)$ for $s \in \{5, 6, 8, 9, 10\}$. For $v = 67$, extend one parallel class in a resolvable TD(4, 5) to obtain a $\{4,5\}$-GDD of type $4^5 1^1$. For $v = 91$, extend one parallel class of a resolvable TD(4, 7) to get a GDD of type $1^1 4^7$. For $v = 139$, use instead a TD(4, 11). The solution for $v = 55$ gives solutions for all $v \geq 175$ by Lemma 3.4.

LEMMA 5.37. $M_{12t+7} - 8 \in \mathrm{Spec}_4(12t+7)$ *for* $t = 6$, *and for* $t \equiv 0, 1 \pmod 4$ *and* $t \geq 5$.

PROOF. Apply the Fundamental Construction to a $\{4,5\}$-GDD of group-type $4^1 5^4$ (puncture the affine plane of order 5), having 5 quadruples and 20 5-blocks. Give every point weight 3, and use five $\{4\}$-GDDs of type 3^4, 19 $\{4\}$-GDDs of type 3^5, and one $\{3,4\}$-GDD of type 3^5 having 10 quadruples. Now fill in groups with four $(22, 7, \{4\})$-IPBDs and one $(19, \{3,4\})$-PBD having 22 quadruples. This establishes that $M_{79} - 8 \in \mathrm{Spec}_4(79)$.

For $t \equiv 0, 1 \pmod 4$, $t \geq 5$, use an ITD(4, 3t, 4) and fill the hole with a $\{3,4\}$-GDD of type 4^4 having 8 quadruples. Fill in groups using a $(3t+7, 7, \{4\})$-IPBD. ■

These results leave only $M_v - 8$ on 127 elements. We proceed as follows using a construction of Bose, Shrikhande and Parker [4]. Using a $\{4,5\}$-GDD of group-type $3^1 7^4$, we form a TD(4, 31) that has a set of spanning TDs, namely one TD(4, 3) and four TD(4, 7)s. Omit one of the TD(4, 7)'s to form an ITD(4, 31, 7). Add three points at infinity. On three of the groups of the ITD, place a $(34, 10, \{4\})$-IPBD whose hole is on the seven points of the hole of the ITD and the three additional points. Now (partially) fill the hole in the ITD using a $(31, 10, \{4\})$-IPBD. At this point, we have a $(127, 34, \{4\})$-IPBD. To get $M_{127} - 8$, use $M_{34} - 2 \in \mathrm{Spec}_4(34)$ and replace the TD(4, 3) by a $\{3,4\}$-GDD of type 3^4 with three quadruples.

For $M_{55} - 4 \in \mathrm{Spec}_4(55)$, we form an ITD(4, 13, 2) adding three points at infinity. We fill the hole with a $\{3\}$-GDD of type 2^4, and delete one block disjoint from this GDD. On each group together with the three extra points, place a $(16, 4; \{4\})$-IPBD with the hole on the three extra points and the point of the deleted block. Fill the final hole with a $(7; \{3\})$-PBD.

In summary,

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LEMMA 5.38. *For* $v \equiv 7 \pmod{12}$ *and* $v \geq 55$, $\mathcal{A}(v) = \mathrm{Spec}_4(v)$ *except possibly for* $M_v - 2$ *when* $v \in \{55, 67, 79, 91\}$, *and* $M_v - 3$ *and* $M_v - 7$ *when* $v = 55$. ∎

5.8. $v \equiv 10 \pmod{12}$

Write $v \equiv 12t + 10$. In analogy with Lemma 5.28, we obtain

LEMMA 5.39. *If there exists a* $\{4\}$*-GDD of group-type* 3^a6^b *with* $a \geq 1$ *and* $a+2b = t$, *and if* $s \in \mathcal{A}(12t+10)$, $s \leq M_{12t+10} - 27a + 27$, *then* $s \in \mathrm{Spec}_4(12t+10)$.

PROOF. Fill groups in the GDD of type 12^a24^b with b $(34, 10, \{3,4\})$-IPBDs, $a-1$ $(22, 10, \{3,4\})$-IPBDs and one $(22, \{3,4\})$-PBD. ∎

The maximum attainable here is quite low compared to the previous three congruence classes. Nevertheless, we can employ truncated ITDs again as follows:

LEMMA 5.40. *Let* $t \geq 4$, $z \in \{0,4,6,8,16\}$, $s_1 \in \mathrm{Spec}_4(3t+1)$ *and* $s_2, s_3, s_4 \in \mathrm{Spec}_4(3t+3)$. *Then*

$$(3t+1)(3t+3) - z + s_1 + s_2 + s_3 + s_4 \in \mathrm{Spec}_4(12t+10).$$

PROOF. similar to Lemma 5.30. ∎

The primary difficulty in this case is that the recursion is using PBDs in the $0 \pmod 3$ class to construct those in the $1 \pmod 3$ class; hence the largest value that we can obtain using Lemma 5.39 is $M_{12t+10} - \lfloor (4.5t+1) \rfloor$ (an easy computation). This leaves an interval of large values to consider that grows as v grows, unlike all of the previous congruence classes considered. To deal with this problem, we use a simple observation, namely that if $M_{12t+7} - s \in \mathrm{Spec}_{(12t+7)}$, then by Lemma 3.4 we have $M_{36t+22} - s \in \mathrm{Spec}_4(36t+22)$, $M_{36t+34} - s \in \mathrm{Spec}_4(36t+34)$ and $M_{36t+46} - s \in \mathrm{Spec}_4(36t+46)$.

We have only to settle the case $v = 82$, apply the induction, and treat the remaining exceptions. We do each in turn.

LEMMA 5.41. $\mathcal{A}(82) \setminus \{M_{82} - 8, M_{82} - 2\} \subseteq \mathrm{Spec}_4(82)$.

PROOF. Take the $\{3,4\}$-GDD of type 15^5 constructed in Lemma 5.33; fill four groups using $(22, 7, \{3,4\})$-IPBDs, and one using a $(22, \{3,4\})$-PBD. This handles all but $M_{82} - s$ for $s \in \{1,2,3,4,6,8,9,11\}$. Lemma 5.32 completes the proof. ∎

At this point, the induction is routine, and leaves only the exceptions $M_v - 8$ and $M_v - 2$ for small values. Given the solutions for $v = 55$ from Lemma 5.36, we need only consider $M_{12t+10} - 8$ for $v \leq 154$. For $v = 118$, extend six parallel classes in a resolvable TD(4, 8) to get a $\{4,5,8\}$-GDD of type 6^14^8 and apply Lemma 5.36. For $v = 130$, extend six parallel classes in a resolvable TD(4, 9).

LEMMA 5.42. $M_{12t+10} - 8 \in \mathrm{Spec}_4(12t+10)$ *for* $t \equiv 0, 3 \pmod 4$ *and* $t \geq 7$.

PROOF. Construct an ITD$(4, 3t, 4)$ and fill the hole using a $\{3,4\}$-GDD of type 4^4 having 8 quadruples. Then fill in groups using a $(3t+10, 10, \{4\})$-IPBD and a $(3t+10, \{3,4\})$-PBD having M_{3t+10} quadruples. ∎

For $M_{82} - 8$, apply weight 4 to a $\{4\}$-GDD of type 5^4. Use $\{4\}$-GDDs except for one GDD of type 4^4 having eight quadruples. Now unplug one of the TD(4, 5)'s in the result, and add two points at infinity. Fill groups with $(22, 7; \{4\})$-IPBDs, and fill the final hole with a $(22; \{3, 4\})$-PBD with the maximum number of quadruples.

To summarize,

LEMMA 5.43. *For* $v \equiv 10 \pmod{12}$ *and* $v \geq 58$, $\mathcal{A}(v) = \mathrm{Spec}_4(v)$ *except possibly for* M_v-2 *when* $v \in \{58, 70, 82, 94\}$, M_v-5 *when* $v = 58$ *and* M_v-8 *when* $v \in \{58, 70\}$. ∎

5.9. Summary

In each congruence class of v (modulo 12), we have established that $\mathcal{A}(v) \subseteq \mathrm{Spec}_4(v)$ for all $v \geq 96$. Together with the necessary conditions from Lemma 2.1, this completes the proof of the Main Theorem.

6. **Applications.** In the development of the proof of the Main Theorem, we have seen substantial connections between the construction of $\{3, 4\}$-PBDs with a specified number of quadruples and many central problems in design theory. Here we comment on a few further connections. First of all, Batten and Totten [2] have classified all $(v, \{n-1, n\})$-PBDs with $v < n^2$, $v \neq 15$; our Main Theorem is in a similar vein. In fact, PBDs are just linear spaces in which the blocks are lines; hence our result has a natural geometric interpretation.

Lindner and Rosa [16] and Rosa and Hoffman [27] determined the possible numbers of repeated blocks in a triple system with $\lambda = 2$ for $v \equiv 1, 3 \pmod 6$, and $v \equiv 0, 4 \pmod 6$, respectively. In a $\{3, 4\}$-GDD with a triples and b quadruples, duplicating each triple, and replacing each quadruple by the four distinct triples on the same points, gives a triple system with a repeated blocks. Hence our Main Theorem can be viewed as the determination of triple systems with $\lambda = 2$ having a prescribed number of repeated blocks and all other blocks in subdesigns of order four.

The general theme of specifying the numbers of blocks of each size is useful in examining extremal problems in design theory; see, for example, [9]. Colbourn and Rödl [10] have shown that if a K-PBD exists, then one can (asymptotically) specify the percentage of blocks of each size, and achieve the specified percentages to any fixed tolerance. Our Main Theorem shows that for $K = \{3, 4\}$, one has a much stronger result.

Since PBDs are basic building blocks in much of combinatorial design theory, the determination of many numerical or extremal properties of designs requires control over the proportion of blocks of each size. Our Main Theorem is the first nontrivial result that shows that one can control the distribution of block sizes completely.

7. **Concluding Remarks.** At the present time, there remain only twenty-two values in doubt for $48 \leq v \leq 96$; we certainly expect that all of the corresponding PBDs exist in this range. However, for smaller values of v, the situation appears to be quite complicated. A complete solution for $v \in \{18, 19, 21, 22\}$ would certainly be useful in clarifying the extent of genuine exceptions, i.e. values in $\mathcal{A}(v) \setminus \mathrm{Spec}_4(v)$.

In the second period, the large number of open cases that remain is largely a consequence of the limitations of recursive constructions. We have succeeded in constructing a large number of designs in this range, but have not attempted an extensive search. Undoubtedly a number of the open cases could be settled, especially those with few quadruples.

For all $v \geq 96$, we have completely determined the possible numbers of triples and quadruples. This is the first interesting case of the general problem of determining distributions of block sizes in PBDs, and suggests that one can obtain quite precise control over that distribution.

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