Canad. Math. Bull. Vol. **21** (1), 1978

MOULTON AFFINE HJELMSLEV PLANES

BY
CATHARINE BAKER

1. **Introduction.** Desarguesian affine Hjelmslev planes (D.A.H. planes) were introduced by Klingenberg in [1] and generalized by Lane and Lorimer in [2]. D.A.H. planes are coordinatized by affine Hjelmslev rings (A.H. rings) which are local rings whose radicals are equal to their sets of two-sided zero divisors and whose principal right ideals are totally ordered. In [5], ordered D.A.H. planes were defined and the induced orderings of their A.H. rings were discussed. In this note an ordered non-Desarguesian A.H. plane is constructed from an arbitrary ordered D.A.H. plane. The existence of such planes ensures that the discussion of ordered non-Desarguesian A.H. planes by J. Laxton in [3] is meaningful. The basic idea employed is essentially the same as the one used in the construction of the classical Moulton plane from the real affine plane (cf. [4]).

2. **Ordered A.H. rings.** An A.H. ring H with radical η is *ordered* if there exists a subset H^+ of H such that $a \in H$ implies exactly one of $a \in H^+$, $a = 0$, $-a \in H^+$ holds; $a, b \in H^+$ implies $a + b \in H^+$; $a, b \in H^+$ and $b \notin \eta$ imply $ab \in H^+$ (cf. [5], 2.2). Let $H^- = \{a \in H \mid -a \in H^+\}$. H has the following properties.

2.1. *Suppose $a \in \eta$. Then $-1 < a < 1$ and for any b such that $-a < b < a$, we have $b \in \eta$* (cf. [5], 6.1).

2.2. *For $b, d \in H^+$, if $b + d \in \eta$, then $b, d \in \eta$.*

Proof. Since $-(b+d) < b, d < b+d$, $b, d \in \eta$ by 2.1.

2.3. *If $m \in H^+ \setminus \eta$ and $p \in H^- \cup \eta$, then $m - p \in H^+ \setminus \eta$.*

Proof. If $p \in H^-$, then since $m, -p \in H^+$, $m - p \notin \eta$ by 2.2. Let $p \in H^+ \cap \eta$. Since $m \notin \eta$ and $p \in \eta$, $m - p \notin \eta$. Suppose $m - p \in H^-$. Then $m, p - m \in H^+$ and $p \in \eta$ imply $m \in \eta$ by 2.2 which is a contradiction.

2.4. The incidence structure with parallelism, $\mathcal{H}$, constructed over an ordered A.H. ring H in the usual manner is an ordered D.A.H. plane (cf. [5], Chapter 3). Thus, $H \times H$ is the set of points of $\mathcal{H}$ and the lines of $\mathcal{H}$ are of the form $[m, n]_1$ where $m \in \eta$, $n \in H$ and $[m, n]_2$ where $m, n \in H$. Here, $[m, n]_1 = \{(ym + n, y) \mid y \in H\}$ and $[m, n]_2 = \{(x, xm + n) \mid x \in H\}$.

Received by the editors February 9, 1977 and in revised form, August 18, 1977.

3. **A Moulton A.H. plane.** We shall consider a new incidence structure with parallelism, $\mathscr{H}'=\langle \mathbb{P}, \mathbb{L}, I, \|\rangle$, where

$$\mathbb{P}=H\times H;$$

$$\mathbb{L}=\{[m,n]_1 \mid m\in\eta, n\in H\}\cup\{[m,n]_2 \mid m\in H^-\cup\eta, n\in H\}\cup \{[m,n]_3 \mid m\in H^+\setminus\eta, n\in H\}.$$

where $[m,n]_1$ and $[m,n]_2$ are defined as in $\mathscr{H}$ and

$$[m,n]_3=\{(x, xm+n) \mid x+nm^{-1}\in H^+\cup\{0\}\}\cup\{(x, xmk+nk) \mid x+nm^{-1}\in H^-\},$$

for a fixed element of $H^+\setminus\eta$ such as $k-1\notin\eta$;
I is set inclusion;
$[m,n]_i \| [p,q]_j$ if and only if $i=j$ and $m=p$.

We define a neighbour relation, $\sim$, on $\mathbb{P}$ by:

$$(a,b)\sim(c,d) \quad \text{if and only if} \quad a-c, b-d\in\eta$$

and a neighbour relation, $\sim$, on $\mathbb{L}$ by:

$$[m,n]_i\sim[p,q]_j \quad \text{if and only if} \quad i=j \quad \text{and} \quad m-p, n-q\in\eta.$$

We shall prove (in sections 6 and 7) that $\mathscr{H}'$ *is a non-Desarguesian* A.H. *plane.*

4. **The ordinary affine plane associated with $\mathscr{H}'$.** Let $\phi: H\to H^*=H/\eta(a\rightsquigarrow a^*)$ be the usual quotient map. The map $\Phi=(\Phi_{\mathbb{P}},\Phi_{\mathbb{L}}):\mathscr{H}\to\mathscr{A}=\mathscr{H}/\sim$, induced by ϕ, is given by $\Phi_{\mathbb{P}}(a,b)=(a^*,b^*)$ and $\Phi_{\mathbb{L}}([m,n]_i)=[m^*,n^*]_i$. Here, $\mathscr{A}=\langle\overline{\mathbb{P}},\overline{\mathbb{L}},\bar{I},\overline{\|}\rangle$ is the ordered ordinary Desarguesian affine plane which is coordinatized by the ordered division ring H^* (cf. [5], Chapter 5). Clearly, $H^{*+}=H^+/(H^+\cap\eta)$ and $H^{*-}=H^-/(H^-\cap\eta)$.

Now consider the incidence structure $\mathscr{A}'=\langle\overline{\mathbb{P}},\overline{\mathbb{L}}',\bar{I},\overline{\|}\rangle$, where

$$\overline{\mathbb{L}}'=\{[0,n^*]_1 \mid n^*\in H^*\}\cup\{[0,n^*]_2 \mid n^*\in H^*\} \cup\{[m^*,n^*]_2 \mid m^*\in H^{*-}, n\in H^*\}\cup\{[m^*,n^*]_3 \mid m^*\in H^{*+}, n\in H^*\}$$

with

$$[0,n^*]_1=\{(n^*,y^*) \mid y^*\in H^*\},$$
$$[m^*,n^*]_2=\{(x^*,x^*m^*+n^*) \mid x^*\in H^*,$$
$$[m^*,n^*]_3=\{(x^*,x^*m^*+n^*) \mid x^*+n^*m^{*-1}\in H^{*+}\cup\{0\}\} \cup\{(x^*,x^*m^*k^*+n^*k^* \mid x^*+n^*m^{*-1}\in H^{*-}\}.$$

$\mathscr{A}'$ is constructed from $\mathscr{A}$ in the same manner as the classical Moulton plane is constructed from the real affine plane; hence $\mathscr{A}'$ is a non-Desarguesian affine plane. (Since $k-1\notin\eta$, $k^*\neq 1$.) In addition, $\mathscr{A}'$ is the image of $\mathscr{H}'$ under the map $\Phi'=(\Phi'_{\mathbb{P}},\Phi'_{\mathbb{L}})$ induced by ϕ. Clearly, Φ' preserves incidence and is onto. Φ' also satisfies the following properties.

4.1. $(a, b) \sim (c, d)$ *if and only if* $\Phi'_P(a, b) = \Phi'_P(c, d)$.

Proof. $(a, b) \sim (c, d)$ if and only if $(a^*, b^*) = (c^*, d^*)$.

4.2. $[m, n]_i \sim [p, q]_j$ *if and only if* $\Phi'_L([m, n]_i) = \Phi'_L([p, q]_j)$.

Proof. $[m, n]_i \sim [p, q]_j$ if and only if $i = j$, $m^* = p^*$ and $n^* = q^*$; i.e., $\Phi'_L([m, n]_i) = \Phi'_L([p, q]_j)$.

5. **Properties of $\mathcal{H}'$.**

5.1. *Any pair of points* (a, b) *and* (c, d), *incident with a line* $[m, n]_2$, *where* $m \in H^+ \setminus \eta$, *of* $\mathcal{H}$, *are incident with a line* $[p, q]_3$ *in* $\mathcal{H}'$.

Proof. If $b, d \in H^+ \cup \{0\}$, then (a, b), (c, d) I $[m, n]_3$ in $\mathcal{H}'$. If $b, d \in H^- \cup \{0\}$, then (a, b), (c, d) I $[mk^{-1}, nk^{-1}]_3$ in $\mathcal{H}'$. We now consider the case where $b, -d \in H^+$.

First, we show that there exists an $f \in H$ such that $(a-c)fk = d(k-1)$; i.e., such that $(a-c)f = d(k-1)k^{-1} = d(1-k^{-1})$. If no such f exists, then there exists $g \in \eta$ such that $(a-c) = d(1-k^{-1})g$. Since (a, b), (c, d) I $[m, n]_2$ in $\mathcal{H}$, we have $b - d = (a-c)m = d(1-k^{-1})gm$. Hence, $b = d((1-k^{-1})gm+1)$; however, $g \in \eta$ implies $d((1-k^{-1})gm+1) \in H^-$; a contradiction since $b \in H^+$.

Take $p = m+f$ and $q = n-af$. We shall show (a, b), (c, d) I $[p, q]_3$ in $\mathcal{H}'$. Since $ap+q = am+af+n-af = b$ and $cpk+qk = cmk+cfk+nk-afk = (cm+n)k-(a-c)fk = d$, $[p, q]_3$ is the required line provided that $p \in H^+ \setminus \eta$.

We now have two possibilities.

Suppose $k > 1$. Since $(a-c)f = d(1-k^{-1})$, where $a-c \in H^+$, $d \in H^-$, $1-k^{-1} \in H^+ \setminus \eta$, then $f \notin H^+ \setminus \eta$. Since $(a-c)f = d(1-k^{-1}) = (cm+b-am)(1-k^{-1})$, we have $b(1-k^{-1}) = (a-c)(f+m(1-k^{-1}))$ with $1-k^{-1} \in H^+ \setminus \eta$ and $b, a-c \in H^+$; hence $f+m(1-k^{-1}) \notin H^- \setminus \eta$. Therefore, $p = (f+m(1-k^{-1}))+mk^{-1} \in H^+ \setminus \eta$.

Suppose $k < 1$. As $(a-c)f = d(1-k^{-1})$, where $a-c \in H^+$, $d \in H^-$, $(1-k^{-1}) \in H^- \setminus \eta$, we have $f \notin H^- \setminus \eta$. Then $p = m+f \in H^+ \setminus \eta$.

5.2. *If two points* (a, b) *and* (c, d) *are incident with some line* $[p, q]_3$ *of* $\mathcal{H}'$, *then there exists a line* $[m, n]_2$ *of* $\mathcal{H}$, *with* $m \in H^+ \setminus \eta$, *through both points.*

Proof. If $b, d \in H^+ \cup \{0\}$, take $[m, n]_2 = [p, q]_2$; if $b, d \in H^- \cup \{0\}$, take $[m, n]_2 = [pk, qk]_2$. If $b, -d \in H^+$, by the argument used in 5.1 there exists $f \in H^- \cup \eta$ such that $(a-c)f = d(1-k^{-1})$. By 2.3 $p-f \in H^+ \setminus \eta$. Therefore, take $[m, n]_2 = [p-f, q+af]_2$.

5.3. *All lines of* $\mathcal{H}'$ *joining two given distinct points are of the same kind.*

Proof. From the properties of $\mathcal{H}$, it is clear that two distinct points cannot be incident with a line of the first kind and a line of the second kind in $\mathcal{H}'$. If the two distinct point are incident with a line of the third kind of $\mathcal{H}'$, then there exists a line $[m, n]_2$ in $\mathcal{H}$ with $m \in H^+ \setminus \eta$, joining these points. These two points

https://doi.org/10.4153/CMB-1978-024-7 Published online by Cambridge University Press

could not be incident with a line of the first kind in $\mathcal{H}$ or in $\mathcal{H}'$. By 2.3 $m-p\in H^+\backslash\eta$, if $p\in H^-\cup\eta$, which implies the two points cannot be incident with any $[p, q]_2$, where $p\in H^-\cup\eta$, in $\mathcal{H}$ and thus, cannot be incident with any line of the second kind in $\mathcal{H}'$.

5.4. *Two points of $\mathcal{H}'$ are neighbours if and only if there exist more than one line of $\mathcal{H}'$ through them.*

Proof. ($\Rightarrow$) Any distinct neighbour points (a, b) and (c, d) are incident wit a least two distinct lines $[m, n]_i$, $[p, q]_i$ in $\mathcal{H}$. Either they are incident with the same two lines in $\mathcal{H}'$ (if these lines exist in $\mathcal{H}'$) or they are incident with $[m, n]_3$, $[p, q]_3$ if $b, d\in H^+\cup\{0\}$, with $[mk^{-1}, nk^{-1}]_3$, $[pk^{-1}, qk^{-1}]_3$ if $b, d\in H^-\cup\{0\}$ or with $[m+f, n-af]_3$, $[p+f, q-af]_3$ if $b, -d\in H^+$, where $(a-c)f=d(1-k^{-1})$ (by 5.1).

($\Leftarrow$) Now assume there exist more than one line of $\mathcal{H}'$ through the distinct points (a, b) and (c, d). By 5.3, these lines are of the same kind. If they also exist in $\mathcal{H}$, then $(a, b)\sim(c, d)$.

If (a, b) and (c, d) are incident with two lines of the third kind in $\mathcal{H}'$, say $[m, n]_3$ and $[p, q]_3$, then (a, b), (c, d) I $[m, n]_2$, $[p, q]_2$ in $\mathcal{H}$, if $b, d\in H^+\cup\{0\}$; (a, b), (c, d) I $[mk, nk]_2$, $[pk, qk]_2$, in $\mathcal{H}$, if $b, d\in H^-\cup\{0\}$; (a, b), (c, d) I $[m-f, n+af]_2$, $[p-f, q+af]_2$ in $\mathcal{H}$, where $(a-c)f=d(1-k^{-1})$ (by 5.1), if $b, -d\in H^+$. Hence, (a, b) and (c, d) are neighbours in $\mathcal{H}$ and thus also in $\mathcal{H}'$.

5.5. *Two lines of $\mathcal{H}'$ are neighbours if and only if for any points incident with either line, there exists a neighbour point incident with the other.*

Proof. Consider two neighbour lines $[m, n]_i$, $[p, q]_j$. If these lines are of the first or second kind, the result is clear. Suppose they are of the third kind. Take any (a, b) I $[m, n]_3$. Then $b=am+n$ if $a+nm^{-1}\in H^+\cup\{0\}$ $[b=amk+nk$ if $a+nm^{-1}\in H^-]$. We have two possibilities: either $a+qp^{-1}\in H^+\cup\{0\}$ or $a+qp^{-1}\in H^-$. First suppose $a+qp^{-1}\in H^+\cup\{0\}$. Then $(a, ap+q)$ I $[p, q]_3$ and $am+n-ap-q=a(m-p)+n-q\in\eta$ [and $amk+nk-ap-q=(am+n)(k-1)+a(m-p)+(n-q)\in\eta$, since $am+n\in\eta$ by 2.2]. Thus the point $(a, ap+q)$ is the required neighbour point if $a+qp^{-1}\in H^+\cup\{0\}$. A similar argument would prove $(a, apk+qk)$ to be the required neighbour point if $a+qp^{-1}\in H^-$.

Now consider two lines $[m, n]_i$, $[p, q]_j$ such that for any point incident with either line, there exists a neighbour point incident with the other line. Take any (a, b), (c, d) I $[m, n]_i$ such that $(a, b)\not\sim(c, d)$. Then there exist (a', b'), (c', d') I $[p, q]_j$ such that $(a', b')\sim(a, b)$ and $(c', d')\sim(c, d)$. Hence,

$$\Phi'_{\mathbb{L}}([m, n]_i)=\Phi'_{\mathbb{P}}((a, b))\Phi'_{\mathbb{P}}((c, d))=\Phi'_{\mathbb{L}}([p, q]_j).$$

By 4.2, $[m, n]_i\sim[p, q]_j$.

https://doi.org/10.4153/CMB-1978-024-7 Published online by Cambridge University Press

5.6. *If two lines* $[m, n]_i$ *and* $[p, q]_j$ *of* $\mathcal{H}'$ *do not intersect, then* $i=j$ *and* $m-p\in\eta$.

Proof. If $[m, n]_i$ and $[p, q]_j$ also appear in $\mathcal{H}$, then $i=j$ and $m-p\in\eta$ by [2], 2.14.

If $i=1$ and $j=3$, then in $\mathcal{H}$, $[m, n]_1\wedge[p, q]_2=((qm+n)(1-pm)^{-1}, (np+q)(1-mp)^{-1})$ and $[m, n]_1\wedge[pk, qk]_2=((qkm+n)(1-pkm)^{-1}, (npk+qk)(1-mpk)^{-1})$ (cf. [2], 2.14), Since $m\in\eta$, $1-mp$, $1-mpk\in H^+\backslash\eta$; therefore, $(np+q)(1-mp)^{-1}$ and $(npk+qk)(1-mpk)^{-1}$ are both in $H^+\cup\{0\}$ [in H^-] if $np+q$ lies in $H^+\cup\{0\}$ [in H^-]. Hence, $((qm+n)(1-pm)^{-1}, (np+q)(1-mp)^{-1})=[m, n]_1\wedge[p, q]_3$, if $np+q\in H^+\cup\{0\}$ and $((qkm+n)(1-pkm)^{-1}, (npk+qk)(1-mpk)^{-1})=[m, n]_1\wedge[p, q]_3$ if $np+q\in H^-$. Thus $[m, n]_1\wedge[p, q]_3\neq\emptyset$.

If $i=2$ and $j=3$, then $m\in H^-\cup\eta$ and $p\in H^+\backslash\eta$, so $p-m$, $pk-m\in H^+\backslash\eta$, by 2.3. By [2], 2.15, in $\mathcal{H}$ $[p, q]_2\wedge[m, n]_2=((n-q)(p-m)^{-1}, (n-q)(p-m)^{-1}p+q)$ and $[pk, qk]_2\wedge[m, n]_2=((n-qk)(pk-m)^{-1}, (n-qk)(pk-m)^{-1}pk+qk)$. However, both $(n-q)(p-m)^{-1}p+q$ and $(n-qk)(pk-m)^{-1}pk+qk$ lie in $H^+\cup\{0\}$ [in H^-] if and only if $n\geq qp^{-1}m[n<qp^{-1}m]$. Therefore, in $\mathcal{H}'$, $[m, n]_2\wedge[p, q]_3=((n-q)(p-m)^{-1}, (n-q)(p-m)^{-1}p+q)$, if $n\geq qp^{-1}m$ and $[m, n]_2\wedge[p, q]_3=((n-qk)(pk-m)^{-1}, (n-qk)(pk-m)^{-1}pk+qk)$, if $n<qp^{-1}m$. Thus, $[m, n]_2\wedge[p, q]_3\neq\emptyset$.

Let $i=j=3$. We may assume, without loss of generality, that $p-m\in H^+\cup\{0\}$. If $m-p\notin\eta$, then in $\mathcal{H}$, $[p, q]_2\wedge[m, n]_2=((n-q)(p-m)^{-1}, (n-q)(p-m)^{-1}p+q)$ and $[pk, qk]_2\wedge[mk, nk]_2=((n-q)(p-m)^{-1}, (n-q)(p-m)^{-1}pk+qk)$, by [2], 2.15. In addition, $(n-q)(p-m)^{-1}p+q$ and $(n-q)(p-m)^{-1}pk+qk$ both lie in $H^+\cup\{0\}$ [in H^-] if and only if $n\geq qp^{-1}m$ [if and only if $n<qp^{-1}m$]. Therefore, in $\mathcal{H}'$, $[m, n]_3\wedge[p, q]_3=((n-q)(p-m)^{-1}, (n-q)(p-m)^{-1}p+q)$ if $n\geq qp^{-1}m$ and $[m, n]_3\wedge[p, q]_3=((n-q)(p-m)^{-1}, (n-q)(p-m)^{-1}pk+qk)$ if $n<qp^{-1}m$. Thus, $[m, n]_3\wedge[p, q]_3\neq\emptyset$ if $m-p\notin n$.

6. In this section, we verify that $\mathcal{H}'$ *is an A.H. plane*; i.e., we show that $\mathcal{H}'$ satisfies the following conditions 6.1–6.4 (cf. [2], 2.2)

6.1. *Any pair of points of* $\mathcal{H}'$ *is joined by a line of* $\mathcal{H}'$, by the properties of $\mathcal{H}$ and by 5.1.

6.2. *Two lines of* $\mathcal{H}'$ *which have a point in common meet exactly once if and only if they are not neighbours.*

Proof. ($\Leftarrow$) Consider any pair of lines $[m, n]_i$, $[p, q]_j$ of $\mathcal{H}'$ which meet in two distinct points (a, b) and (c, d). By 5.3, $i=j$; $i, j\in\{1, 2, 3\}$. If $i=1$ or 2, then $[m, n]_i\sim[p, q]_i$, by the properties of $\mathcal{H}$. Let $i=3$. Then by 5.2, (a, b), (c, d) are incident with lines of the second kind in $\mathcal{H}$.

If $b, d\in H^+\cup\{0\}$ [$b, d\in H^-\cup\{0\}$; $b, -d\in H^+$], then (a, b), (c, d) I $[m, n]_2$,

https://doi.org/10.4153/CMB-1978-024-7 Published online by Cambridge University Press

$[p, q]_2$ in $\mathcal{H}$ $[(a, b), (c, d)\ I\ [mk, nk]_2, [pk, qk]_2$ in $\mathcal{H}$; $(a, b), (c, d)\ I\ [m-f, n+af]_2, [p-f, q+af]_2$ in $\mathcal{H}$ where $(a-c)f = d(1-k^{-1})]$. Thus, $m-p,\ n-q \in \eta$.

($\Rightarrow$) Any pair of neighbour lines of the first or second kind in $\mathcal{H}'$ which have a point in common, meet more than once by the property 6.2 applied to $\mathcal{H}$. Consider two neighbour lines $[m, n]_3$ and $[p, q]_3$ and some (a, b) incident with both of them. Clearly, both $a+qp^{-1} \in H^+ \cup \{0\}$ and $a+nm^{-1} \in H^+ \cup \{0\}$ if and only if $b \in H^+ \cup \{0\}$. If $b \in H^+ \cup \{0\}$ [if $b \in H^-$], then $(a, b) = (a, am+n) = (a, ap+q)$ $[(a, b) = (a, amk+nk) = (a, apk+qk)]$. Since $m-p \in \eta$, there exists $c \in H^+ \cap \eta [c \in H^- \cap \eta]$ such that $c(m-p) = 0$. Then $(a+c,\ (a+c)m+n)$ $[(a+c,\ (a+c)mk+nk)]$ is also incident with both $[m, n]_3$ and $[p, q]_3$.

6.3. *For each flag* $((a, b), [m, n]_i) \in \mathbb{P} \times \mathbb{L}$, *there exists a unique line* $L((a, b), [m, n]_i)$ *such that* $(a, b)\ I L\ ((a, b), [m, n]_i)$ *and* $L((a, b), [m, n]_i) \parallel [m, n]_i$.

Proof. The result is clear if $i=1$ or 2. If $i=3$, the required line is $[m, b-am]_3$ if $b \in H^+ \cup \{0\}$ or $[m, bk^{-1}-am]_3$, if $b \in H^-$. The uniqueness of such a line is readily verified.

6.4. *There exist an ordinary affine plane* $\mathcal{A}'$ *and an epimorphism* $\Phi' = (\Phi'_{\mathbb{P}}, \Phi'_{\mathbb{L}})\colon \mathcal{H}' \to \mathcal{A}'$ *with the following properties.*

1. $(a, b) \sim (c, d)$ *in* $\mathcal{H}'$ *if and only if* $\Phi'_{\mathbb{P}}(a, b) = \Phi'_{\mathbb{P}}(c, d)$.
2. $[m, n]_i \sim [p, q]_j$ *in* $\mathcal{H}'$ *if and only if* $\Phi'_{\mathbb{L}}([m, n]_i) = \Phi'_{\mathbb{L}}([p, q]_j)$.
3. *If* $[m, n]_i \wedge [p, q]_j = \emptyset$, *then* $\Phi'_{\mathbb{L}}([m, n]_i) \parallel \Phi'_{\mathbb{L}}([p, q]_j)$.

Proof. The ordinary affine plane $\mathcal{A}'$ and the epimorphism Φ' defined in 4 satisfy these conditions by 4.1, 4.2 and 5.6.

Therefore, $\mathcal{H}'$ is an A.H. plane.

7. Since $\mathcal{A}'$ is not Desarguesian, $\mathcal{H}'$ cannot be Desarguesian.

8. **An ordering of** $\mathcal{H}'$. First we give an ordering of $\mathcal{H}$. We define a ternary relation ρ on the points of $\mathcal{H}$ by $(P_1, P_2, P_3) \in \rho$ $(P_i = (a_i, b_i),\ i = 1, 2, 3)$ if $P_1, P_2, P_3\ I\ [m, n]_1$ and $b_1 < b_2 < b_3$ or $b_1 > b_2 > b_3$ or if $P_1, P_2, P_3\ I\ [m, n]_2$ and $a_1 < a_2 < a_3$ or $a_1 > a_2 > a_3$. It is easily verified that ρ satisfies the order axioms 01–07 given in [5]; in fact, ρ is equivalent to the ordering of $\mathcal{H}$ induced by the ordering of H in [5]. In particular, we have the following result.

8.1. *In* $\mathcal{H}$, *if* $P_i = (a_i, b_i)\ I\ [m, n_i]_1$ $[[m, n_i]_2;\ [m, n_i]_2]$, $i \in \{1, 2\}$ *and* $P_1, P_2\ I\ [p, q]_2$ $[[p, q]_1;\ [p, q]_2$ *where* $p-m \notin \eta]$, *then* $a_1 < a_2\ [b_1 < b_2;\ a_1 < a_2$ *when* $p-m \in H^+$ *and* $a_2 < a_1$ *when* $p-m \in H^-]$ *if and only if* $n_1 < n_2$.

Proof. Assume $n_1 < n_2$. By [2], 2.14 [2.14; 2.15], $a_2 - a_1 = (n_2 - n_1)(1-pm)^{-1} > 0$ since $1-pm \in H^+ \backslash \eta [b_2 - b_1 = (n_2 - n_1)(1-pm)^{-1} > 0$ since $1-pm \in H^+ \backslash \eta$; $a_2 - a_1 = (n_2 - n_1)(p-m)^{-1}$ which is greater than zero if $p-m \in H^+$ and less than zero if $p-m \in H^-]$.

https://doi.org/10.4153/CMB-1978-024-7 Published online by Cambridge University Press

Conversely, if $a_1 < a_2[b_1 < b_2;\ a_1 < a_2$ with $p-m \in H^+$ or $a_2 < a_1$ with $p-m \in H^-]$, then $n_1 \not\geq n_2$ otherwise we obtain a contradiction.

8.2. In a similar manner, we may define a relation ρ' on $\mathcal{H}'$ by putting ρ' equal to ρ on the points of lines of the first or second kind and for $P_i = (a_i, b_i)\, I\, [m, n]_3$ $(i \in \{1, 2, 3\})$, $(P_1, P_2, P_3) \in \rho'$ if and only if $a_1 < a_2 < a_3$ or $a_1 > a_2 > a_3$. The relation ρ' clearly satisfies axioms 01–06 (cf. [5]), so it remains to verify 07. Since ρ is an ordering of $\mathcal{H}$, any non-degenerate parallel projection which does not involve lines of the third kind will preserve the ordering ρ'. We verify that the parallel projections involving lines of the third kind preserve order in the following sequence of lemmas. We shall prove the lemmas in one direction; the converses then follow as in 8.1.

8.3. *Let* $P_i = (a_i, b_i)\, I\, [m, n_i]_1$ $[[m, n_i]_2]$, $i \in \{1, 2\}$ *and* $P_1, P_2\, I\, [p, q]_3$. *Then* $a_1 < a_2$ *if and only if* $n_1 < n_2$.

Proof. Assume $a_1 < a_2$. If $a_1, a_2 \geq -qp^{-1}$ or $a_1, a_2 \leq -qp^{-1}$, then the result follows from 8.1. Therefore, let $a_1 < -qp^{-1} < a_2$. Then $n_2 - n_1 = a_2(1-pm) - qm - a_1(1-pkm) + qkm > -qp^{-1}((1-pm)-(1-pkm)) - qm + qkm = 0$ since $1-pm,\ 1-pkm \in H^+ \setminus \eta$ $[n_2 - n_1 = a_2(p-m) + q + a_1(pk-m) - qk > -qp((p-m)-(pk-m)) + q - qk = 0$ since $p-m,\ pk-m \in H^+ \setminus \eta$ by 2.3] by 6.4.

8.4. *Let* $P_i = (a_i, b_i)\, I\, [m, n_i]_3$, $i \in \{1, 2\}$ *and* $P_1, P_2\, I\, [p, q]_1$ $[[p, q]_2]$. *Then* $b_1 < b_2 [a_2 < a_1]$ *if and only if* $n_1 < n_2$.

Proof. Assume $n_1 < n_2$. If $n_1, n_2 \geq -qm$ or $n_1, n_2 \leq -qm$ $[b_1, b_2 \geq 0$ or $b_1, b_2 \leq 0]$, then the result follows from 8.1. If $n_1 < -qm < n_2$, then by 6.4, $b_2 - b_1 = (qm + n_2)(1-pm)^{-1} - (qmk + n_1 k)(1-pmk)^{-1} > 0$ since $1-pm$, $1-pmk \in H^+ \setminus \eta$. [First, suppose $b_1 < 0 < b_2$. Then $-a_2 p < q < -a_1 p$ which implies $0 < n_2 - n_1 < (a_2 - a_1)(p-m)$ if $k > 1$ and $0 < n_2 - n_1 < (a_2 - a_1)(pk^{-1} - m)$ if $k < 1$. Since $p-m,\ pk^{-1} - m \in H^- \setminus \eta$, $a_2 < a_1$. Now suppose $b_2 < 0 < b_1$. Then $-a_1 p < q < -a_2 p$; hence $0 < n_2 - n_1 < (a_2 - a_1)(pk^{-1} - m)$ if $k > 1$ and $0 < n_2 - n_1 < (a_2 - a_1)(p-m)$ if $k < 1$. Thus, $a_2 < a_1$.]

8.5. *Let* $P_i = (a_i, b_i)\, I\, [m, n_i]_3$, $i \in \{1, 2\}$ *and* $P_1, P_2\, I\, [p, q]_3$ *where* $m - p \notin \eta$. *Then* $n_1 < n_2$ *if and only if* $a_1 < a_2$ *when* $p - m \in H^+$ *and* $a_2 < a_1$ *when* $p - m \in H^-$.

Proof. Assume $n_1 < n_2$. If $n_1, n_2 \geq qp^{-1}m$ or $n_1, n_2 \leq qp^{-1}m$, the result follows by 8.1. Therefore, let $n_1 < qp^{-1}m < n_2$. Then by 6.4, $a_1 - a_2 = (n_1 - q)(p-m)^{-1} - (n_2 - q)(p-m)^{-1} = (n_1 - n_2)(p-m)^{-1}$. Thus, $a_1 > a_2$ if $p - m \in H^-$ and $a_1 < a_2$ if $p - m \in H^+$.

8.6. It is now clear that non-degenerate parallel projections preserve ρ'. Therefore, $\mathcal{H}'$ is a non-Desarguesian A.H. plane with ordering ρ'.

9. **An example.** Let $H=\mathbb{R}\times\mathbb{R}$. If $a\in H$, we may write $a=(A_1, A_2)$, where A_1, $A_2\in\mathbb{R}$. Define two binary operations, addition and multiplication on H by $(A_1, A_2)+(B_1, B_2)=(A_1+B_1, A_2+B_2)$ and $(A_1, B_2)\cdot(B_1, B_2)=(A_1B_1, A_1B_2+A_2B_1)$, for any $A_1, A_2, B_1, B_2\in\mathbb{R}$. Then H is an A.H. ring with unit $(1, 0)$ and unique maximal ideal $\eta=\{(0, A)\mid A\in\mathbb{R}\}$. If we define $H^+=\{(A_1, A_2)\in H\mid A_1>0\}\cup\{(0, A_2)\mid A_2>0\}$ then H becomes an ordered A.H. ring. The incidence structure $\mathcal{H}$ constructed in the usual manner over H is an ordered D.A.H. plane. The incidence structures $\mathcal{H}'$, constructed as in 3 are ordered non-Desarguesian A.H. planes. If k is taken to be $(2, 0)$, then $\mathcal{H}'$ is the ordered non-Desarguesian A.H. plane whose associated ordinary affine plane is the classical Moulton plane.

References

1. Klingenberg, W.: *Desarguessche Ebenen mit Nachbarelementen.* Abh. Math. Sem. Univ. Hamburg **20** (1955), 97–111.

2. Lane, N. D. and Lorimer, J. W.: *Desarguesian affine Hjelmslev planes.* J. reine angew. Math. **278/279** (1975), 336–352.

3. Laxton, J. A. A.: *Ordered non-Desarguesian Affine Hjelmslev Planes.* M.Sc. thesis, McMaster University, Hamilton (1976).

4. Moulton, F. R.: *A simple non-Desarguesian plane geometry.* Trans. Amer. Math. Soc. **3** (1902), 192–195.

5. Thomas, L. A.: *Ordered Desarguesian affine Hjelmslev-planes.* M.Sc. thesis, McMaster University, Hamilton (1975).

Dept. of mathematics
McMaster University
Hamilton, Ontario