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Tame or wild Toeplitz shifts

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Abstract. We investigate tameness of Toeplitz shifts. By introducing the notion of extended Bratteli–Vershik diagrams, we show that such shifts with finite Toeplitz rank are tame if and only if there are at most countably many orbits of singular fibres over the maximal equicontinuous factor. The ideas are illustrated using the class of substitution shifts. A body of elaborate examples shows that the assumptions of our results cannot be relaxed.

Key words: Toeplitz shifts, tame dynamical systems, Bratteli–Vershik systems, almost automorphy, constant length substitutions
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1. *Introduction*

Tameness is known for its many facets related to deep theorems in topology, Banach space theory and model theory, such as Rosenthal's ℓ^1-embedding theorem or the BFT-dichotomy. It was introduced to dynamics by Köhler, under the name *regularity* [**34**], and the joint efforts of the community helped to shed light on general structural properties of tame dynamical systems [**19**, **24**–**26**, **30**, **33**], see also [**27**] for an exposition and numerous further references. The opposite of tame is wild, but with mathematical modesty and to not overuse the word wild, the community simply calls wild dynamical systems non-tame. One fascinating phenomenon is that non-tame systems are not just not tame, but qualitatively very far from tame systems. This is most visibly reflected in the following characterization: a $\mathbb{Z}$-action defined by a homeomorphism T on a compact metrizable space X is *tame* if the cardinality of its Ellis semigroup $E(X, T)$ is at most that of the continuum c, whereas if it is non-tame, $E(X, T)$ must contain a copy of $\beta\mathbb{N}$.

To put our results in context, it is important to say a few words about the relation between tameness and almost automorphy—a classical notion extensively studied by Veech [**40**].

Huang realized that tame minimal $\mathbb{Z}$-actions are necessarily uniquely ergodic and *almost automorphic* [30], that is, the maximal equicontinuous factor $\mathcal{Z}$ contains a point which has a unique pre-image under the factor map $\pi : X \to \mathcal{Z}$. Such a point is called *regular* and correspondingly, points $z \in \mathcal{Z}$ for which $\pi^{-1}(z)$ consists of more than one element are called *singular*. Recently, Glasner extended Huang's result to minimal actions of general groups possessing an invariant measure [25]. Based on these results, it was proved in [19] that tame minimal systems are actually regularly almost automorphic, which means that the set of regular points has full (Haar) measure.

The vanishing, in measure, of the singular points is thus a necessary condition for tameness, but it is far from being sufficient. For shift dynamical systems, the set of singular points is the union of the $\mathbb{Z}$-orbits of its *discontinuity points*: these are the points in the maximal equicontinuous factor whose fibre contains two points which disagree on their 0-coordinate. A binary almost automorphic shift whose maximal equicontinuous factor is an irrational rotation on the circle is non-tame if its set of discontinuity points is a Cantor set [19, Proposition 3.3]. Note that among such systems, one easily finds examples for which the set of singular points is a zero measure set. Furthermore, it is possible to construct non-tame almost automorphic systems for which the set of singular points consists of a single orbit [22]. However, in this case, the pre-image of a singular point under the factor map has to be uncountable. These results suggest that non-tameness is related to the smallness of the difference between X and its maximal equicontinuous factor, where smallness is computed either via a measure or via cardinality, either in X or its maximal equicontinuous factor. Our results largely affirm this suggestion but emphasize that this relation is generally speaking more subtle.

We investigate the notion of tameness for the class of Toeplitz shifts, which are almost automorphic extensions of odometers (procyclic group rotations). In this class, Oxtoby found a first example of a minimal system which is not uniquely ergodic [37]. Jacobs and Keane defined and studied Toeplitz shifts systematically in [32], recognising their close relation to Toeplitz's constructions in [39]. Toeplitz shifts have since enjoyed ample attention due to their dynamical diversity and their relevance in measurable and topological dynamics, see e.g. [3, 4, 9, 12, 13, 15, 23, 31, 36, 42] as well as [10] for a survey and further references. Gjerde and Johansen [23] represent Toeplitz shifts as Bratteli–Vershik systems where the Bratteli diagrams are appropriately constrained, and it is this representation that is particularly useful to us. A Toeplitz shift has *finite Toeplitz rank* if it has such a representation for which the number of vertices at each level of the Bratteli diagram is uniformly bounded. Note that, on the face of it, the class of finite Toeplitz rank systems is possibly smaller than the class of Toeplitz shifts of finite topological rank, that is, the class of Toeplitz shifts which have some proper Bratteli–Vershik representation (which not necessarily satisfies the constraints of Gjerde and Johansen) with a bounded number of vertices at each level. It is known that the class of finite Topelitz rank systems is a subset of the class of shifts with non-superlinear complexity, [8, Corollary 6.7]. We show the following theorem.

THEOREM 1.1. *Let (X, T) be a Toeplitz shift of finite Toeplitz rank. Then (X, T) is tame if and only if its maximal equicontinuous factor has only countably many singular points.*

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This result should be contrasted with the work of Aujogue [1], who shows that a family of tiling systems is tame despite having uncountably many singular points.

As a corollary to Theorem 1.1, we get the following necessary and sufficient criterion for tameness of substitution shifts of constant length. Note that, in this case, the property of having only finitely many orbits of singular points can be easily read off from an associated graph introduced in [5]. We elaborate on this in the main body of this work.

THEOREM 1.2. *Let θ be a primitive aperiodic substitution of constant length. The associated shift (X_θ, T) is tame if and only if θ has a coincidence and the maximal equicontinuous factor contains only finitely many orbits of singular points.*

Against the background of Theorem 1.1, one might be tempted to guess that the presence of uncountably many singular fibres implies non-tameness for general Toeplitz shifts, that is, also for those of infinite rank. However, it turns out that in spite of Theorem 1.1, neither the cardinality of the singular points nor that of the fibres decide whether (X, T) is tame or otherwise. We show the following theorem.

THEOREM 1.3. *There is a tame binary Toeplitz shift whose set of discontinuity points is a Cantor set and so its maximal equicontinuous factor has uncountably many singular points. Moreover, there exist tame as well as non-tame binary Toeplitz shifts with a unique singular orbit whose fibres are uncountable.*

The first example shows that the aforementioned sufficient criterion for non-tameness of almost automorphic shifts over irrational rotations of the circle, namely that the set of *discontinuity points* is a Cantor set, does not generalize when we replace the circle by a totally disconnected set. The proof of the second part of Theorem 1.3 is based on a slightly refined version of the constructions carried out in [22], where the possible non-tameness of systems with a unique singular orbit was already observed. Somewhat surprisingly, our constructions allow us to deduce the fact that it is possible for a minimal system (X, T) to be non-tame even if it is forward tame, that is, the corresponding forward motion is tame (see §1.1 for definitions).

In light of Theorem 1.3, it is not straightforward to identify a property that implies non-tameness for Toeplitz shifts. Whilst stipulating that a Cantor set of singular fibres exists is not sufficient, additionally controlling the points in such a set of fibres does the trick. This control is granted by a property we refer to as *thickness*. Its definition necessitates extending the Bratteli–Vershik diagrams associated to Toeplitz shifts, see Definitions 2.7 and 2.10. Our main result, from which Theorems 1.1 and 1.2 follow, is the following theorem.

THEOREM 1.4. *Every thick Toeplitz shift is non-tame.*

Let us close the introduction by pointing out an interesting connection between tameness of substitution shifts and *amorphic complexity*, a topological invariant which was introduced to detect complex behaviour in the zero entropy regime [21]. In the case of symbolic systems, it coincides with the box dimension of the maximal equicontinuous factor, the box dimension being determined through an averaging metric which is defined by the dynamics. For constant length substitution shifts on a binary alphabet, Theorem 1.2 and [20, Theorem 1.1] yield that (X_θ, T) is tame if and only if its amorphic complexity is 1.

The work is organized as follows. In the remainder of this section, we collect some basic notions which are needed throughout the article. In §2, we give the Bratteli–Vershik and Toeplitz background needed and prove Theorem 1.4. We take some time to explain our results specified to the family of substitution shifts as this is an important family where the proofs are simpler and motivational for the following notions. We recall the general construction of semicocycle extensions in §3, and we obtain criteria for tameness of almost automorphic shifts by investigating their discontinuity points and their formulation as semicocycle extensions in §4. While our examples and main results solely deal with symbolic shifts on finite alphabets, it turns out that the extra effort due to treating general almost automorphic systems in that section is almost negligible. Finally, the first part of Theorem 1.3 is proven in §5 and the second part is dealt with in the last section.

1.1. *Basic notions and notation.* Most of this section is standard, see e.g. [2, 6, 28]. We provide additional references for less standard material in the text.

We denote by $\mathbb{N}$ the positive integers and by $\mathbb{N}_0$ the non-negative integers. A *dynamical system* is a continuous $\mathbb{Z}$-action (or $\mathbb{N}$-action) on a compact metric space X. Such a system is specified by a pair (X, T), where T is a continuous self-map on X. Clearly, T is invertible when dealing with a $\mathbb{Z}$-action. Further, every $\mathbb{Z}$-action restricts to two $\mathbb{N}$ actions, its so-called *forward motion* given by positive powers of T, and its *backward motion* given by negative powers of T. Notions such as *subsystem, minimality, (topological) factor, extension, conjugacy* etc. have their standard meaning.

A $\mathbb{Z}$-action is called *equicontinuous* if the family $\{T^n : n \in \mathbb{Z}\}$ is equicontinuous. It is well known, see e.g. [10], that a minimal equicontinuous system $(\mathcal{Z}, S)$ is a *minimal rotation*, that is, there is a continuous abelian group structure on $\mathcal{Z}$ and an element $g \in \mathcal{Z}$ such that the homeomorphism S is given by adding g, $S(z) = z + g$. We hence refer to such a system by $(\mathcal{Z}, +g)$. Minimality implies that g is a *topological generator* of $\mathcal{Z}$, that is, $\{ng : n \in \mathbb{Z}\}$ is dense in $\mathcal{Z}$.

An equicontinuous factor of (X, T) is *maximal* if any other equicontinuous factor of (X, T) factors through it. Here, (X, T) is an *almost one-to-one extension* of a system (Y, S) if the associated factor map $\pi : X \to Y$ is *almost one-to-one*, that is, if $\{x \in X : \pi^{-1}(\{\pi(x)\}) = \{x\}\}$ is G_δ-dense in X. A system (X, T) is *almost automorphic* if it is an almost one-to-one extension of a minimal equicontinuous factor. Almost automorphic systems are necessarily minimal and for minimal systems, $\{x \in X : \pi^{-1}(\{\pi(x)\}) = \{x\}\}$ is a dense G_δ if it is non-empty. Given an almost automorphic system $\pi : (X, T) \to (\mathcal{Z}, +g)$, we call the points $z \in \mathcal{Z}$ with a unique π-preimage *regular*, and those which have multiple preimages *singular*. Correspondingly, we call a π-fibre $\pi^{-1}(z)$ *regular* if it is a singleton and otherwise *singular*. A point $x \in X$ in a regular fibre is also called an *injectivity point*.

In §3, we discuss a natural representation of almost automorphic $\mathbb{Z}$-actions as (bilateral) shifts. Here, by *shift*, we mean a subsystem of the system $(K^{\mathbb{Z}}, \sigma)$ given by the set of

K-valued bilateral sequences $(x_n)_{n\in\mathbb{Z}} \in K^{\mathbb{Z}}$, equipped with the product topology, where K is a compact metric space and σ is the left shift, $\sigma(x)_n = x_{n+1}$.

The theory of topological independence allows for an alternative characterization of (non-)tameness which turns out to be particularly convenient in explicit computations and does not explicitly involve the Ellis semigroup [33]. Given a system (X, T) and subsets $A_0, A_1 \subseteq X$, we say that $J \subseteq \mathbb{Z}$ is an *independence set* for (A_0, A_1) if for each finite subset $I \subseteq J$ and every choice function $\varphi \in \{0, 1\}^I$, there exists $x \in X$ such that $T^i(x) \in A_{\varphi(i)}$ for each $i \in I$. By combining the results from [33] and the aforementioned shift representation of almost automorphic systems, we obtain the following characterization of non-tameness which we actually understand as its definition in the following proposition.

PROPOSITION 1.5. (Cf. [33, Proposition 6.4] and [22, Proposition 3.1]) *A shift $(X, \sigma) \subseteq (K^{\mathbb{Z}}, \sigma)$ is non-tame (or wild) if and only if there are disjoint compact sets $V_0,\ V_1 \subseteq K$ and a sequence of integers $(t_n)_{n\in\mathbb{N}}$ such that for each choice function $\varphi \in \{0, 1\}^{\mathbb{N}}$, there is $(x_n)_n \in X$ for which $x_{t_n} \in V_{\varphi(n)}$ for all $n \in \mathbb{N}$.*

We call the infinite sequence (t_n) of the proposition an *independence sequence* for the pair (V_0, V_1) and note that its elements must be pairwise distinct. In the terminology of [33], the elements of (t_n) form an independence set for the pair of cylinder sets

$$[V_i] = \{x \in X : x_0 \in V_i\} \quad (i = 0, 1).$$

In line with the above characterization of non-tameness, we call a shift (X, σ) *forward non-tame* if it allows for an infinite independence sequence (for some disjoint cylinder sets) of positive integers. Similar to the characterization of non-tameness of $\mathbb{Z}$-actions given in the introduction, this is the case if and only if the Ellis semigroup of the $\mathbb{N}$-action given by the forward motion of (X, σ) contains a copy of $\beta\mathbb{N}$.

Remark 1.6. In the case of symbolic shifts, that is, when $K = \{a_0, \ldots, a_n\}$ is finite, it is important to note that V_0 and V_1 can be chosen to be singletons $\{a_k\}$ and $\{a_\ell\}$, respectively, and we simply write $[a_k], [a_\ell]$ for the corresponding cylinder sets. Indeed, with [33, Proposition 6.4], we have that if there exist V_0 and V_1 as in Proposition 1.5, then there are $x \in [V_0]$ and $y \in [V_1]$ such that for any pair of neighbourhoods U_x and U_y of x and y, respectively, there is an infinite independence set. One may hence choose disjoint cylinder sets $U_x = [* \cdots * a_k * \cdots *]$ and $U_y = [* \cdots * a_\ell * \cdots *]$ (where $*$ is a placeholder and $a_k \neq a_\ell$) to see that actually, we have an independence sequence for $(\{a_k\}, \{a_\ell\})$.

In particular, to prove tameness (or forward tameness) of binary shifts, that is, where $K = \{a, b\}$, it suffices to show that $([a], [b])$ does not allow for an infinite independence set (of positive integers).

2. *Toeplitz shifts*

Our main goal in this work is to characterize those Toeplitz shifts which are tame. In the first two parts of this section, we define Toeplitz shifts, and in particular a simple and ubiquitous subclass, which is defined by (primitive, aperiodic) substitutions of constant length which have a coincidence. While our results are far more general, we will later use examples from this class to illustrate our constructions. These constructions are carried out in §2.4, where we turn to the Bratteli–Vershik representation of Toeplitz shifts to prove Theorems 1.1, 1.2 and 1.4.

2.1. *Odometers and Toeplitz shifts.* Given a sequence (ℓ_n) of natural numbers, we work with the group

$$\mathbb{Z}_{(\ell_n)} := \prod_n \mathbb{Z}/\ell_n\mathbb{Z},$$

where the group operation is given by coordinate-wise addition with carry. For a detailed exposition of equivalent definitions of $\mathbb{Z}_{(\ell_n)}$, we refer the reader to [10]. Endowed with the product topology over the discrete topology on each $\mathbb{Z}/\ell_n\mathbb{Z}$, the group $\mathbb{Z}_{(\ell_n)}$ is a compact metrizable topological group, where the unit $z = \ldots 001$, which we simply write as $z = 1$, is a topological generator. We write elements (z_n) of $\mathbb{Z}_{(\ell_n)}$ as left-infinite sequences $\ldots z_2 z_1$, where $z_n \in \mathbb{Z}/\ell_n\mathbb{Z}$, so that addition in $\mathbb{Z}_{(\ell_n)}$ has the carries propagating to the left as usual in $\mathbb{Z}$. If $\ell_n = \ell$ is constant, then $\mathbb{Z}_{(\ell_n)} = \mathbb{Z}_\ell$ is the classical ring of ℓ-adic integers.

With the above notation, an *odometer* is a dynamical system $(\mathcal{Z}, +1)$, where $\mathcal{Z} = \mathbb{Z}_{(\ell_n)}$ for some sequence (ℓ_n). A *Toeplitz shift* is a symbolic shift (X, σ), $X \subseteq \mathcal{A}^{\mathbb{Z}}$ with $\mathcal{A}$ finite, which is an almost automorphic extension of an odometer and hence minimal.

2.2. *Constant length substitutions.* A special class of almost automorphic extensions of odometers is the class of *primitive aperiodic constant length substitutions which possess a coincidence.* Since we illustrate our theory mainly with examples from this class, and because the proof of our main result simplifies for this class, we give the reader both a brief exposition of substitutions and also a flavour of our main result for this important class of almost automorphic extensions.

Let $\mathcal{A}$ be a finite set, referred to as an *alphabet*. A *substitution of (constant) length* ℓ over $\mathcal{A}$ is a map $\theta : \mathcal{A} \to \mathcal{A}^\ell$. We can write such a substitution as follows: there are ℓ maps $\theta_i : \mathcal{A} \to \mathcal{A}$, $0 \le i \le \ell - 1$ such that $\theta(a) = \theta_0(a) \mid \cdots \mid \theta_{\ell-1}(a)$ for all $a \in \mathcal{A}$, where $\mid$ is to separate the concatenated letters.

We use concatenation to extend θ to a map on finite and infinite words in $\mathcal{A}$. We say that θ is *primitive* if there is some $k \in \mathbb{N}$ such that for any $a, a' \in \mathcal{A}$, the word $\theta^k(a)$ contains at least one occurrence of a'. We say that a finite word is *allowed* for θ if it appears as a subword in some $\theta^k(a)$, $a \in \mathcal{A}$, $k \in \mathbb{N}$.

Let $X_\theta \subseteq \mathcal{A}^{\mathbb{Z}}$ be the set of bi-infinite sequences all of whose finite subwords are allowed for θ. Then (X_θ, σ) is the *substitution shift* defined by θ. Primitivity of θ implies that (X_θ, σ) is minimal. We say that a primitive substitution is *aperiodic* if X_θ does not comprise σ-periodic sequences. This is the case if and only if X_θ is an infinite space.

The shift (X_θ, σ) of a primitive aperiodic substitution of constant length ℓ factors onto the odometer $(\mathbb{Z}_\ell, +1)$. Indeed, for any $n \ge 1$, the space X_θ can be partitioned into ℓ^n clopen subsets $\sigma^i(\theta^n(X_\theta))$, $i = 0, \ldots, \ell^n - 1$, and the factor map is given by $X_\theta \ni x \mapsto \ldots z_2 z_1 \in \mathbb{Z}_\ell$, where z_n is the unique i such that $x \in \sigma^i(\theta^n(X_\theta))$ [7]. The maximal equicontinuous factor of (X_θ, σ) is therefore a covering of $(\mathbb{Z}_\ell, +1)$ and the degree of this covering is called the *height* of the substitution. The height h is always finite. Given θ, there is a primitive aperiodic substitution θ', referred to as the *pure base* of θ, which is of the same length ℓ, has height 1 and is such that (X_θ, σ) is a $\mathbb{Z}/h\mathbb{Z}$-suspension over $(X_{\theta'}, \sigma)$. That is, $X_\theta \cong X_{\theta'} \times \mathbb{Z}/\sim$ with $(x, n+h) \sim (\sigma(x), n)$ and the action is induced by $\mathrm{id} \times (+1)$. There is an explicit construction of θ' which, in fact, equals θ if $h = 1$. Clearly, the maximal equicontinuous factor of the pure base system is $(\mathbb{Z}_\ell, +1)$. For all details, see [7].

Let θ have as pure base the substitution θ' defined on the alphabet $\mathcal{A}'$. We say that θ has a *coincidence* if for some $k \in \mathbb{N}$ and some $i_1, \ldots, i_k \in \{0, \ldots, \ell - 1\}$, we have $|\theta'_{i_1} \ldots \theta'_{i_k}(\mathcal{A}')| = 1$. The importance of this notion lies in the theorem of Dekking stating that the substitution shift (X_θ, σ) is almost automorphic if and only if θ has a coincidence [7].

We shall see below that the question of whether (X_θ, σ) is tame or not is governed by the cardinality of the set of orbits of singular points in the maximal equicontinuous factor of (X_θ, σ). Since $X_\theta \cong X_{\theta'} \times \mathbb{Z}/\sim$, the orbits of singular points in the maximal equicontinuous factor of (X_θ, σ) are in one-to-one correspondence with the orbits of singular points in the maximal equicontinuous factor of the pure base system $(X_{\theta'}, \sigma)$. We may therefore just determine the cardinality of the latter. There is an effective procedure which achieves this [5]. We recapitulate a slightly modified version here in the only case which concerns us, which is when θ has a coincidence and, by going over to the pure base of the substitution if needed, its height is 1.

Consider the graph $\mathcal{G}_\theta$ whose vertices are the sets

$$\{\mathcal{A}\} \cup \{A := \theta_{w_1} \cdots \theta_{w_k}(\mathcal{A}) \colon k \geq 1,\ w_1, \ldots, w_k \in \{0, 1, \ldots, \ell - 1\},\ \text{and } |A| > 1\}$$

and whose edges are defined as follows: if A and B are vertices in $\mathcal{G}_\theta$, then there is an oriented edge from B to A, labelled i, if and only if $\theta_i(A) = B$. An infinite path in $\mathcal{G}_\theta$ defines a point $\ldots z_2 z_1 \in \mathbb{Z}_\ell$, where z_i is the label of the ith edge in the path.

We define $\hat{\Sigma}_\theta$ to be the set of all sequences $(z_i) \in \mathbb{Z}_\ell$ obtained as above from infinite paths in $\mathcal{G}_\theta$. If θ is minimal, aperiodic and has a coincidence, then $\hat{\Sigma}_\theta$ is non-empty. The $\mathbb{Z}$-orbit of $\hat{\Sigma}_\theta$ under $+1$ equals $\{z \in \mathbb{Z}_\ell : |\pi^{-1}(z)| > 1\}$, which is the set of singular points in $\mathbb{Z}_\ell$. This set is a proper subset of $\mathbb{Z}_\ell$, as (X_θ, σ) is almost automorphic.

Recall that a *cycle* on an oriented graph is a finite path which is closed and minimal in the sense that it is not the concatenation of smaller closed paths.

LEMMA 2.1. *Let θ be a primitive aperiodic substitution of constant length with pure base θ'. Its maximal equicontinuous factor contains either finitely many, or uncountably many orbits of singular points. The latter is the case if and only if $\mathcal{G}_{\theta'}$ contains two distinct cycles which share a common vertex.*

Proof. Observe that besides the paths corresponding to the orbit through 0, two infinite paths in $\mathcal{G}_{\theta'}$ belong to the same $\mathbb{Z}$-orbit of $\mathbb{Z}_\ell$ if they differ only on a finite initial segment, that is, if they are tail equivalent. Therefore, the statement comes down to showing that there are finitely many or uncountably many distinct infinite paths up to tail equivalence in $\mathcal{G}_{\theta'}$.

Clearly, $\mathcal{G}_{\theta'}$ must contain cycles as it contains infinite paths. If we have a vertex in two different cycles then, starting from that vertex, we can follow through the two cycles in any order we wish and therefore the number of paths in $\mathcal{G}_{\theta'}$ is uncountable.

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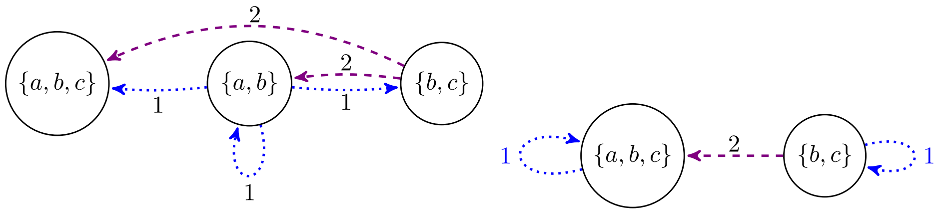

FIGURE 1. The graph $\mathcal{G}_\theta$ for the substitutions from Example A (left) and Example B (right). We used blue dotted and violet dashed lines for better comparison with Figure 2.

Now assume that there is no vertex in two different cycles. An infinite path must visit some vertex infinitely often. As this vertex is not part of more than one cycle, the path must eventually follow the same cycle. As there are only finitely many vertices and edges, there can only be finitely many cycles. Hence, up to tail equivalence, there are only finitely many infinite paths in $\mathcal{G}_{\theta'}$. □

Let us anticipate the following important consequence of Lemma 2.1 combined with Theorem 2.18 and the discussion in §2.3.4.

THEOREM 2.2. *Let θ be a primitive aperiodic substitution of constant length with pure base θ'. Then (X_θ, σ) is tame if and only if it has a coincidence and $\mathcal{G}_{\theta'}$ does not contain two distinct cycles which share a common vertex.*

Example A. Let θ be the substitution

$$\begin{array}{rcl} a & \mapsto & aaca \\ b & \mapsto & abba \\ c & \mapsto & aaba, \end{array}$$

on the alphabet $\mathcal{A} = \{a, b, c\}$. It is primitive, aperiodic and has trivial height. Its graph $\mathcal{G}_\theta$ is depicted on the left-hand side in Figure 1. Note that $\{a, c\}$ is not a vertex in $\mathcal{G}_\theta$ because it cannot be expressed as $\{a, c\} = \theta_{w_1} \cdots \theta_{w_k}(\mathcal{A})$ for any word $w_1 \cdots w_k$. Here, $\mathcal{G}_\theta$ has two different cycles at $\{a, b\}$, so by Theorem 2.2, (X_θ, σ) is non-tame.

Example B. We modify slightly the above example and define θ as

$$\begin{array}{rcl} a & \mapsto & aaca \\ b & \mapsto & abba \\ c & \mapsto & acba, \end{array}$$

on the alphabet $\mathcal{A} = \{a, b, c\}$. It still is primitive, aperiodic and has trivial height. The graph $\mathcal{G}_\theta$ is shown on the right in Figure 1, and as it has only one cycle about any vertex, by Theorem 2.2, (X_θ, σ) is tame.

2.3. *The Bratteli–Vershik representation of a Toeplitz shift.* In this section, we briefly discuss the Bratteli–Vershik representation of Toeplitz shifts. As Bratteli–Vershik systems are well documented in the literature, we keep this to a minimum. Interested readers may consult classical references on Bratteli–Vershik systems, such as [29]. In particular, since we will only be concerned with Bratteli–Vershik systems that are conjugate to Toeplitz shifts, we refer the reader to the work by Gjerde and Johansen [23]; unless stated otherwise, we adopt the latter's notational conventions.

2.3.1. *Bratteli–Vershik systems.* A *Bratteli diagram* is an infinite graph $B = (V, E)$, where the vertex set $V = \bigsqcup_{n\geq 0} V_n$ and the edge set $E = \bigsqcup_{n\geq 0} E_n$ are equipped with a *range map* $r\colon E \to V$ and a *source map* $s\colon E \to V$ such that:

(1) $V_0 = \{v_0\}$ is a singleton;
(2) V_n and E_n are finite sets;
(3) $r(E_n) = V_{n+1}, s(E_n) = V_n$;
(4) $r^{-1}(v) \neq \emptyset$ for all $v \neq v_0$.

The pair (V_n, E_n) or just V_n is called the *nth level* of the diagram B. A finite or infinite sequence of edges $(\gamma_n : \gamma_n \in E_n)$ such that $r(\gamma_n) = s(\gamma_{n+1})$ is called a *finite* or *infinite path*, respectively. The source map and the range map extend to paths in the obvious way. For a Bratteli diagram B, let X_B be the set of infinite paths $(\gamma_n)_{n\geq 0}$ starting at the *top vertex* v_0. Given a path $(\gamma_n)_{n\geq 0}$ and $m \geq 0$, we call $(\gamma_n)_{n\geq m}$ a *tail of* (γ_n) and $(\gamma_n)_{n\leq m}$ a *head of* (γ_n). Two paths (γ_n) and (γ'_n) are called *tail equivalent* (or *cofinal*) if they share a common tail.

We shall constantly use the *telescoping* procedure. Let B be a Bratteli diagram and $n_0 = 0 < n_1 < n_2 < \cdots$ be a strictly increasing sequence of integers. The *telescoping of B to* (n_k) is the Bratteli diagram B', whose k-level vertex set is $V'_k := V_{n_k}$, and where the set of edges between $v \in V'_k$ and $w \in V'_{k+1}$ are in one-to-one correspondence with the set of paths in B between $v \in V_{n_k}$ and $w \in V_{n_{k+1}}$. There is then an obvious bijection between X_B and $X_{B'}$.

A Bratteli diagram B has *rank d* if there is a telescoping B' of B such that B' has exactly d vertices at each level. We say that B is *simple* if for any level m, there is $n > m$ such that for any two vertices $v \in E_n$ and $w \in V_m$, there is a path with source w and range v. This is equivalent to saying that by telescoping, we can arrive at a Bratteli diagram B' such that any two vertices in consecutive levels are connected by an edge.

Recall also that the path space X_B comes with a totally disconnected compact metrizable topology, and if B is simple and $|E_n| > 1$ infinitely often, then X_B is a Cantor set.

Definition 2.3. We say that $B = (V, E)$ has the *equal path number property* if there is a sequence $(\ell_n)_{n\geq 0}$ such that for each $v \in V_{n+1}$, there are ℓ_n edges with range v. We call $(\ell_n)_{n\geq 0}$ the *characteristic sequence* of B.

An *ordered* Bratteli diagram is a Bratteli diagram together with a total order on each set of edges which end at the same vertex. In other words, for each $r^{-1}(v)$, $v \in V$, the order naturally defines a bijection $\omega : r^{-1}(v) \to \{0, \ldots, |r^{-1}(v)| - 1\}$. We refer to $\omega(e)$ also as the *label* of e. Under certain circumstances, given in detail in [23], this order induces a *proper order* ω on X_B. In a nutshell, the *successor* of an infinite path $(\gamma_n)_{n\geq 0} \in X_B$ with respect to ω, when this exists, is a tail-equivalent path whose order labelling corresponds to an addition of 1 to that of $(\gamma_n)_{n\geq 0}$. The notion of a predecessor is defined analogously. The order being *proper* means that there is a unique path (the *maximal* path) which has no successor in this order, and a unique path (the *minimal* path) which has no predecessor. In this case, one can define the *Vershik* map $\varphi_\omega : X_B \to X_B$, which sends a non-maximal path to its successor and which sends the unique maximal path to the unique minimal path. It is a homeomorphism and thus defines a dynamical system (X_B, φ_ω) referred to as a *Bratteli–Vershik system*. If (B, ω) is a properly ordered Bratteli diagram and $B' = (V', E')$ is the telescoping of B to levels (n_k), then the order ω defines a natural proper order ω' on B', and (X_B, φ_ω) is topologically conjugate to $(X_{B'}, \varphi_{\omega'})$. Properly ordered simple Bratteli diagrams define minimal Bratteli–Vershik systems. Conversely, any Cantor minimal dynamical system (X, T) is conjugate to a Bratteli–Vershik system where B is simple [29]; the latter is called a Bratteli–Vershik representation of (X, T). Not every such dynamical system has a Bratteli–Vershik representation with finite rank, but if this is the case, one says that (X, T) has *finite topological rank*. More precisely, the topological rank of such a system is the smallest rank among its Bratteli–Vershik representations.

2.3.2. *Toeplitz Bratteli–Vershik systems.* We now focus on Bratteli diagrams which have the equal path number property. Here, if $v \in V_n$, then $|r^{-1}(v)| = \ell_n$. Recall that a Bratteli–Vershik system is *expansive* if and only if there is $k \in \mathbb{N}$ such that for distinct $x, y \in X_B$, the head of length k of $\phi_\omega^n(x)$ differs from that of $\phi_\omega^n(y)$ for some $n \in \mathbb{Z}$. Downarowicz and Maass [14] show that a simple properly ordered Bratteli–Vershik system with finite topological rank is expansive if and only if its topological rank is strictly larger than 1. A simple properly ordered Bratteli–Vershik system with topological rank 1 is an odometer.

THEOREM 2.4. [23] *The family of expansive, simple, properly ordered Bratteli–Vershik systems with the equal path number property coincides with the family of Toeplitz shifts up to conjugacy.*

In view of this result, we call an expansive simple properly ordered Bratteli–Vershik system with the equal path number property a *Toeplitz Bratteli–Vershik system*. Moreover, we say that a Toeplitz shift has *finite Toeplitz rank* if it is conjugate to a Toeplitz Bratteli–Vershik system which has finite rank. Note that having finite Toeplitz rank is stronger than having a Toeplitz shift with finite topological rank as we cannot rule out that a Toeplitz system with infinite Toeplitz rank has a Bratteli–Vershik representation of finite rank but without the equal path number property.

LEMMA 2.5. [23] *Let (X_B, φ_ω) be a Toeplitz Bratteli–Vershik system with characteristic sequence (ℓ_n). The level-wise application of the edge order map*

$$\omega : (X_B, \varphi_\omega) \to (\mathbb{Z}_{(\ell_n)}, +1), \omega((\gamma_n)) = (\omega(\gamma_n))$$

is a factor map to the maximal equicontinuous factor.

For Bratteli diagrams that have the equal path number property, it is standard to describe the ordering ω using a sequence of constant length morphisms. To describe the ordering of the edge set E_n between V_n and V_{n+1}, we use the morphism

$$\theta^{(n)} : V_{n+1} \to V_n^{\ell_n} \tag{2.1}$$

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which, when written as a concatenation of maps $\theta^{(n)} = \theta_0^{(n)}|\cdots|\theta_{\ell_n-1}^{(n)}$, where $\theta_i^{(n)}: V_{n+1} \to V_n$ (similarly as in §2.2), is given by

$$\theta_i^{(n)}(v) = s(e), \quad \text{with } e \in \omega^{-1}(i) \cap r^{-1}(v),$$

that is, the ith morphism reads the source of the unique edge with range v and label i.

If (B, ω) has the equal path number property and $B' = (V', E')$ is the telescoping of B to levels (n_k), then B' also has the equal path number property, and the corresponding morphism

$$\theta'^{(k)} : V'_{k+1} \to V_k'^{\ell'_k} \tag{2.2}$$

is given by the composition

$$\theta'^{(k)} = \theta^{(n_k)} \cdots \theta^{(n_{k+1}-1)}.$$

We can again write $\theta'^{(k)}$ as a succession of maps $\theta_i'^{(k)} : V'_{k+1} \to V'_k$ which are each a composition of maps $\theta_{i_n}^{(n)} : V_{n+1} \to V_n$, $n_k \le n < n_{k+1}$, where the labels i_n are those of the edges in (B, ω) which constitute the edge $e \in E'_k$ with label i. More precisely,

$$\theta_i'^{(k)} = \theta_{i_{n_k}}^{(n_k)} \cdots \theta_{i_{n_{k+1}-1}}^{(n_{k+1}-1)} \quad \text{where } i = \sum_{n=n_k}^{n_{k+1}-1} i_n \prod_{m=n_k}^{n-1} \ell_m$$

(with the understanding that $\prod_{m=n_k}^{n_k-1} \ell_m = 1$). For instance, if we telescope level n with level $n+1$, we get $\theta'^{(n)} = \theta^{(n)}\theta^{(n+1)}$ and

$$\theta'^{(n)}_{i+j\ell_n} = \theta_i^{(n)}\theta_j^{(n+1)} \quad \text{for } 0 \le i < \ell_n,\ 0 \le j < \ell_{n+1}.$$

By telescoping if necessary, we can assume that $\ell_n > 1$ for each n. For otherwise, $\ell_n = 1$ for almost all n and this implies that X_B is not a Cantor space.

2.3.3. *Shift interpretation.* There is a strong connection between expansive Bratteli–Vershik systems and shifts [14]. Suppose (X_B, φ_ω) is expansive and $k \in \mathbb{N}$ is such that the heads of length k suffice to separate orbits of distinct elements in X_B. Given a path $x \in X_B$, associate the bi-infinite sequence whose nth entry consists of the head of length k of $\varphi_\omega^n(x)$. Let (X_k, σ) be the shift whose space consists of all such sequences of length-k heads. Then (X_k, σ) and (X_B, ϕ_ω) are conjugate. By telescoping the diagram to the kth level, we may assume that $k = 1$. Given a Toeplitz Bratteli–Vershik system, we may (and will throughout this article) therefore assume that

$$p_1 : (X_B, \varphi_\omega) \to (X_1, \sigma), \quad (\gamma_\ell)_{\ell \ge 0} \mapsto ((\varphi_\omega^n(\gamma))_0)_{n \in \mathbb{Z}} \tag{2.3}$$

is a conjugacy.

If there is only one edge between the top vertex v_0 and each vertex in V_1, then the range map r is a bijection between E_0 and V_1 so that (X_B, ϕ_ω) is conjugate to a shift over the alphabet V_1. We denote the respective conjugacy also by r. Observe that this situation can always be enforced by insertion of an extra level. Namely, we introduce an intermediate vertex set $\mathcal{V}$ between V_0 and V_1, which is in one-to-one correspondence

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with E_0. We introduce one edge from v_0 to each vertex of $\mathcal{V}$ and then for each $e \in E_0 = \mathcal{V}$, an edge with source e and range $r(e) \in V_1$ with the same order label as e. Clearly, the resulting Bratteli–Vershik system is topologically conjugate to the old one. For Toeplitz Bratteli–Vershik systems, this means that $\ell_0 = 1$.

The following lemma is elementary to verify; its proof follows from the built-in recognizability of Bratteli–Vershik systems, by which we mean that for each n, the towers defined by the first n levels of B form a partition of $p_1(X_B)$, where p_1 is defined in equation (2.3) and which we assume, without loss of generality, to be a conjugacy. Given a sequence (x_n) and $n < m$, we denote by $x_{[n,m[}$ the finite word $x_n x_{n+1} \cdots x_{m-1}$.

LEMMA 2.6. *Let (X_B, φ_ω) be a Toeplitz Bratteli–Vershik system with characteristic sequence (ℓ_n) where $\ell_0 = 1$. Set $\ell^{(n)} = \prod_{k=0}^{n-1} \ell_k$. For all $\gamma \in X_B$ and $n \in \mathbb{N}$, we have*

$$r \circ p_1(\gamma)_{[-z^{(n)}, \ell^{(n)} - z^{(n)}[} = \theta^{(1)} \cdots \theta^{(n)} \circ r(\gamma_n),$$

where $z^{(n)} = \sum_{k=0}^{n-1} \omega(\gamma_k) \ell^{(k)}$. In particular,

$$\{x_0 : x \in r \circ p_1(\omega^{-1}(z))\} \subseteq \bigcap_n \theta^{(1)}_{z_1} \cdots \theta^{(n)}_{z_n}(V_{n+1}). \tag{2.4}$$

2.3.4. *Toeplitz Bratteli–Vershik diagrams for constant length substitutions.* A *stationary* Toeplitz Bratteli–Vershik system is one where for all $n \geq 1$, $V_n = V_1$, $E_n = E_1$, and the order structure on E_n is the same as that on E_1. In this case, the morphisms $\theta^{(n)}$ of equation (2.2) all agree and hence define a single morphism $\theta : V_1 \to V_1^{\ell_1}$. If the range map is a bijection between E_0 and V_1 (there is a single edge between v_0 and each of the vertices of V_1), we can identify the space X_1 with the substitution shift space X_θ of θ. In other words, a stationary Toeplitz Bratteli–Vershik system for which $E_0 \cong V_1$ defines a primitive substitution of constant length.

The converse, associating a Toeplitz Bratteli–Vershik system to a primitive substitution θ of constant length ℓ over an alphabet $\mathcal{A}$, is subtle. The natural approach [**41**], which consists of defining a stationary Bratteli–Vershik system by setting $V_n = \mathcal{A}$ and defining the edges and their order with $\theta^{(n)} = \theta$ as in equation (2.2), works well if all substitution words start with the same letter and end with the same letter, that is, $\theta_0(a)$ and $\theta_{\ell-1}(a)$ are independent of a. Indeed, if that is the case, then the order on the Bratteli diagram is proper. However, for general θ, the respectively defined order may fail to be proper and the arguments in [**41**] only give a measurable Bratteli–Vershik representation. While there are classical methods to rewrite the substitution to obtain a stationary Bratteli–Vershik representation for (X_θ, σ) [**16**, **17**], those procedures do not necessarily give a Toeplitz Bratteli–Vershik representation. Instead, one needs to follow the approach of [**23**] to obtain a properly ordered Toeplitz Bratteli–Vershik system such that (X_1, σ) equals (X_θ, σ).

2.4. *The extended Bratteli diagram.* In this section, we introduce the notion of the extended Bratteli diagram and its essential thickness, concepts which are fundamental for the proofs of Theorems 1.4 and 1.1. The extended Bratteli diagram can be seen as a generalization of the graph $\mathcal{G}_\theta$, introduced in §2 for constant length substitution shifts.

Definition 2.7. Let (B, ω) be an ordered Bratteli diagram with the equal path number property and characteristic sequence $(\ell_n)_{n\geq 0}$. The extended Bratteli diagram is an infinite graph which satisfies the properties (1)–(3) of a Bratteli diagram, but not necessarily property (4). The *extended Bratteli diagram* $\tilde{B} = (\tilde{V}_n, \tilde{E}_n)$ associated to B has the following vertices and edges.

(1) The level n vertex set $\tilde{V}_n$ is the set of all non-empty subsets of V_n.
(2) For $0 \leq i < \ell_n$, $\tilde{E}_n$ contains an edge labelled i with source $A \in \tilde{V}_n$ and range $B \in \tilde{V}_{n+1}$ if and only if $\theta_i^{(n)}(B) = A$.

Identifying singleton sets with the element they contain, we see that $\tilde{B}$ contains B as a sub-diagram. The labelling of the edges defines an order on $\tilde{B}$ which extends the order on B. We also consider the space $X_{\tilde{B}}$ of infinite paths over $\tilde{B}$ starting at the top vertex v_0. Clearly, $X_{\tilde{B}}$ contains X_B. The edge order map ω from Lemma 2.5 extends to a map from $X_{\tilde{B}}$ to the maximal equicontinuous factor $\mathbb{Z}_{\ell_n}$ which we denote by the same letter ω. Due to the lack of property (4), the vertices in the extended diagram need not to have any outgoing edges. Infinite paths ignore such vertices and we call vertices *extendable* if they are traversed by a path in $X_{\tilde{B}}$.

To a path γ in $X_{\tilde{B}}$, we associate the sequence of maps

$$\theta_\gamma^{(n)} := \theta_{\omega(\gamma)_n}^{(n)} : \tilde{V}_{n+1} \to \tilde{V}_n \tag{2.5}$$

and the sequence of subsets $A_n := s(\gamma^{(n)}) \subseteq V_n$. Then $\theta_{\omega(\gamma)_n}^{(n)}(A_{n+1}) = A_n$. Recall that we can arrange for $\ell_0 = 1$ in the characteristic sequence of the original Toeplitz Bratteli system. This implies that in the extended Bratteli diagram, the top vertex is linked to any vertex of $\tilde{V}_1$ by exactly one edge.

We remark that in the case where the ordered diagram B is *stationary*, that is, $E_n = E_1$ and $\theta^{(n)} = \theta^{(1)}$ for $n \geq 1$, then the graph $\mathcal{G}_{\theta^{(1)}}$ defined in §2 is an abbreviated form of the extended Bratteli diagram $(\tilde{B}, \tilde{\omega})$. The extended Bratteli diagram will also be stationary and so can be described by the edge and order structure of its first level. The only other difference is that in $\mathcal{G}_{\theta^{(1)}}$, we chose to exclude vertices indexing one-element sets, as $\mathcal{G}_{\theta^{(1)}}$ is only to identify the singular fibres.

Note that since $|\theta_i^{(n)}(A)| \leq |A|$ for each n and i, a path in $X_{\tilde{B}}$ must pass through vertices of non-decreasing cardinality. This motivates the following definition.

Definition 2.8. If a path of $X_{\tilde{B}}$ eventually goes through vertices $A_n \in \tilde{V}_n$ with $|A_n| = k < \infty$ for all n large, we will say that the path has *thickness* k. Otherwise, we say that the path has *infinite thickness*.

If the diagram has finite rank, then there are no paths with infinite thickness. We denote by $X_{\tilde{B}}^k$ the infinite paths of thickness $k \in \mathbb{N} \cup \{\infty\}$. Note that the sub-diagram $X_{\tilde{B}}^1$ corresponds to the original path space X_B.

The following lemma tells us that $z \in \mathbb{Z}_{(\ell_n)}$ is singular if and only if it is the image of a path of thickness $k > 1$. Let $\mathrm{thk}(\gamma)$ denote the thickness of the path γ.

LEMMA 2.9. *Let (X_B, φ_ω) be a Toeplitz Bratteli–Vershik system with extended path space $X_{\tilde{B}}$. Let $z \in \mathbb{Z}_{(\ell_n)}$. Then,*

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$$|\{\gamma \in X_B : \omega(\gamma) = z\}| = \sup\{\mathrm{thk}(\tilde{\gamma}) : \tilde{\gamma} \in X_{\tilde{B}}, \omega(\tilde{\gamma}) = z\}.$$

In particular, the set of singular points in $\mathbb{Z}_{(\ell_n)}$ coincides with the union $\bigcup_{j\geq 2} \omega(X^j_{\tilde{B}})$, and the rank of the Toeplitz Bratteli–Vershik system is an upper bound for the maximal number of elements in a fibre of the factor map to the maximal equicontinuous factor.

Proof. Recall the definition of the maps $\theta^{(n)}_{\tilde{\gamma}}$ and subsets $A_n := s(\gamma^{(n)}) \subseteq V_n$ associated to a path $\tilde{\gamma} \in X_{\tilde{B}}$ in equation (2.5). If $\mathrm{thk}(\tilde{\gamma}) \geq k$, then there exists n_0 such that $|A_n| \geq k$ for $n \geq n_0$. Since $\theta^{(n)}_{\tilde{\gamma}}(A_{n+1}) = A_n$, there are at least k paths γ in the original path space X_B such that $\omega(\tilde{\gamma}) = \omega(\gamma)$. This shows the inequality '$\geq$'.

Now suppose that $|\{\gamma \in X_B : \omega(\gamma) = z\}| \geq k$, so that there are at least k distinct paths $\gamma \in X_B$ with the same edge labels. We need to make sure that there is at least one n such that they go through k different vertices at level n. Note that if two paths with equal edge labels agree on a vertex at level n, then their head agrees up to level n. Thus, k distinct paths must at some level go through k distinct vertices. This implies that there is an A_n with $|A_n| \geq k$ which is a vertex of a path in $X_{\tilde{B}}$ which has edge labels z. Thus, $\sup\{\mathrm{thk}(\tilde{\gamma}) : \tilde{\gamma} \in X_{\tilde{B}}, \omega(\tilde{\gamma}) = z\} \geq k$. □

2.5. *Thick Toeplitz shifts.* A pair of *parallel edges* in $\tilde{E}_n$ is a pair of edges $(e_1, e_2) \in \tilde{E}_n \times \tilde{E}_n$ with the same source and range but distinct labels according to the order. A *double path* in $X_{\tilde{B}}$ is a pair of paths consisting of parallel edges at each level $n > 0$. We write $\bar{\gamma} = (\bar{\gamma}_n) = (\gamma_{n,1}, \gamma_{n,2})$ to denote a double path.

Definition 2.10. The largest k such that $X^k_{\tilde{B}}$ is uncountable is called the *essential thickness* of (X_B, φ_ω). We say that (X_B, φ_ω) is *thick* if its essential thickness k is strictly larger than 1 and finite, and if there is a double path of thickness k. A Toeplitz shift is *thick* if it has a thick Toeplitz Bratteli–Vershik representation.

Example A. (Continued) To illustrate the above notions, we apply them to the first substitution in Example A. While this example is not sensitive to some of the subtleties that we will meet later (because of its stationarity), it can at least be described explicitly.

Recall that $\theta : \{a, b, c\} \to \{a, b, c\}^4$ is the substitution

$$\begin{array}{ccc} a & \mapsto & aaca \\ b & \mapsto & abba \\ c & \mapsto & aaba. \end{array}$$

Since all substitution words begin and end on a, the approach of [41] to define the Toeplitz Bratteli–Vershik system works here and it is not difficult to derive the extended system as well. The extended Bratteli diagram is stationary and we have drawn one level in Figure 2. We follow the convention of reading levels from top to bottom. Note that the vertices $\{a, b, c\}$ and $\{a, c\}$ have no outgoing edges, so no infinite path will go through them and, in particular, there are no paths of thickness 3. There are uncountably many paths of thickness 2, namely those which keep going through vertices $\{a, b\}$ or $\{b, c\}$. Hence, all singular fibres consist of two elements and the essential thickness is 2.

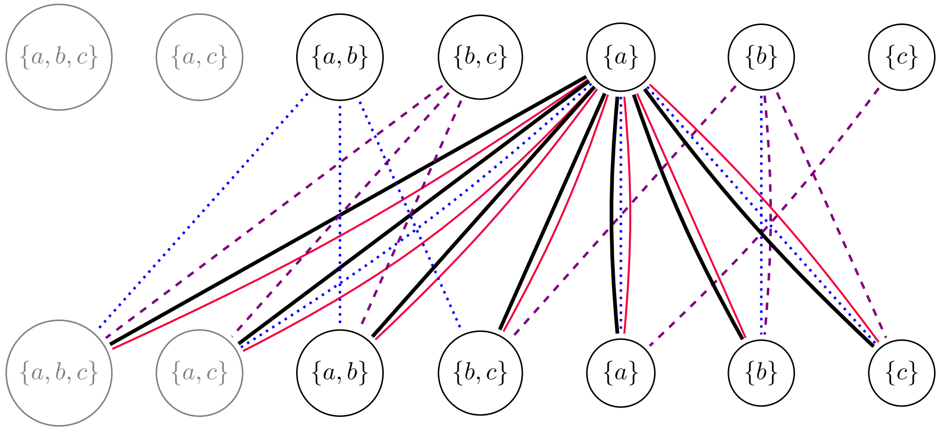

FIGURE 2. One level of the stationary extended Bratteli diagram of Example A. The order is indicated through colour: black, blue dotted, violet dashed and red edges correspond to order label 0, 1, 2 and 3, respectively. The grey vertices are not extendable. Red edges are finer than black edges if viewed without colour.

If we telescope the extended Bratteli diagram to even levels, we will find that there are two edges between two consecutive vertices $\{a, b\}$. These two edges form a pair of parallel edges and consequently the telescoped diagram admits a double path of thickness 2. In particular, the Toeplitz Bratteli–Vershik system is thick.

Notice the connections between Figure 2 and the graph $\mathcal{G}_\theta$ in Figure 1. The infinite paths on $\mathcal{G}_\theta$ correspond to paths in the stationary extended Bratteli diagram which start at the top vertex v_0 (the first level consists of one edge between v_0 and each of the seven vertices of $\tilde{V}_1$) and go downwards without ever passing through a vertex which is a singleton nor through a vertex which does not have an outgoing edge. The fact that the telescoped extended Bratteli diagram admits a double path of thickness 2 going through the vertices $\{a, b\}$ is equivalent to the fact that the vertex $\{a, b\}$ of $\mathcal{G}_\theta$ belongs to two distinct cycles.

Example B. (Continued) It is not difficult to derive an extended Bratteli diagram for the substitution of Example B as well. What one will find is that there is one edge between two consecutive vertices $\{a, b, c\}$ and one edge between two consecutive vertices $\{b, c\}$. It follows that the diagram has thickness 3. However, there is only one path which goes infinitely often through $\{a, b, c\}$ and only countably many which go infinitely often through $\{b, c\}$. It follows that the essential thickness of the diagram is 1. The system is therefore not thick.

Example C. Oxtoby [37] described a family of minimal binary Toeplitz shifts that are not uniquely ergodic and hence cannot be tame. We describe the Bratteli–Vershik representations for the one-sided versions of this family to maximize the similarity to his original description. Given a sequence (ℓ_n) of natural numbers, define the substitutions

$$a \overset{\theta^{(n)}}{\mapsto} ab^{\ell_n - 1}$$
$$b \overset{\theta^{(n)}}{\mapsto} aa^{\ell_n - 1},$$

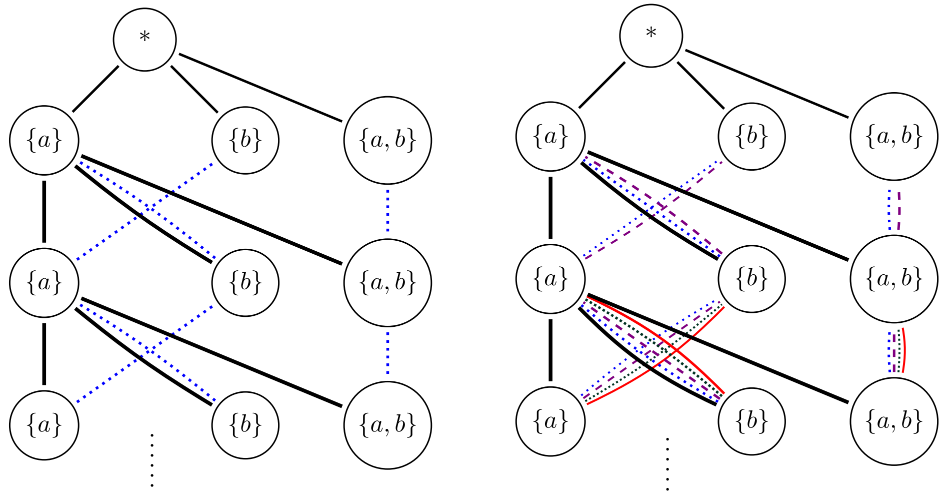

FIGURE 3. On the right, we see the first levels of the extended Bratteli diagram of the one-sided shifts for Example C with $\ell_1 = 3$ and $\ell_2 = 5$. The order is indicated through colour: black, blue dotted, violet dashed, green densely dotted and red edges correspond to order label 0, 1, 2, 3 and 4, respectively. Red edges are finer than black edges if viewed without colour. As more levels are added, there are increasingly many edges between vertices labelled $\{a, b\}$ in consecutive levels. This is to be contrasted with the one-sided *period-doubling* substitution shift (on the left), where $\ell_n = 2$ for all n, and which is tame (again, black and blue dotted edges correspond to order label 0 and 1, respectively). It has thickness one.

and define an ordered Bratteli diagram with the sequence $\{\theta^{(n)}\}$ as in equation (2.1); see Figure 3 for two examples, the one on the left with $\ell_1 = \ell_2 = 2$, the second with $\ell_1 = 3$, $\ell_2 = 5$.

Note that these ordered Bratteli diagrams each have two maximal paths and one minimal path; this means that we cannot define a Vershik map which is a homeomorphism. Nevertheless, we can still define a continuous Vershik map by sending the two maximal paths to the unique minimal path. This one-sided Bratteli–Vershik system is conjugate to the one-sided shift defined by Oxtoby. Oxtoby showed that if (ℓ_n) grows fast enough, in particular if $\sum(\ell_{k-1}/\ell_k) < 1$, then the resulting system is not uniquely ergodic and thus it cannot be tame. On the right-hand side of Figure 3, one clearly sees the beginning of a double path of thickness two. This should be contrasted with the stationary figure on the left, which is a one-sided representation of the *period-doubling* substitution shift and which has thickness one, and so is tame, as we will see below.

Recall that $x \in X$ is a *condensation point* if every neighbourhood of x is uncountable. Let $\mathcal{C} \subseteq X$ be the set of its condensation points. The Cantor–Bendixson theorem gives that for second countable spaces, $X \backslash \mathcal{C}$ is countable.

LEMMA 2.11. *Let (X_B, φ_ω) be a Toeplitz Bratteli–Vershik system.*

(1) *If all paths in $X_{\tilde{B}}$ have finite thickness, the maximal equicontinuous factor contains uncountably many singular points if and only if there is $j > 1$ such that $X_{\tilde{B}}^j$ is uncountable.*

(2) *If $X^j_{\tilde{B}}$ contains a double path, then it is uncountable.*

(3) *If $X_{\tilde{B}}$ has finite rank and $X^j_{\tilde{B}}$ is uncountable, then $X^j_{\tilde{B}}$ contains a double path.*

Proof. For $z \in \mathbb{Z}_{(\ell_n)}$ and $n \in \mathbb{N}$, let $B^z_n \subseteq V_n$ be of maximal cardinality among those elements of $\tilde{V}_n$ which are traversed by some path $\tilde{\gamma} \in X_{\tilde{B}}$ with $\omega(\tilde{\gamma}) = z$. Observe that B^z_n is uniquely determined because if $B, B' \in V_n$ are traversed by such a $\tilde{\gamma}$, then $B \cup B'$ is extendable and there is such a $\tilde{\gamma}$ going through $B \cup B'$. This shows that the supremum in the formula of Lemma 2.9 is attained at some path $\tilde{\gamma}$. Since all paths in $X_{\tilde{B}}$ have finite thickness, Lemma 2.9 implies that the restriction of ω to X_B is finite-to-one, and this implies that the restriction of ω to $X^j_{\tilde{B}}$ is finite-to-one. Hence, if $X^j_{\tilde{B}}$ is uncountable, then its image under ω must be uncountable. The converse follows from Lemma 2.9, which tells us that the singular points of $\mathbb{Z}_{(\ell_n)}$ are given by the image of $X_{\tilde{B}} \setminus X^1_{\tilde{B}} = \bigcup_{j \geq 2} X^j_{\tilde{B}}$.

Suppose that $X^j_{\tilde{B}}$ contains a double path. Then there is n_0 such that for all $n \geq n_0$, there is $A_n \subseteq V_n$ containing j elements such that between A_n and A_{n-1}, there are at least 2 edges in the extended Bratteli diagram. The set of paths in $X^j_{\tilde{B}}$ obtained by choosing one of the two edges at each level is uncountable.

Suppose now that $X^j_{\tilde{B}}$ is uncountable. Then the set of condensation points $X^j_{\tilde{B}} \cap \mathcal{C}$ of $X^j_{\tilde{B}}$ is uncountable. Since $X^j_{\tilde{B}} \cap \mathcal{C}$ has no isolated points, for any given path $\gamma \in X^j_{\tilde{B}} \cap \mathcal{C}$ and $n \geq 0$, there are infinitely many distinct paths in $X^j_{\tilde{B}} \cap \mathcal{C}$ which agree with γ on its head of length n. Let K be the rank of the Toeplitz Bratteli–Vershik system and pick some $m_1 \geq 1$. There is $m > m_1$ and $K + 1$ paths of $X^j_{\tilde{B}} \cap \mathcal{C}$ which agree with γ's head of length m_1 but pairwise disagree on the head of length m. Since infinitely often $|V_n| = K$, the pigeon hole principle requires that two of the distinct paths must meet a common vertex of level $m_2 > m$. Telescoping the levels m_1 through m_2, the part of these two paths between level m_1 and level m_2 defines a parallel edge. Iterating this procedure (m_2 playing the role of m_1 etc.) proves the statement. □

The following corollary can be seen as a generalization of Lemma 2.1 to all Toeplitz shifts with finite Toeplitz rank.

COROLLARY 2.12. *Let (X_B, φ_ω) be a Toeplitz Bratteli–Vershik system with finite Toeplitz rank. The maximal equicontinuous factor contains uncountably many singular points if and only if (X_B, φ_ω) is thick.*

Proof. Finite rank implies that all paths in $X_{\tilde{B}}$ are of finite thickness. Now, part (1) of Lemma 2.11 gives that we have uncountably many singular points if and only if the essential thickness is strictly greater than 1. Part (3) of the same lemma gives that if the essential thickness is k, then there is a double path of thickness k. □

Given a choice function $\varphi \in \{0, 1\}^{\mathbb{N}_0}$ and a double path $\bar{\gamma}$, let $\varphi(\bar{\gamma})$ be the (single edge) path $(\gamma_{n,\varphi(n)})$. Such a single edge path defines a sequence of maps $\theta^{(n)}_{\varphi(\bar{\gamma})}$, see equation (2.5). The second part of Lemma 2.11 implies that the essential thickness is an upper bound for the maximal thickness a double path can have. From the next results, we obtain more delicate information: the size of the image $|\theta^{(m)}_{\varphi(\bar\gamma)} \cdots \theta^{(n)}_{\varphi(\bar\gamma)}(V_{n+1})|$ of sufficiently long finite paths is also bounded above by the essential thickness.

In the proof of the next statement, we denote the set of all finite and infinite paths in $\tilde B$ which start at the top vertex v_0 by $Y_{\tilde B}$. Observe that $Y_{\tilde B}$ can naturally be seen as a compact space, where finite paths are seen as infinite paths which eventually pass through a placeholder vertex.

LEMMA 2.13. *Let (X_B, φ_ω) be a Toeplitz Bratteli–Vershik system with essential thickness k. Consider $z^0, z^1 \in \mathbb{Z}_{(\ell_n)}$ with $z^0_n \neq z^1_n$ for each n. Then for all but at most countably many $\varphi \in \{0,1\}^{\mathbb{N}_0}$, we have*

$$\text{for all } m \in \mathbb{N} \quad \text{there exists } n_0 > m \quad \text{for all } n \geq n_0\colon |\theta^{(m)}_{z_m^{\varphi(m)}} \cdots \theta^{(n)}_{z_n^{\varphi(n)}}(V_{n+1})| \leq k.$$

Proof. We only have to consider the case of finite k as the statement is trivially true otherwise. Suppose there are uncountably many φ such that

$$\text{there exists } m \in \mathbb{N} \quad \text{for all } n_0 > m \quad \text{there exists } n \geq n_0\colon |\theta^{(m)}_{z_m^{\varphi(m)}} \cdots \theta^{(n)}_{z_n^{\varphi(n)}}(V_{n+1})| > k.$$

Then for each such φ, there are arbitrarily large n and paths $\gamma^{\varphi;n} \in Y_{\tilde B}$ of length n with $\omega(\gamma^{\varphi;n}_\ell) = z_\ell^{\varphi(\ell)}$ for $\ell = 0, \ldots, n-1$ which traverse a subset of V_m of size bigger than k. Due to the compactness of $Y_{\tilde B}$, there must hence be an infinite path γ^φ (that is, an element of $X_{\tilde B}$) with $\omega(\gamma^\varphi_\ell) = z_\ell^{\varphi(\ell)}$ ($\ell \in \mathbb{N}_0$) which traverses a subset of V_m of size bigger than k. It follows that $X^{>k}_{\tilde B}$ is uncountable, contradicting our assumption that k is the essential thickness. □

Note that if a double path $\bar\gamma$ has thickness $k < \infty$, then there exists m_0 such that also the opposite inequality is true. More precisely, for uncountably many $\varphi \in \{0,1\}^{\mathbb{N}_0}$, we have

$$\text{there exists } m \quad \text{for all } n \geq m\colon |\theta^{(m)}_{\varphi(\bar\gamma)} \cdots \theta^{(n)}_{\varphi(\bar\gamma)}(V_{n+1})| \geq k.$$

Indeed, the contrary, that is, the assumption that for all but at most countably many φ, we have

$$\text{for all } m \quad \text{there exists } n \geq m\colon |\theta^{(m)}_{\varphi(\bar\gamma)} \cdots \theta^{(n)}_{\varphi(\bar\gamma)}(V_{n+1})| < k,$$

implies that all but at most countably $\varphi(\bar\gamma)$ belong to $X^{<k}_{\tilde B}$, which is a contradiction.

We immediately obtain the following corollary.

COROLLARY 2.14. *Let (X_B, φ_ω) be a thick Toeplitz Bratteli–Vershik system with essential thickness k. Possibly, after telescoping, there exists a choice function φ such that $|\theta^{(m)}_{\varphi(\bar\gamma)}(V_{m+1})| = k$ for all large enough m.*

2.6. *Toeplitz systems and non-tameness.* In this section, we prove one of our main results, Theorem 2.17. It applies to all finite rank Toeplitz systems with finite Toeplitz rank, as stated in Theorem 2.18.

As before, we identify subsets $A_n \subseteq V_n$ with vertices $A_n \in \tilde V_n$. For the convenience of the reader, we provide a proof of the next statement which is reminiscent of [**10**, Theorem 13.1].

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LEMMA 2.15. *Consider a Toeplitz Bratteli–Vershik system with characteristic sequence* (ℓ_n) *and let* (A_n) *be some sequence of extendable vertices* $A_n \subseteq V_n$. *Suppose that the set of singular points of* $\mathbb{Z}_{(\ell_n)}$ *has Haar measure* 0.

By telescoping, we can ensure that

$$\lim_{n\to\infty} \frac{|\{i \in [0, \ell_n - 1] : |\theta_i^{(n)}(A_{n+1})| \geq 2\}|}{\ell_n} = 0.$$

Proof. Observe that the Haar probability measure μ of the set $D \subseteq \mathbb{Z}_{\ell_n}$ of singular points is

$$\mu(D) = \lim_{n\to+\infty} \frac{1}{\prod_{k=0}^{n} \ell_k} |\{z_n \cdots z_0 : z \in D\}| = 0.$$

Let $n \geq m \in \mathbb{N}_0$ and $w = w_n \cdots w_m$, with $0 \leq w_i \leq \ell_i - 1$. If there is an extendable $A_{n+1} \subseteq V_{n+1}$ such that $|\theta_{w_m}^{(m)} \cdots \theta_{w_n}^{(n)}(A_{n+1})| \geq 2$, then w is a subword of some singular point $z \in D$, see Lemma 2.9. Hence, given any sequence of extendable vertices $A_n \in \tilde{V}_n$ and any $m \geq 0$, we have

$$\limsup_{n\to+\infty} \frac{1}{\prod_{k=m}^{n} \ell_k} |\{w_n \cdots w_m : |\theta_{w_m}^{(m)} \cdots \theta_{w_n}^{(n)}(A_{n+1})| \geq 2\}|$$

$$\leq \limsup_{n\to+\infty} \frac{1}{\prod_{k=m}^{n} \ell_k} |\{z_n \cdots z_m : z \in D\}| \leq \prod_{k=0}^{m-1} \ell_k \cdot \limsup_{n\to+\infty} \frac{1}{\prod_{k=0}^{n} \ell_k} |\{z_n \cdots z_0 : z \in D\}| = 0.$$

Telescoping from level m to n, the statement follows. □

Recall that for systems with finite topological rank, Lemma 2.11 tells us that essential thickness $k > 1$ is equivalent to the existence of a double path of thickness k. Therefore, the following technical proposition applies to all finite rank Toeplitz systems with uncountably many singular points.

PROPOSITION 2.16. *Let* (X_B, φ_ω) *be a thick Toeplitz Bratteli–Vershik system. Suppose that the set of singular points has Haar measure* 0. *Possibly, after telescoping, there exist:*

(1) *for any* $n \geq 1$, *an arithmetic progression* $j_0^{(n)}, j_1^{(n)}, j_2^{(n)} \in \{0, \ldots, \ell_n - 1\}$ *(that is,* $j_2 - j_1 = j_1 - j_0$*), and sets* $A_n \subseteq V_n$, *such that* $\theta_{j_1^{(n)}}^{(n)}$ *and* $\theta_{j_2^{(n)}}^{(n)}$ *restrict to the same bijection from* A_{n+1} *to* A_n, *while* $B_n := \theta_{j_0^{(n)}}^{(n)}(V_{n+1})$ *is a proper subset of* A_n;

(2) *for any* $n > 1$, *an* $i_n \in \{0, \ldots, \ell_n - 1\}$ *such that* $\theta_{i_n}^{(n)}(V_{n+1})$ *is contained in* $\{a \in A_n : \theta_{j_1^{(n)}}^{(n-1)}(a) \notin B_{n-1}\}$.

Proof. Let $k > 1$ be the essential thickness of the Toeplitz Bratteli–Vershik system and $\bar{\gamma}$ be a double path of thickness k. It is a sequence of parallel edges $\bar{\gamma}_n = (\gamma_{n,1}, \gamma_{n,2})$, that is,

$$s(\gamma_{n,1}) = s(\gamma_{n,2}) = A_n \subseteq V_n, \quad r(\gamma_{n,1}) = r(\gamma_{n,2}) = A_{n+1} \subseteq V_{n+1}$$

such that, for large enough n, $|A_n| = k$, and furthermore, the maps $\theta_{\omega(\gamma_{n,1})}^{(n)}$ and $\theta_{\omega(\gamma_{n,2})}^{(n)}$ each map A_{n+1} bijectively to A_n. Note that, by definition, all A_{n+1} are extendable.

Take the nth parallel edge $(\gamma_{n,1}, \gamma_{n,2})$ of our double path and set

$$\Delta_n(\gamma_{n,1}, \gamma_{n,2}) := \omega(\gamma_{n,2}) - \omega(\gamma_{n,1}).$$

Telescoping with the next level $n+1$ will produce four parallel edges. It is crucial to observe that at least two of these four edges have the same value of Δ_n as the two above. Hence, we can apply Lemma 2.15 to conclude that, possibly after telescoping, there is j in $(\omega(\gamma_{n,1}) + \Delta_n\mathbb{Z}) \cap \{0, \ldots, \ell_n - 1\}$ such that $|\theta_j^{(n)}(A_{n+1})| < k$. We next need to find such a j where moreover $\theta_j^{(n)}(A_{n+1}) \subseteq A_n$.

By Corollary 2.14, we can find for each n large enough, $\kappa_n \in \{0, \ldots, \ell_n - 1\}$ such that $\theta_{\kappa_n}^{(n)}(A_{n+1}) = A_n$ and $|\theta_{\kappa_n}^{(n)}(V_{n+1})| = k$, and hence $\theta_{\kappa_n}^{(n)}(V_{n+1}) = A_n$. Define $\psi_j^{(n)} : \tilde{V}_{n+2} \to \tilde{V}_{n-1}$ through

$$\psi_j^{(n)} := \theta_{\kappa_{n-1}}^{(n-1)} \theta_j^{(n)} \theta_{\kappa_{n+1}}^{(n+1)}$$

and note that $\psi_j^{(n)}(A_{n+2}) \subseteq A_{n-1}$. Moreover, the inclusion is proper if $|\theta_j^{(n)}(A_{n+1})| < k$, while $\psi_j^{(n)}$ is a bijection if $j = \omega(\gamma_{n,1})$ or $j = \omega(\gamma_{n,2})$. Thus, we can find $j_0^{(n)}, j_1^{(n)}, j_2^{(n)} \in (\omega(\gamma_{n,1}) + \Delta_n\mathbb{Z}) \cap \{0, \ldots, \ell_n - 1\}$ such that $j_2^{(n)} - j_1^{(n)} = j_1^{(n)} - j_0^{(n)}$ and $B_{n-1} = \psi_{j_0^{(n)}}^{(n)}(A_{n+2})$ is a proper subset of A_{n-1}, while $\psi_{j_1^{(n)}}^{(n)}(A_{n+2}) = A_{n-1}$ and $\psi_{j_2^{(n)}}^{(n)}(A_{n+2}) = A_{n-1}$. We now telescope the three floors together and thus the $\psi_{j_0^{(n)}}^{(n)}, \psi_{j_1^{(n)}}^{(n)}, \psi_{j_2^{(n)}}^{(n)}$ can be realized as $\theta_{j_0^{(n)}}^{(n)}, \theta_{j_1^{(n)}}^{(n)}$ and $\theta_{j_2^{(n)}}^{(n)}$.

This shows the first statement except for the fact that we only know that $\theta_{j_1^{(n)}}^{(n)}$ and $\theta_{j_2^{(n)}}^{(n)}$ restrict to bijections $f_1^{(n)}$ and $f_2^{(n)}$ from A_{n+1} to A_n, and it remains to show that, possibly after telescoping, $f_1^{(n)} = f_2^{(n)}$. It is convenient to identify the A_{n+1} with A_1. We do this using the bijection $\tau^{(n)} := f_1^{(1)} \cdots f_1^{(n)}$. With

$$I_n := \tau^{(n-1)} f_2^{(n)} \tau^{(n)^{-1}}, \tag{2.6}$$

our aim is thus to show that, possibly after telescoping, all I_n are the identity.

Let further $f_0^{(n)} : A_{n+1} \to A_n$ be the restriction of $\theta_{j_0^{(n)}}^{(n)}$ to A_{n+1}; it is non-surjective with image B_n. If we telescope level n with level $n+1$, we get nine compositions $f_i^{(n)} f_j^{(n+1)}$ for the three possible values of i and j. Let us take a closer look at two sets of choices for them.

Consider first the maps $f_0^{(n)} f_1^{(n+1)}, f_1^{(n)} f_1^{(n+1)}, f_2^{(n)} f_1^{(n+1)}$. These correspond, after telescoping of the two levels, to the restriction to $A^{(n+1)}$ of maps $\theta_{j_0^{(n)}}^{(n)}, \theta_{j_1^{(n)}}^{(n)}$ and $\theta_{j_2^{(n)}}^{(n)}$, with $j_2^{(n)} - j_1^{(n)} = j_1^{(n)} - j_0^{(n)}$, and $f_1^{(n)} f_1^{(n+1)}, f_2^{(n)} f_1^{(n+1)}$ are bijections while $f_0^{(n)} f_1^{(n+1)}$ is not. It is quickly seen that under this choice, the map I_n after telescoping coincides with the map before telescoping. In other words, if we replace $f_2^{(n)}$ with $f_2^{(n)} f_1^{(n+1)}$ in equation (2.6), then

$$\tau^{(n-1)} f_2^{(n)} f_1^{(n+1)} \tau^{(n+1)^{-1}} = I_n.$$

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Now consider the maps $f_0^{(n)} f_0^{(n+1)}$, $f_1^{(n)} f_1^{(n+1)}$, $f_2^{(n)} f_2^{(n+1)}$. Again these correspond, after telescoping of the two levels, to the restriction to A_{n+1} of maps $\theta^{(n)}_{j_0^{(n)}}$, $\theta^{(n)}_{j_1^{(n)}}$ and $\theta^{(n)}_{j_2^{(n)}}$, with $j_2^{(n)} - j_1^{(n)} = j_1^{(n)} - j_0^{(n)}$, and $f_1^{(n)} f_1^{(n+1)}$, $f_2^{(n)} f_2^{(n+1)}$ are bijections while $f_0^{(n)} f_0^{(n+1)}$ is not. The telescoping, however, affects the map I_n. The new map $\tilde{I}_n$ becomes

$$\tilde{I}_n = \tau^{(n-1)} f_2^{(n)} f_2^{(n+1)} {\tau^{(n+1)}}^{-1} = I_n I_{n+1}.$$

As A_1 is finite, the sequence I_n admits a constant subsequence $g^{(n_k)} = g$. Telescoping the levels from n_k to $n_{k+1} - 1$ in the first way described above, we arrive at a situation where all I_n coincide with g. Let N be the order of g. Now, telescoping N consecutive levels together in the second way above, we arrive at a situation where all I_n are equal to the identity. While all this telescoping has an effect on $\theta^{(n)}_{j_0^{(n)}}$, it does not change its crucial property, namely that it maps A_{n+1} to a proper subset of A_n, and that $j_0^{(n)}$, $j_1^{(n)}$, $j_2^{(n)}$ form an arithmetic progression.

It remains to show the second property. Since the order ω is proper, we can assume, by telescoping if necessary, that for each n, $|\theta_0^{(n)}(V_{n+1})| = |\theta^{(n)}_{\ell_n - 1}(V_{n+1})| = 1$. Take $a \in A_n$ such that $\theta^{(n-1)}_{j_1^{(n)}}(a) \notin B_{n-1}$. By minimality (and perhaps further telescoping), we find i_n such that $\theta^{(n)}_{i_n}(V_{n+1}) = \{a\}$. □

THEOREM 2.17. *Every thick Toeplitz shift is non-tame.*

Proof. Let (X_B, φ_ω) be a thick Bratteli–Vershik representation of the given Toeplitz shift which we assume to be conjugate to (X_1, σ), see §2.3.3. We will construct an infinite independence set for two cylinder sets in X_1 which we define in the proof.

Observe that if the set of singular points in $\mathcal{Z}$ has positive Haar measure, then (X_B, φ_ω) is non-tame [**19**, Theorem 1.2]. We hence assume the set of singular points in $\mathcal{Z}$ to have zero Haar measure so that we can apply Proposition 2.16 in the following.

Let $\varphi \in \{0, 1\}^{\mathbb{N}_0}$ be a choice function. We use the notation of the proof of Proposition 2.16 and let h_n be the restriction of $\theta^{(n)}_{i_n}$ to A_{n+1}. Recall that $f_1^{(n-1)} h_n(A_{n+1}) \subseteq A_{n-1} \backslash B_{n-1}$. Set

$$z = \cdots i_{2n+2} j^{(2n+1)}_{\varphi_n} \cdots \cdots i_2 j^{(1)}_{\varphi_0},$$

$t_0 = 0$, $t_1 = (j_1^{(2)} - i_2)\ell_1 + \Delta_1$, and for $n \geq 2$,

$$t_n = t_{n-1} + (j_1^{(2n)} - i_{2n}) \prod_{j=1}^{2n-1} \ell_j + \Delta_{2n-1} \prod_{j=1}^{2n-2} \ell_j.$$

Choose $x \in \omega^{-1}(z)$. By equation (2.4), we have $x_0 \in \theta^{(1)}_{j^{(1)}_{\varphi_0}} \theta^{(2)}_{i_2}(A_3) = f^{(1)}_{\varphi_0} h_2(A_3)$. If $\varphi_0 = 0$, then

$$f_0^{(1)} h_2(A_3) \subseteq \operatorname{im} f_0^{(1)} = B_1,$$

whereas if $\varphi_0 = 1$, then

$$f_1^{(1)} h_2(A_3) \subseteq A_1 \backslash B_1.$$

Furthermore,

$$z + t_1 = \cdots i_4 j_{\varphi_1}^{(3)} j_1^{(2)} j_{\varphi_0+1}^{(1)}.$$

Hence, taking into account that $f_1^{(n)} = f_2^{(n)}$, we have

$$x_{t_1} \in \theta^{(1)}_{j^{(1)}_{\varphi_0+1}} \theta^{(2)}_{j_1^{(2)}} \theta^{(3)}_{j^{(3)}_{\varphi_1}} \theta^{(4)}_{i_4}(A_5) = f_1^{(1)} f_1^{(2)} f_{\varphi_1}^{(3)} h_4(A_5).$$

Since

$$f_{\varphi_1}^{(3)} h_4(A_5) = \begin{cases} f_0^{(3)} h_4(A_5) \subseteq B_3 & \text{if } \varphi_1 = 0, \\ f_1^{(3)} h_4(A_5) \subseteq A_3 \backslash B_3 & \text{if } \varphi_1 = 1, \end{cases}$$

it follows that

$$\begin{array}{ll} x_{t_1} \in \tau^{(2)}(B_3) & \text{if } \varphi_1 = 0, \\ x_{t_1} \in A_1 \backslash \tau^{(2)}(B_3) & \text{if } \varphi_1 = 1. \end{array}$$

Similarly, we find for all $n \geq 2$,

$$\begin{array}{ll} x_{t_n} \in \tau^{(2n)}(B_{2n+1}) & \text{if } \varphi_n = 0, \\ x_{t_n} \in A_1 \backslash \tau^{(2n)}(B_{2n+1}) & \text{if } \varphi_n = 1. \end{array}$$

By finiteness of A_1, there is a subsequence of B_{2n+1} such that $\tau^{2n}(B^{2n+1})$ is constant, say equal to B. Restricting to choice functions with support in this subsequence, we obtain an independence sequence for the cylinder set $[B]$ and its complement in $[A_1]$. □

THEOREM 2.18. *Let (X, σ) be a Toeplitz shift of finite Toeplitz rank. Then (X, σ) is non-tame if and only if its maximal equicontinuous factor has uncountably many singular points.*

Proof. Let (X_B, φ_ω) be a finite rank Toeplitz Bratteli–Vershik representation of (X, σ). If the maximal equicontinuous factor of (X_B, φ_ω) has uncountably many singular fibres then, by Corollary 2.12, it is thick. By Theorem 2.17, (X_B, φ_ω) and hence (X, σ) is non-tame.

Conversely, suppose that (X, σ) has countably many singular fibres. By Lemma 2.9, all fibres are finite. The result follows from [**22**, Lemma 3.2]—see also Corollary 4.4 below. □

Remark 2.19. In fact, the same techniques and a little more care will give us a slightly stronger version of Theorems 2.17 and 2.18. Namely, the statements also hold true if one replaces the word non-tame by either forward non-tame or backward non-tame. What one has to ensure in these situations is that the sequence of times $(t_n)_{n\geq 0}$ is either always positive or always negative. The point in the proof where one has to be more careful is in the statement and proof of Proposition 2.16: in the case of forward tameness, for example, one would require that $i_0 < j_0 < j_1 < j_2$, so that each t_n as defined in the proof of Theorem 2.17 is strictly positive. To achieve this requires simply further telescoping.

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For non-tame constant length substitution shifts with a coincidence, an independence set can be explicitly computed, as the following example illustrates.

Example A. (Continued) We illustrate Proposition 2.16 and Theorem 2.17 with our the substitution in Example A. The graph of Figure 2 has no parallel edges of thickness 2, but if we telescope two levels together, we get a pair of parallel edges between the vertex $\{a, b\}$ above and the vertex $\{a, b\}$ below. For substitutions, telescoping amounts to taking powers of the substitution. We therefore need to work (at least) with the second power θ^2 of the substitution:

$$\begin{array}{ccc} a & & aaca\ aaca\ aaba\ aaca \\ b & \mapsto & aaca\ abba\ abba\ aaca \\ c & & aaca\ aaca\ abba\ aaca \end{array}$$

We see that the restrictions of ${\theta^2}_5$ and ${\theta^2}_9$ to $\{a, b\}$ are both equal to the identity. Furthermore, ${\theta^2}_1$ is the projection onto letter a. (In the coloured version, the images of ${\theta^2}_1$, ${\theta^2}_5$ and ${\theta^2}_9$ are marked blue, and that of ${\theta^2}_{10}$ is in red.) We may therefore chose $\theta^{(1)} = \theta^2$, $A_2 = \{a, b\} = A_1$ and the arithmetic progression $j_0 = 1,\ j_1 = 5,\ j_2 = 9$. Since here the extended Bratteli diagram is stationary, all $\theta^{(n)}$ and A_n can be taken to be equal. As ${\theta^2}_{10}$ is the projection onto the letter b, we may take $i_1 = 10$. Therefore, the independence sequence from the proof of Theorem 2.17 is given by $t_0 = 0$, and $t_n = t_{n-1} - 5 \cdot 16^{2n-1} + 4 \cdot 16^{2n-2}$ for $n \geq 1$.

Example D. As is the case with many properties, tameness is not preserved under strong orbit equivalence (see [**23**] for a definition). Take the two substitutions

$$\begin{array}{ccc} a & \mapsto & aabaa \\ b & \mapsto & abbaa \end{array}$$

and

$$\begin{array}{ccc} a & \mapsto & aaaba \\ b & \mapsto & abbaa. \end{array}$$

Then both substitution shifts are conjugate to the stationary Bratteli–Vershik system they define, see §2.3.4. Furthermore, since the underlying unordered Bratteli diagrams are identical, these two shifts are strong orbit equivalent. However, by computing the graphs $\mathcal{G}_\theta$, we see that the first has one orbit of singular fibres, whereas the second has uncountably many. In the language of this section, the second has essential thickness two. Hence, by Theorem 2.18, the second shift is non-tame. As the first Toeplitz system has one singular fibre, it is tame by Theorem 2.18.

3. *Almost automorphy and semicocycle extensions*

Any almost automorphic system can be realized as a *semicocycle extension* of its maximal equicontinuous factor. Conversely, semicocycle extensions are a useful tool to construct almost automorphic systems. Below we will use this tool to construct interesting examples of Toeplitz shifts which underline the necessity of the conditions formulated in Theorem 2.18.

In this section, we recall the relevant details from the literature, essentially following the discussion in [22, §2] where actions of general discrete groups are considered; the interested reader may also consult [10, 11]. Here, we simplify and restrict to $\mathbb{Z}$-actions. We mention that for the main purposes of this article, where we are concerned with almost automorphic *symbolic* extensions, the concept of *separating covers*, as used in [36, 38], could also be employed. Nevertheless, we believe that the semicocycle approach is not only more flexible but also more transparent.

Recall that $\mathcal{Z}$ is a compact metrizable monothetic group with topological generator g so that $(\mathcal{Z}, +g)$ is a minimal rotation. Given $\hat{z} \in \mathcal{Z}$ and a compact metrizable space K, we call a map $f : \hat{z} + \mathbb{Z}g \to K$ a (K-valued) *semicocycle* over the pointed dynamical system $(\mathcal{Z}, +g, \hat{z})$ if it is continuous in the subspace topology on $\hat{z} + \mathbb{Z}g \subseteq \mathcal{Z}$.

Given a semicocycle $f : \hat{z} + \mathbb{Z}g \to K$, we consider the closure of its graph,

$$F := \overline{\{(z, k) \in \mathcal{Z} \times K : z \in \hat{z} + \mathbb{Z}g,\ k = f(z)\}} \subseteq \mathcal{Z} \times K.$$

We use the letter F also to denote the map

$$F : \mathcal{Z} \to 2^K, \quad F(z) = \{k \in K : (z, k) \in F\} \tag{3.1}$$

which we refer to as the *section* function. Here, 2^K denotes the set of subsets of K equipped with the topology induced by the Hausdorff metric. Since $(\mathcal{Z}, +g)$ is minimal and K is compact, $F(z)$ is non-empty for every $z \in \mathcal{Z}$.

As we do not assume that f is uniformly continuous, its graph closure F is not necessarily the graph of a function, that is, $F(z)$ may contain more than one point. In this case, we call a point $z \in \mathcal{Z}$ a *discontinuity point* of f. We collect the discontinuities of f in the set

$$D_f = \{z \in \mathcal{Z} : |F(z)| > 1\} \subseteq \mathcal{Z}.$$

By definition, we have $(\hat{z} + \mathbb{Z}g) \cap D_f = \emptyset$ and f admits a continuous extension to the complement of D_f, which we also denote by f,

$$f : \mathcal{Z} \setminus D_f \to K, \quad f(z) = k_z,$$

where k_z is the unique point in $F(z)$. Note that if we take any other point $\hat{z} \in \mathcal{Z}$ whose orbit does not contain a discontinuity point and define $f' : \hat{z} + \mathbb{Z}g \to K$ with the above extension through $f'(\hat{z} + ng) = f(\hat{z} + ng)$, then we get the same graph closure F and the same set of discontinuities, $D_{f'} = D_f$.

LEMMA 3.1. *Let f be a K-valued semicocycle for a minimal rotation $(\mathcal{Z}, +g)$ and let F be the associated section function as in equation (3.1). Then F is continuous on D_f^c. In particular, given $w \in D_f^c$, for any neighbourhood $W \subseteq K$ of $f(w)$, there exists a neighbourhood $U \subseteq \mathcal{Z}$ of 0 such that for all $z' \in U + w$, we have $F(z') \subseteq W$.*

Proof. Choose a compatible metric d on K and denote by d_H the associated Hausdorff metric on 2^K. Let $w \in \mathcal{D}_f^c$ so that w is a point of continuity of f, that is, $F(w) = \{f(w)\}$. Then for any $\varepsilon > 0$, there is a neighbourhood $U \subseteq \mathcal{Z}$ of 0 such that for all $z' \in (U + w) \cap D_f^c$, we have $d(f(z'), f(w)) \leq \varepsilon$. This implies $d_H(F(z'), F(w)) \leq \varepsilon$ first for all

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$z' \in (U+w) \cap D_f^c$, but then also for all $z' \in U+w$, as $F(z') \subseteq \overline{\bigcup_{\tilde{z} \in (U+w)\cap D_f^c} F(\tilde{z})}$ by definition of F as the graph closure. The statement follows. $\square$

Recall that our shifts can be defined over a compact and not necessarily finite set; see §1.1.

Definition 3.2. Let f be a K-valued semicocycle over $(\mathcal{Z}, +g, \hat{z})$, and let X_f be the shift-orbit closure of the sequence $\hat{f} = (\hat{f}_n) \in K^{\mathbb{Z}}$,

$$\hat{f}_n := f(\hat{z} + ng).$$

We call (X_f, σ) the *shift associated to f*.

Any element of X_f is hence a limit of a sequence of translates of $\hat{f}$. Put differently, for each $x \in X_f$, there is $(n_k) \in \mathbb{Z}^{\mathbb{N}}$ such that for every $m \in \mathbb{Z}$, we have $x_m = \lim_{k\to\infty} f(\hat{z} + (n_k + m)g)$. The group $\mathcal{Z}$ acts on $\mathcal{Z} \times K$ by left translation in its first factor. We say that a semicocycle f over $(\mathcal{Z}, +g, \hat{z})$ is *separating* (note the slight terminological deviation from [10, 11, 22] where the phrase *invariant under no rotation* is used instead of the term *separating* (which we take from [36, 38])) if the stabiliser in $\mathcal{Z}$ of its graph F is trivial, that is, if $z \in \mathcal{Z}$ satisfies $F(z + y) = F(y)$ for all y, then $z = 0$.

THEOREM 3.3. [11, Theorem 5.2] *The shift associated to a separating semicocycle over $(\mathcal{Z}, +g, \hat{z})$ is an almost automorphic extension of $(\mathcal{Z}, +g)$. The equicontinuous factor map $\pi : (X_f, \sigma) \to (\mathcal{Z}, +g)$ satisfies*

$$x_n \in F(\pi(x) + ng) \quad \textit{for each } (x_n)_{n\in\mathbb{Z}} \in X_f \textit{ and } n \in \mathbb{Z}.$$

Accordingly, we call (X_f, σ) a *semicocycle extension of* $(\mathcal{Z}, +g)$ defined by *f*. Recall that the set of singular points in $\mathcal{Z}$ of the semicocycle extension is the set of $z \in \mathcal{Z}$ such that $|\pi^{-1}(z)| > 1$. As any two distinct sequences must differ on at least one index, we see that $\pi^{-1}(z)$ contains more than one point if and only if a translate of z is a discontinuity point for the semicocycle. The singular points of the semicocycle extension are therefore given by $D_f + \mathbb{Z}g$.

The following theorem says that any almost automorphic system is conjugate to a shift over a compact alphabet, notably, one obtained by a semicocycle.

THEOREM 3.4. (Cf. [10, Theorem 6.4], [11, Theorem 5.2], [22, Theorem 2.5]) *Consider a topological dynamical system (X, T). The following statements are equivalent.*

(i) *(X, T) is almost automorphic.*

(ii) *The maximal equicontinuous factor of (X, T) is a minimal rotation and (X, T) is conjugate to a semicocycle extension of this minimal rotation.*

(iii) *(X, T) is conjugate to an almost automorphic shift.*

From the above, the implications (ii) ⇒ (iii) ⇒ (i) are clear. Let us briefly comment on (i) ⇒ (ii) by describing how to obtain a realization of (X, T) as a semicocycle extension. The maximal equicontinuous factor of a minimal $\mathbb{Z}$-action is a minimal rotation which we

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denote here as $(\mathcal{Z}, +g)$. Given a regular point $\hat{z} \in \mathcal{Z}$ with its unique pre-image $\hat{x}$ under π, the function $f : \hat{z} + \mathbb{Z}g \to X$,

$$f(\hat{z} + ng) := T^n(\hat{x})$$

is continuous, hence a semicocyle over $(\mathcal{Z}, +g, \hat{z})$. The conjugacy between (X, T) and (X_f, σ) comes about as any element $x \in X$ defines a sequence in $(s_n) \in X^{\mathbb{Z}}$ through $s_n = T^n(x)$ and the sequence $\hat{f}$ defined by f is the image of $\hat{x}$ under this map.

The set of discontinuity points D_f coincides with the set of singular points of $\mathcal{Z}$. As the set of discontinuity points must be contained in the set of singular points, and here they coincide, we call the above semicocycle *maximal*.

If we are given an almost automorphic shift, then we may realize it as a semicocycle extension also in a different way.

Definition 3.5. Let $(X, \sigma) \subseteq (K^{\mathbb{Z}}, \sigma)$ be an almost automorphic shift with maximal equicontinuous factor map $\pi : (X, \sigma) \to (\mathcal{Z}, +g)$. We call

$$D = \{z \in \mathcal{Z} : \text{there exists } x, y \in \pi^{-1}(z) \text{ with } x_0 \neq y_0\}$$

the set of *discontinuity points* of the shift and the continuous map $f^{\mathrm{can}} : \mathcal{Z} \setminus D \to K$,

$$f^{\mathrm{can}}(z) = x_0, x \in \pi^{-1}(z)$$

the *canonical semicocycle* of the shift.

Indeed, if we choose a regular point $\hat{z} \in \mathcal{Z}$ with its unique pre-image $\hat{x}$ under π, we see that f^{can} is the continuous extension of $\hat{z} + ng \to \hat{x}_n$ to $\mathcal{Z} \setminus D$ and hence a semicocycle over $(\mathcal{Z}, +g, \hat{z})$. Thus, D is the set of discontinuity points of the canonical semicocycle.

The evaluation map $ev_0 : X \to K$, $ev_0(x) = x_0$, relates f^{can} to the maximal semicocycle f through $ev_0 \circ f(\hat{z} + ng) = \hat{x}_n$. The pointwise extension of ev_0 to $X^{\mathbb{Z}} \to K^{\mathbb{Z}}$ restricts to a conjugacy between $X_f \subseteq X^{\mathbb{Z}}$ and $X \subseteq K^{\mathbb{Z}}$. As ev_0 is continuous, the set of discontinuity points D of the shift is contained in D_f, but it is often substantially smaller. In fact, the set D is associated to the concrete realization of the shift and is not invariant under conjugacy, as can be seen in the following example.

Example E. Let $\mathcal{Z} = S^1 = \mathbb{R}/2\pi\mathbb{Z}$, the circle, and $g \in S^1$ be an irrational angle (incommensurate with 2π). The dynamical system $(S^1, +g)$ is a minimal rotation. Let $\alpha \in \mathbb{Z}g$, $\alpha \neq 0$, and $\beta \in S^1 \backslash \mathbb{Z}g$. The characteristic function $\chi_{[0,\alpha)}$ on the half-open interval $[0, \alpha)$ is continuous on the $\mathbb{Z}$-orbit of β and thus $f = \chi_{[0,\alpha)}$ defines a semicocycle over $(S^1, +g, \beta)$ which can easily be seen to be separating. Clearly, f is discontinuous at exactly two points, namely $D_f = \{0, \alpha\}$. Indeed, the sections $F(0)$ and $F(\alpha)$ contain two points, $\{0, 1\}$, whereas all other sections contain only one point. In particular, different choices for $\alpha \in \mathbb{Z}g \backslash \{0\}$ lead to different semicocycle extensions, that is, different symbolic dynamical systems which, in particular, have different sets of discontinuities $D = \{0, \alpha\}$. Note, however, that all these semicocycle extensions are topologically conjugate dynamical systems, as one can see using their description as rotations on the Cantor set obtained by disconnecting the circle S^1 along the $\mathbb{Z}$-orbit of g [18].

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We mention as an aside that the above semicocycle occurs frequently in the description of the physics of quasicrystalline condensed matter. In particular, the function $V(n) = \chi_{[0,\alpha)}(\beta + ng)$ serves as the potential in the so-called Kohmoto model which describes the motion of a particle in a one-dimensional quasicrystal which can be obtained from the above data β and α by means of the cut and project method [**35**].

In the context of constant length substitutions, we have the following description of the discontinuity points.

LEMMA 3.6. *Let θ be a primitive aperiodic substitution of constant length which has a coincidence and trivial height. The set of discontinuities D of (X_θ, σ) coincides with the set $\hat{\Sigma}_\theta$ from §2.2.*

Proof. Here, D^c is the set of points $z \in \mathbb{Z}_\ell$ for which all points in the fibre $\pi^{-1}(z)$ have the same entry at the 0th index. This happens if and only if one block of z entries, say $z_n \cdots z_k$, is such that $|\theta_{z_k} \cdots \theta_{z_n}(\mathcal{A})| = 1$. The result follows. $\square$

The corresponding result for more general Toeplitz shifts reads as follows. We leave its simple proof to the reader. Here we again assume that the given Toeplitz shift is conjugate, via p_1 as in equation (2.3), to the top level shift (X_1, σ) of the Bratteli–Vershik representation.

PROPOSITION 3.7. *Consider a Toeplitz shift with maximal equicontinuous factor $(\mathbb{Z}_{\ell_n}, +1)$ and Toeplitz Bratteli–Vershik representation (X_B, φ_ω). Then $z \in \mathbb{Z}_{\ell_n}$ is a discontinuity point if and only if $z = \omega(\gamma)$ for some path $\gamma = (\gamma_n)$ of the extended Bratteli diagram for which $r(\gamma_n)$ has at least 2 elements, for all $n \geq 0$.*

4. *Criteria for tameness of almost automorphic shifts*

In this part, we derive some general criteria for tameness and non-tameness of minimal dynamical systems. Recall that due to [**25**, **30**], every tame minimal system is necessarily almost automorphic. Accordingly, we restrict to the study of almost automorphic shifts $(X, \sigma) \subseteq (K^{\mathbb{Z}}, \sigma)$ (or $(K^{\mathbb{N}}, \sigma)$) in this section, see also Theorem 3.3. We denote their maximal equicontinuous factors throughout by $(\mathcal{Z}, +g)$ and the corresponding factor map by π.

To study tameness of the shift (X, σ), we need to establish some terminology. Recall from Proposition 1.5 that non-tameness manifests itself through the existence of a pair V_a, V_b of (non-empty) disjoint compact subsets of K, together with an *independence sequence* (t_n) for V_a, V_b, that is, a sequence $(t_n)_{n\in\mathbb{N}}$ in $\mathbb{Z}$ (or $\mathbb{N}$ if we regard forward tameness) such that for any *choice function* $\varphi \in \{a, b\}^{\mathbb{N}}$, we can find $x^\varphi \in X$ with $x^\varphi_{t_n} \in V_{\varphi(n)}$ for all $n \in \mathbb{N}$. Given a choice function φ, a sequence (t_n) and a pair $V_a, V_b \subseteq K$, we say that x^φ *realizes* φ along (t_n) for V_a, V_b if $x^\varphi_{t_n} \in V_{\varphi(n)}$ for all $n \in \mathbb{N}$. Given a pair $V_a, V_b \subseteq K$ and a sequence $(t_n)_{n\in\mathbb{N}}$ with $t_n g \to z$, we denote, for $E \subseteq \mathcal{Z}$,

$$\Sigma^{V_a,V_b}_{(t_n)}(E) = \{\varphi \in \{a, b\}^{\mathbb{N}} : \text{there exists } x^\varphi \in X \text{ for all } n \in \mathbb{N} : x^\varphi_{t_n} \in V_{\varphi(n)}, \pi(x^\varphi) \in E - z\},$$

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the set of all choice functions which are realized along (t_n) by points $x^\varphi \in X$ for which $\pi(x^\varphi) \in E - z$. With this notation, $(t_n)_{n\in\mathbb{N}}$ is an independence sequence for the pair (V_a, V_b) if $\Sigma^{V_a,V_b}_{(t_n)}(\mathcal{Z}) = \{a, b\}^{\mathbb{N}}$. To simplify the notation, we also write $\Sigma_{(t_n)}(E)$ for $\Sigma^{V_a,V_b}_{(t_n)}(E)$ if the dependence on the pair (V_a, V_b) is either clear from the context or one can take any pair of disjoint compact subsets of K. For instance, if we have a binary shift, that is, $K = \{a, b\}$, we only need to consider $V_a = \{a\}$ and $V_b = \{b\}$, cf. Remark 1.6.

In the next set of results, D denotes the set of discontinuities as defined in Definition (3.5). We begin with a proposition which is implicit in the proof of [22, Lemma 3.2].

PROPOSITION 4.1. *Let $(X, \sigma) \subseteq (K^{\mathbb{Z}}, \sigma)$ be an almost automorphic shift and let $V_a, V_b \subseteq K$ be closed non-empty disjoint subsets. Let $(t_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{Z}$ with $t_n g \to z$. The set of choice functions which are realized by points $x \in X$ along (t_n) with $\pi(x) \notin D - z$ is at most countable, that is, $\Sigma^{V_a,V_b}_{(t_n)}(D^c)$ is at most countable.*

Proof. Let f be the canonical semicocycle and let F be its associated section function. Suppose x realizes the choice function φ. All limit points of $(\sigma^{t_n}(x))$ belong to the fibre of $\pi(x) + z$ which we assume to be a point of continuity of f. Since $V_a \cap V_b = \emptyset$, there exists $i \in \{a, b\}$ such that $f(\pi(x) + z) \in V_i^c$. As V_i^c is open, we can apply Lemma 3.1 to $W = V_i^c$ to guarantee that there exists a neighbourhood $U \subseteq \mathcal{Z}$ of 0 such that $F(w + \pi(x) + z) \subseteq V_i^c$ for all $w \in U$. Therefore, if n is large enough such that $t_n g - z \in U$, we have $x_{t_n} = (\sigma^{t_n}(x))_0 \in F(\pi(x) + t_n) \subseteq V_i^c$. Hence, for all large n, $x_{t_n} \notin V_i$. This means that for all large n, $\varphi(n)$ is constant. The set of eventually constant choice functions is countable. □

As a matter of fact, we can improve Proposition 4.1. To that end, we first observe the following lemma.

LEMMA 4.2. *Let $(X, \sigma) \subseteq (K^{\mathbb{Z}}, \sigma)$ be an almost automorphic shift and let $V_a, V_b \subseteq K$ be closed non-empty disjoint subsets. Let $(t_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{Z}$ with $t_n g \to z$. Let $w \in \mathcal{Z}$. If $\{t_n g + w : n \in \mathbb{N}\} \cap D$ is finite, then $\Sigma^{V_a,V_b}_{(t_n)}(\{z + w\})$ is finite.*

Proof. By definition, $\Sigma^{V_a,V_b}_{(t_n)}(\{z + w\})$ is the set of choice functions which are realized along (t_n) by points in the fibre $\pi^{-1}(w)$. If $t_n g + w \in D^c$, then all points in $\pi^{-1}(w)$ have the same value at t_n. Since only finitely many $w + t_n g$ are discontinuity points, we can thus realize only finitely many choices through points of $\pi^{-1}(w)$. □

Recall that the *derived set* of a subset of a metrizable space is the set of all its accumulation points. We have the following strengthening of Proposition 4.1 which is particularly useful when D' is just a singleton, see §6.

PROPOSITION 4.3. *Let $(X, \sigma) \subseteq (K^{\mathbb{Z}}, \sigma)$ be an almost automorphic shift and let $V_a, V_b \subseteq K$ be closed non-empty disjoint subsets. Let $(t_n)_{n\in\mathbb{N}} \subseteq \mathbb{Z}$ be a sequence with $t_n g \to z$ and let D' denote the derived set of D. Then $\Sigma^{V_a,V_b}_{(t_n)}(D'^c)$ is at most countable.*

Proof. The answer to the question of whether $\Sigma^{V_a,V_b}_{(t_n)}(D'^c)$ is at most countable or not does not change if we discard multiple occurrences of elements of $(t_n)_{n\in\mathbb{N}}$. By possibly

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going over to a subsequence, we may therefore assume without loss of generality that the elements of $(t_n)_{n\in\mathbb{N}}$ are pairwise distinct. Let $w \in \mathcal{Z}$ be such that $z + w \in D \setminus D'$, so that $z + w$ is an isolated point of D. Hence, $t_n g + w \notin D$ for sufficiently large n, that is, $\{t_n g + w : n \in \mathbb{N}\} \cap D$ is finite. Moreover, since X is compact, the set $D \setminus D'$ is at most countable. By Lemma 4.2, $\Sigma_{(t_n)}^{V_a,V_b}(D \setminus D')$ is at most countable. The result follows with Proposition 4.1. □

In a similar way, we obtain the following useful criterion for forward tameness. For tameness of the full $\mathbb{Z}$-action, it has to be tested for the forward and the backward motion.

COROLLARY 4.4. [22, Lemma 3.2] *Let (X, σ) be an almost automorphic shift with maximal equicontinuous factor $(\mathcal{Z}, +g)$. Suppose that the set discontinuity points $D \subseteq \mathcal{Z}$ is countable. If $z + \mathbb{N}g \cap D$ is finite for each $z \in \mathcal{Z}$, then (X, σ) is forward tame.*

PROPOSITION 4.5. *Let $(X, \sigma) \subseteq (K^{\mathbb{Z}}, \sigma)$ be an almost automorphic shift and $V_a, V_b \subseteq K$ be closed non-empty disjoint subsets. Let $(t_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{Z}$ with $t_n g \to z$. Given $E \subseteq \mathcal{Z}$, we have $\overline{\Sigma_{(t_n)}^{V_a,V_b}(E)} \subseteq \Sigma_{(t_n)}^{V_a,V_b}(\overline{E})$.*

Proof. Let $\varphi \in \{a, b\}^{\mathbb{N}}$ be the limit of the sequence $(\varphi^{(m)})_m \subseteq \Sigma_{(t_n)}^{V_a,V_b}(E)$. For each m, choose $x^{(m)} \in E - z$ which realizes $\varphi^{(m)}$. Let x be a limit of some subsequence of $(x^{(m)})_m$. We claim that x realizes φ. Indeed, given n, there exists m_n such that $\varphi^{(m)}(n) = \varphi(n)$ for $m \geq m_n$ and hence $x_{t_n}^{(m)} \in V_{\varphi(n)}$ for $m \geq m_n$. Since the convergence of $x^{(m)}$ to x is pointwise and V_a and V_b are compact, this implies that $x_{t_n} \in V_{\varphi(n)}$ for each $n \in \mathbb{N}$. □

The above results lead to the following necessary condition for non-tameness.

COROLLARY 4.6. *Let $(X, \sigma) \subseteq (K^{\mathbb{Z}}, \sigma)$ be an almost automorphic shift and V_a, $V_b \subseteq K$ be closed non-empty disjoint subsets. If $(t_n)_{n\in\mathbb{N}}$ is an independence sequence for (V_a, V_b) with $t_n g \to z$, then $\Sigma_{(t_n)}^{V_a,V_b}(D) = \Sigma_{(t_n)}^{V_a,V_b}(D') = \{a, b\}^{\mathbb{N}}$.*

Proof. As $(t_n)_n$ is an independence sequence, its elements are pairwise distinct and so we may apply Propositions 4.1 and 4.3 to see that $\Sigma_{(t_n)}(D)$ and $\Sigma_{(t_n)}(D')$ are dense in $\{a, b\}^{\mathbb{N}}$. Since D and D' are closed, the result follows from Proposition 4.5. □

LEMMA 4.7. *Let $(X, \sigma) \subseteq (K^{\mathbb{Z}}, \sigma)$ be an almost automorphic shift and V_a, $V_b \subseteq K$ be closed non-empty disjoint subsets. Let $\tilde{V}_a$, $\tilde{V}_b$ be closed disjoint subsets of K such that $V_i \subseteq \operatorname{int}\tilde{V}_i$. Let $(t_n)_{n\in\mathbb{N}} \subseteq \mathbb{Z}$ be such that $t_n g \to z$. Let $E \subseteq \mathcal{Z}$ be compact with $(E - z + t_n g) \subseteq D^c$ for all n. Then there is a sequence $(\tau_n)_{n\in\mathbb{N}} \subseteq \mathbb{Z}$ with $\tau_n g \to 0$ and $\Sigma_{(\tau_n)}^{\tilde{V}_a,\tilde{V}_b}(E) \supseteq \Sigma_{(t_n)}^{V_a,V_b}(E)$. Moreover, $(\tau_n)_{n\in\mathbb{N}}$ can be chosen strictly negative and it can also be chosen strictly positive.*

Proof. Let $f : D^c \to K$ be the canonical semicocycle and $F : \mathcal{Z} \to 2^K$ its associated section function. We apply Lemma 3.1 to $\operatorname{int}\tilde{V}_i$ to obtain that, given $w \in D^c$ such that $f(w) \in \operatorname{int}\tilde{V}_i$, there exists a neighbourhood U of $0 \in \mathcal{Z}$ such that for all $z' \in U + w$, we have $F(z') \subseteq \operatorname{int}\tilde{V}_i$. Let $A \subseteq D^c$ be a compact subset. As f is continuous on D^c, also $A_i := f^{-1}(V_i) \cap A$ is compact for $i = a, b$. This implies that we can choose the neighbourhood

U of $0 \in \mathcal{Z}$ uniformly for all $w \in A$ so that, for all $z' \in U + A_i$ and $i = a, b$, we have $F(z') \subseteq \mathrm{int}\tilde{V}_i$.

Applying the above to $A = E - z + t_n g$, with $n \in \mathbb{N}$, we conclude that there exists a neighbourhood U_n of $0 \in \mathcal{Z}$ such that

$$F(z') \subseteq \tilde{V}_i \quad \text{for all } z' \in U_n + ((E - z + t_n g) \cap f^{-1}(V_i)). \tag{4.1}$$

As the dynamics on $\mathcal{Z}$ is forward minimal, there exist $t'_n \in \mathbb{Z}^+$ such that $t'_n g \in -U_n + z$. Define $\tau_n = t_n - t'_n$. Without loss of generality, we may assume that $\limsup_n U_n = \{0\}$ so that $t'_n g \to z$. Hence, $\tau_n g$ tends to $0 \in \mathcal{Z}$ as $n \to +\infty$.

Let $\varphi \in \Sigma^{V_a,V_b}_{(t_n)}(E)$. There is $x \in \pi^{-1}(E - z)$ realising φ along (t_n). Let $w = \pi(x) + z$. As $f(w - z + t_n g) = x_{t_n} \in V_{\varphi(n)}$, we have

$$F(z') \subseteq \tilde{V}_{\varphi(n)} \quad \text{for all } z' \in U_n + w - z + t_n g \tag{4.2}$$

with U_n as in equation (4.1).

Let $y \in \pi^{-1}(w)$. Then $y_{\tau_n} = (\sigma^{\tau_n}(y))_0 \in F(w + \tau_n g)$. We have

$$w + \tau_n g = w - t'_n g + t_n g \in U_n + w - z + t_n g$$

which implies—by equation (4.2)—that $y_{\tau_n} \in F(w + \tau_n g) \subseteq \tilde{V}_{\varphi(n)}$. We thus have shown that for all n, $y_{\tau_n} \in \tilde{V}_{\varphi(n)}$. This means that φ is realized by y along (τ_n) for $\tilde{V}_a$, $\tilde{V}_b$, or, in other words, $\varphi \in \Sigma^{\tilde{V}_a,\tilde{V}_b}_{(\tau_n)}(E)$.

For the last part, note that we can choose t'_n to rise fast enough such that $t_n - t'_n \to -\infty$. Or we can use backward minimality and choose $t'_n \in \mathbb{Z}^-$ to fall fast enough so that $t_n - t'_n \to +\infty$. □

For later reference, we apply the above results to binary shifts, that is, $K = \{a, b\}$. Note that in this case, we are bound to choose $V_a = \tilde{V}_a = \{a\}$ and $V_b = \tilde{V}_b = \{b\}$ in Lemma 4.7.

THEOREM 4.8. *Let $(X, \sigma) \subseteq (\{a, b\}^{\mathbb{Z}}, \sigma)$ be a non-tame almost automorphic shift with maximal equicontinuous factor $(\mathcal{Z}, +g)$.*

(1) *There is an independence set (t_n) for the pair $\{a\}, \{b\}$ with $t_n g \to z$ such that all choice functions are realized along (t_n) by elements of $D' - z$.*
(2) *If for each $z \in \mathcal{Z}$, there are only finitely many $t \in \mathbb{Z}$ such that $(D - z + tg) \cap D \neq \emptyset$, then there exists an independence set (t_n) with $t_n g \to 0$ such that all choice functions are realized along (t_n) by elements of D. Moreover, (t_n) can be chosen strictly negative and it can also be chosen strictly positive.*

Proof. The first statement is Proposition 4.3 applied to the only possible pair of non-empty disjoint subsets of $\{a, b\}$. As for the second, let $(t_n) \subseteq \mathbb{N}$ be an independence set such that $t_n g \to z$. By Corollary 4.6, all choice functions are realized along (t_n) by points of $\pi^{-1}(D - z)$. Taking a subsequence of $(t_n) \subseteq \mathbb{N}$, we can suppose that $(D - z + t_n g) \cap D \neq \emptyset$ for all n. As K is finite, all its subsets are clopen and so we may apply Lemma 4.7 to find an independence set (τ_n) with $\tau_n g \to 0$ such that all choice functions are realized along (τ_n) by elements of D. We may choose (τ_n) strictly negative or strictly positive. □

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5. *A tame Toeplitz shift with uncountably many singular fibres*

In this section, we construct an almost automorphic tame extension of the odometer $\mathcal{Z} := \mathbb{Z}_{(4^n)}$ by means of a semicocycle whose discontinuity set D is uncountable. Note that Theorem 2.18 implies that the Toeplitz shift we construct here does not have finite Toeplitz rank.

5.1. *Set of discontinuities D and its properties.* Given $z \in \mathcal{Z} = \mathbb{Z}_{(4^n)}$, we denote by $\mathrm{head}_i(z) = z_i \cdots z_1$ the *head* of length i. We use the *common head length* function

$$L(z, z') := \sup\{i \in \mathbb{N} : \mathrm{head}_i(z) = \mathrm{head}_i(z')\}$$

and further set $L(z, A) = \sup\{L(z, z') : z' \in A\}$ for $A \subseteq \mathcal{Z}$. Recall that the topology of $\mathbb{Z}_{(4^n)}$ has a base of clopen sets U_w labelled by finite words $w = w_k \cdots w_1$, $w_i \in \{0, \ldots, 4^i - 1\}$, where

$$U_w = \{z \in \mathbb{Z}_{(4^n)} : \mathrm{head}_k(z) = w\}.$$

In the following, integers $t \in \mathbb{Z}$ are identified with their natural representation in $\mathcal{Z}$ and we write $z_n(t)$ for the nth entry in that representation.

We recursively define a chain of subsets D^i of $\mathcal{Z}$. The subset D^i will contain $m_i := 2^i$ elements which are all of the form $z = \cdots z_2 z_1$ with $z_i = 3^{k_i}$ a power of 3. We write $\overline{\cdot}$ to indicate infinite repetition to the left.

(0) $D^0 = \{\overline{3^0}\}$.

($\tilde{0}$) $\tilde{D}^0 = \{\overline{3^1}3^0\}$.

(1) $D^1 = D^0 \cup \tilde{D}^0 = \{\overline{3^0}, \overline{3^1}3^0\}$.

(i+1) Suppose that we have constructed D^i and ordered its elements (in some fashion)

$$D^i = \{d^{(i,0)}, \ldots, d^{(i,m_i-1)}\}.$$

For each $l \in \{0, \ldots, m_i - 1\}$, we define $\tilde{d}^{(i,l)} \in \tilde{D}^i$ to be

$$\tilde{d}^{(i,l)} = \overline{3^{m_i+l}}\ \mathrm{head}_{m_i+l} d^{(i,l)}$$

and $D^{i+1} := D^i \cup \tilde{D}^i$.

We may order D^{i+1} in a way where we first take the elements of D^i and then those of $\tilde{D}^i$. For example, this gives

$$D^2 = \{\overline{3^0}, \overline{3^1}3^0, \overline{3^2}3^0 3^0, \overline{3^3}3^1 3^1 3^0\}$$

and

$$\begin{aligned} D^3 = &\{\overline{3^0},\ \overline{3^1}3^0,\ \overline{3^2}3^0 3^0,\ \overline{3^3}3^1 3^1 3^0\} \\ &\cup \{\overline{3^4}3^0 3^0 3^0 3^0,\ \overline{3^5}3^1 3^1 3^1 3^1 3^1 3^0,\ \overline{3^6}3^2 3^2 3^2 3^2 3^2 3^0 3^0,\ \overline{3^7}3^3 3^3 3^3 3^3 3^3 3^3 3^1 3^1 3^0\}. \end{aligned}$$

Define

$$D = \overline{\bigcup_{i \in \mathbb{N}} D^i}.$$

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We denote by $\mathrm{Head}_m := \{\mathrm{head}_m(z) : z \in D\}$, the heads of length m of elements of D and by $q : \mathrm{Head}_{m+1} \to \mathrm{Head}_m$ the obvious surjective map given by shortening the head by one element. A word $w \in \mathrm{Head}_m$ is *(left) special* if it does not have a unique preimage under q.

PROPOSITION 5.1. *The set D satisfies the following properties.*

P1 *Let $z, z' \in D$. If $z_n = z'_n$, then $\mathrm{head}_n(z) = \mathrm{head}_n(z')$.*

P2 *Any $\mathbb{Z}$-orbit in $\mathbb{Z}_{(4^n)}$ hits D at most once. Moreover, no $d \in D$ satisfies $d_n = 0$ or $d_n = 4^n - 1$ for any n.*

P3 *For each m, there is a unique special word $w_m \in \mathrm{Head}_m$.*

P4 *D is uncountable.*

Proof. Property P1 is established by direct inspection of the sets D^i.

As for the first part of property P2, note that distinct elements of D cannot be tail equivalent, due to property P1. The second part is immediate.

To see property P3, notice that Head_m has exactly m elements, namely Head_{2^i+l} consists of the first $2^i + l$ elements of D^{i+1}, $l = 0, \ldots, 2^i - 1$. Hence, q is 1-1 on all but one element.

Finally, property P4 follows since no point of D^i remains isolated (in D) so that D is a non-empty compact set without isolated points. Such a set is uncountable. □

LEMMA 5.2. *For all $z \in \mathcal{Z} \setminus \mathbb{Z}$, there is at most one $d \in D$ and one $t \in \mathbb{Z}$ such that $d + t + z \in D$. In particular, for all $z \in \mathcal{Z}$, there is at most one $t \in \mathbb{Z}$ such that $(D + z + t) \cap D \neq \emptyset$.*

Proof. Let $z \in \mathcal{Z} \setminus \mathbb{Z}$. Suppose first that the condition $d + t + z \in D$ and $d + t' + z \in D$ can be satisfied by one $d \in D$ but perhaps two distinct t, t'. Then the orbit of $d + t + z$ intersects D twice which contradicts property P2.

Now suppose there are distinct d and d' in D and $t, t' \in \mathbb{Z}$ with both $d + t + z$ and $d' + t' + z \in D$. Any $d \in D$ is a sequence of powers of 3, $d = (3^{p_n})_{n\in\mathbb{N}}$ with $p_n \leq n - 1$ and hence

$$d_n \leq 3^{n-1} \quad \text{for each } n \in \mathbb{N} \text{ and all } d \in D. \tag{5.1}$$

We write

$$d = (3^{p_n})_{n\in\mathbb{N}}, \quad d' = (3^{p'_n})_{n\in\mathbb{N}}, \quad d + t + z = (3^{q_n}), \quad d' + t' + z = (3^{q'_n}) .$$

Suppose first that $t, t' \geq 0$. Then there is M such that for all $n \geq M$, we have $z_{n-1}(t) = z_{n-1}(t') = 0$ (recall that $z_n(t)$ is the nth entry of t understood as an element of $\mathcal{Z}$). Let c_n and c'_n be the carry over from the $n - 1$th to the nth coordinate in the addition of d, t, z and the addition of d', t', z', respectively. We claim that $c_n = c'_n$ if $n \geq M$. Suppose for a contradiction that $c_n = 1$ while $c'_n = 0$. Then $c_n = 1$ means that $d_{n-1} + z_{n-1}(t) + z_{n-1} + c_{n-1} \geq 4^{n-1}$, while $c'_n = 0$ means that $d'_{n-1} + z_{n-1}(t') + z_{n-1} + c'_{n-1} = 3^{q'_{n-1}}$. The first inequality together with equation (5.1) implies $z_{n-1} \geq 4^{n-1} - 3^{n-2} - 1$. We thus get

$$3^{q'_{n-1}} = d'_{n-1} + z_{n-1}(t') + z_{n-1} + c'_{n-1} \geq d'_{n-1} + 4^{n-1} - 3^{n-2} - 1 > 3^{n-2},$$

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which contradicts the assumption that $(3^{q'_n}) \in D$ because of equation (5.1). It follows that, indeed, $c_n = c'_n$ if $n \geq M$.

Since $z \notin \mathbb{Z}$, we have $d \neq d + t + z$ for all $t \in \mathbb{Z}$. As also $d \neq d'$, property P1 implies that there is M' such that $p_n \neq q_n$ and $p_n \neq p'_n$ for all $n \geq M'$. Choose n larger than M and M'. As $z_n(t) = z_n(t') = 0$, we get $3^{q_n} = d_n + z_n + c_n = 3^{p_n} + z_n + c_n$ and likewise $3^{q'_n} = 3^{p'_n} + z_n + c'_n$. Using $c_n = c'_n$, this gives

$$3^{p_n} + 3^{q'_n} = 3^{p'_n} + 3^{q_n},$$

which contradicts the assumption that $p_n \notin \{q_n, p'_n\}$. Hence, the first statement follows for $t, t' \geq 0$. Now if $s = \min\{t, t'\} < 0$, we replace z in the above argument by $z - s$ to conclude.

To prove the second statement, suppose that $(D + z + t) \cap D \neq \emptyset$. This means that there exists $d \in D$ such that $d + z + t \in D$. If $z \in \mathbb{Z}$, then property P2 implies $t = -z$. Otherwise, d and t are uniquely determined by z according to the first statement. In particular, $(D + z + t') \cap D \neq \emptyset$ implies $t' = t$. □

5.2. *The semicocycle and its extension.* We next turn to the construction and discussion of the desired semicocycle extension. Define $f : \mathcal{Z} \setminus D \to \{a, b\}$ by

$$f(z) = \begin{cases} a & \text{if } L(z, D) \text{ is odd,} \\ b & \text{otherwise.} \end{cases} \tag{5.2}$$

In other words, $f(z) = a$ if the largest possible head-overlap an element $d \in D$ can have with z has odd length.

LEMMA 5.3. *The function f is continuous. Accordingly, given a point $z \in \mathcal{Z}$ with $z + \mathbb{Z} \cap D = \emptyset$, we have that f is a semicocycle over $(\mathcal{Z}, +1, z)$.*

Furthermore, the set of discontinuities of this semicocycle coincides with D so that f is separating.

Proof. Given $w \in \mathcal{Z}$, the function $z \mapsto L(z, w)$ is continuous on $\mathcal{Z} \setminus \{w\}$ and cannot be continuously extended to all of $\mathcal{Z}$. This gives that f is continuous and likewise that D is the set of discontinuities of the respective semicocycle over $(\mathcal{Z}, +1, z)$ for any appropriate z.

To prove that f is separating, let $d, d' \in D$ and assume that $D + z = D$ with $z \notin \mathbb{Z}$. Then $d + z \in D$ and $d' + z \in D$, which by Lemma 5.2, implies $d = d'$. However, D is not a singleton, which is a contradiction. By the same lemma, $(D + t) \cap D \neq \emptyset$ with $t \in \mathbb{Z}$ implies $t = 0$. Hence, the stabiliser in $\mathbb{Z}_{(4^n)}$ of D is trivial. □

As discussed in §3, we hence obtain a semicocycle extension (X_f, σ) of $(\mathcal{Z}, +1)$.

5.3. *Absence of independence.* By Remark 1.6, (X_f, σ) is forward tame if the cylinder sets $[a]$, $[b]$ do not have an infinite independence set in $\mathbb{N}$. This is what we now show.

As before, given $t \in \mathbb{N}$, we let $\ldots z_2(t) z_1(t)$ denote its representation in $\mathcal{Z}$. Define

$$m(t) = \min\{n \in \mathbb{N} : z_n(t) \neq 0\},$$
$$M(t) = \min\{n \in \mathbb{N} : z_m(t) = 0 \text{ for all } m \geq n\}.$$

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LEMMA 5.4. *Let $t \in \mathbb{N}$ and $z \in D$. For all $z' \in D$, we have* $\mathrm{head}_{M(t)+1}(z+t) \neq \mathrm{head}_{M(t)+1}(z')$. *In particular, $L(z+t, D) \leq M(t)$.*

Proof. When adding t to z, denote by c_l the carry over from the $l-1$st entry to the lth entry. These carry overs are determined successively, starting with $l=2$. However, whatever the carry over $c_{M(t)}$ is, as $z_{M(t)}(t)=0$ and $z_{M(t)} < 4^{M(t)}-1$, the carry over $c_{M(t)+1}$ must be 0. Hence, $z_l = (z+t)_l$ for all $l \geq M(t)+1$.

Suppose for a contradiction that $\mathrm{head}_{M(t)+1}(z+t) = \mathrm{head}_{M(t)+1}(z')$ for some $z' \in D$. Then, by the above, $z_{M(t)+1} = (z+t)_{M(t)+1} = z'_{M(t)+1}$. As $z, z' \in D$, this implies that $\mathrm{head}_{M(t)+1}(z) = \mathrm{head}_{M(t)+1}(z')$ (property P1). Hence, $\mathrm{head}_{M(t)+1}(z) = \mathrm{head}_{M(t)+1}(z+t)$ so that $z_l = (z+t)_l$ for all $l \leq M(t)+1$. This implies $z+t=z$, which is a contradiction as $t > 0$. □

COROLLARY 5.5. *Let $t \in \mathbb{N}$ and $z, z' \in D$. If $\mathrm{head}_{M(t)}(z) = \mathrm{head}_{M(t)}(z')$, then $L(z+t, D) = L(z'+t, D)$.*

Proof. Suppose first that $L(z+t, D) < M(t)$. Then for all $z'' \in D$, there is $n < M(t)$ with $z''_n \neq (z+t)_n$. As $\mathrm{head}_{M(t)}(z+t) = \mathrm{head}_{M(t)}(z'+t)$, this implies $L(z+t, z'') = L(z'+t, z'')$ and thus $L(z+t, D) = L(z'+t, D)$. Likewise, $L(z'+t, D) < M(t)$ implies $L(z+t, D) = L(z'+t, D)$. By Lemma 5.4, the only other possibility is $L(z+t, D) = L(z'+t, D) = M(t)$. □

PROPOSITION 5.6. *Suppose $0 < t_1 < t_2$ and $M(t_1) < m(t_2)$. Then there exists a choice function $\varphi\colon \{1,2\} \to \{a,b\}$ which cannot be realized along t_1, t_2 by elements $x \in X_f$ with $\pi(x) \in D$, that is, there is no $x \in \pi^{-1}(D)$ with $x_{t_i} \in [\varphi(i)]$ for $i=1,2$, where π denotes the factor map onto $(\mathcal{Z}, +1)$.*

Proof. We assume that $m(t_2)$ is even; the other case has a similar proof.

Let $z \in D$ and suppose that $w := \mathrm{head}_{m(t_2)-1}(z)$ is not the special element of $\mathrm{Head}_{m(t_2)-1}$. Let $z' \in D$. If $L(z, z') < m(t_2)-1$, then also $L(z+t_2, z') < m(t_2)-1$ (as $\mathrm{head}_{m(t_2)-1}(z) = \mathrm{head}_{m(t_2)-1}(z+t_2)$). However, if $L(z,z') \geq m(t_2)-1$, then we must even have $L(z,z') \geq m(t_2)$. Indeed, as w is not special, $\mathrm{head}_{m(t_2)-1}(z) = \mathrm{head}_{m(t_2)-1}(z')$ implies $\mathrm{head}_{m(t_2)}(z) = \mathrm{head}_{m(t_2)}(z')$. Now $L(z,z') \geq m(t_2)$ implies that $z'_{m(t_2)} \neq (z+t_2)_{m(t_2)}$ and hence $L(z+t_2, z') \leq m(t_2)-1$. Hence, the largest possible head-overlap an element of D can have with $z+t_2$ is $m(t_2)-1$. However, this overlap can be obtained by using $z'=z$, as $L(z, z+t_2) = m(t_2)-1$. Since $m(t_2)-1$ is an odd number, we have $f(z+t_2) = a$ (observe that $f(z+t_2)$ is well defined as $z+t_2 \notin D$ due to property P2). Therefore, $f(z+t_2) = b$ implies that $\mathrm{head}_{m(t_2)-1}(z) = w_{m(t_2)-1}$, the special element of $\mathrm{Head}_{m(t_2)-1}$.

Let $z, z' \in D$. Suppose that $f(z+t_2) = f(z'+t_2) = b$. We just saw that this implies that the heads of length $m(t_2)-1$ of z and z' are both equal to the special element $w_{m(t_2)-1}$ so that, in particular, $\mathrm{head}_{m(t_2)-1}(z) = \mathrm{head}_{m(t_2)-1}(z')$. As $M(t_1) < m(t_2)$, this implies $\mathrm{head}_{M(t_1)}(z) = \mathrm{head}_{M(t_1)}(z')$ and hence, by Corollary 5.5, $f(z+t_1) = f(z'+t_1)$. This means that we have no choice for the first element. One of the two choice functions $\varphi(1)=a, \varphi(2)=b$ or $\varphi'(1)=b, \varphi'(2)=b$ cannot be realized along t_1, t_2 by elements of $\pi^{-1}(D)$. □

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THEOREM 5.7. *(X_f, σ) is tame.*

Proof. Suppose that (X_f, σ) is non-tame. Lemma 5.2 guarantees that the assumptions of the second part of Theorem 4.8 are satisfied so that we can then find an independence set $(\tau_n) \subseteq \mathbb{N}$ such that $\tau_n \to 0$ in $\mathcal{Z}$ and every choice function is realized along (τ_n) by elements of D. As $\tau_n \to 0$, we have $m(\tau_n) \to +\infty$. Hence, we can find two elements $t_1 = \tau_{n_1}$ and $t_2 = \tau_{n_2}$ which satisfy $M(t_1) < m(t_2)$. Proposition 5.6 says that not all choice functions for (t_1, t_2) can be realized by elements of D. This contradicts the fact that all choice functions can be realized along (τ_n) by elements of D. Hence, (X_f, σ) is tame. $\square$

6. *Toeplitz shifts with a unique singular orbit*

In this section, we study a class of Toeplitz shifts which have a single orbit of singular fibres. While the set of singular points in the maximal equicontinuous factor is thus countable, the singular fibres themselves will be uncountable and this will give rise to the possibility that the shift is non-tame. In fact, this was already observed in [22]. However, by refining the construction from [22], we will show that an uncountable singular fibre does not ensure non-tameness, just as uncountably many singular orbits do not ensure non-tameness, as we saw in Theorem 5.7. Further, we will see that minimal non-tame systems can still be forward (or likewise backward) tame.

Specifically, given any language $\mathcal{L}$ on a two-letter alphabet, we construct a binary Toeplitz shift (X, σ), whose singular points are $\mathbb{Z}$, and such that there is a positive sequence (t_n) of integers with $\Sigma_{(t_n)}(\{0\}) = \mathcal{L}$. Then, taking $\mathcal{L}$ to be the language of a Sturmian shift, or alternatively $\{a, b\}^{\mathbb{N}}$, we find that X can be either forward tame, or not. Independently of $\mathcal{L}$, however, all of the constructed systems will be backward tame.

6.1. *The semicocycle extension.*

Consider a double sequence $(l_i^n)_{n\in\mathbb{N}, i\in\mathbb{N}_0} \subseteq \mathbb{N}_0$ with the following properties:

(R1) $(l_0^n)_{n\in\mathbb{N}}$ is strictly increasing;
(R2) for all $n \in \mathbb{N}$, $(l_i^n)_{i\in\mathbb{N}_0}$ is strictly increasing;
(R3) for all $n \in \mathbb{N}$, there is $i_n \in \mathbb{N}_0$ such that $l_{i+i_n}^n = l_{2i}^{n+1}$.

An example is given by

$$l_i^1 = 2^i - 1, \quad l_{2i}^{n+1} = l_{i+2^{n-1}}^n, \quad l_{2i+1}^{n+1} = \frac{1}{2}(l_{i+2^{n-1}}^n + l_{i+1+2^{n-1}}^n)$$

with first values

0	1	2	3	4	5	6	7	8	9	10	11
l_0^1	l_1^1		l_2^1				l_3^1				
	l_0^2	l_1^2	l_2^2		l_3^2		l_4^2				l_5^2
			l_0^3	l_1^3	l_2^3	l_3^3	l_4^3		l_5^3		l_6^3
							l_0^4	l_1^4	l_2^4	l_3^4	l_4^4

Consider a right-extendable language $\mathcal{L} = \bigcup_{n\in\mathbb{N}} \mathcal{L}^n \subseteq \{a, b\}^*$ over the alphabet $\{a, b\}$ where $\mathcal{L}^n = \mathcal{L} \cap \{a, b\}^n$ and $\mathcal{L}^1 = \{a, b\}$ (that is, $\mathcal{L}$ is not trivial). Right-extendable means that any word $w \in \mathcal{L}^n$ has an extension to the right $wc \in \mathcal{L}^{n+1}$ where $c \in \{a, b\}$. Similarly as in the previous section, if w has two extensions in $\mathcal{L}^{n+1}$, it is called *right special*. We denote the set $\{n, n+1, \ldots, m-1\}$ by $[n, m)$.

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LEMMA 6.1. *Let* $(l_i^n)_{n\in\mathbb{N},i\in\mathbb{N}_0}$ *satisfy the conditions (R1)–(R3) above, and let* $\mathcal{L}$ *be a right-extendable binary language. Then there exists a family* $(f^n)_{n\in\mathbb{N}}$ *of functions* $f^n : \mathbb{N}_0 \to \{a, b\}$ *which satisfy the following conditions for all* $N \in \mathbb{N}$.

(1) *For each* $i \in \mathbb{N}_0$, *the function*

$$x \mapsto f^1(x)f^2(x)\cdots f^N(x) \in \{a, b\}^N$$

is constant on $[l_i^N, l_{i+1}^N)$ *and the constant value is a word* $w^{(N,i)} \in \mathcal{L}^N$.

(2) *Each word* $w \in \mathcal{L}^N$ *arises infinitely often in the above way, that is, for each* $i_0 \in \mathbb{N}$, *there is* $i \geq i_0$ *with* $w = w^{(N,i)} := f^1(x)f^2(x)\cdots f^N(x)$ *for* $x \in [l_i^N, l_{i+1}^N)$.

Proof. Let f^1 be any function which takes infinitely often each of the values a and b, and is constant on all $[l_i^1, l_{i+1}^1)$. Then the above is satisfied for $N = 1$.

Now, suppose we have already defined $f^1, \ldots, f^N$ satisfying the above properties. Then define f^{N+1} as follows: given $i \geq i_N \in \mathbb{N}_0$, let $w^{N,i} \in \mathcal{L}^N$ be the word such that $w^{(N,i)} = f^1(x)f^2(x)\cdots f^N(x)$ for $x \in [l_i^N, l_{i+1}^N)$. Let $j = i - i_N$. If $w^{(N,i)}$ is right special, set

$$f^{N+1}(x) = \begin{cases} a & \text{if } x \in [l_{2j}^{N+1}, l_{2j+1}^{N+1}), \\ b & \text{if } x \in [l_{2j+1}^{N+1}, l_{2j+2}^{N+1}), \end{cases}$$

otherwise

$$f^{N+1}(x) = c \quad \text{if } x \in [l_{2j}^{N+1}, l_{2j+2}^{N+1}) = [l_i^N, l_{i+1}^N),$$

where c is the unique letter extending $w^{(N,i)}$ to the right. This way, we have defined $f^{N+1}(x)$ for all $x \geq l_{i_N}^N$. We extend it arbitrarily for lower values of x. It follows from condition (R3) and the assumptions on $f^1, \ldots, f^N$ that $x \mapsto f^1(x)f^2(x)\cdots f^N(x)f^{N+1}(x)$ is constant on the intervals $[l_j^{N+1}, l_{j+1}^{N+1})$ and takes a value in $\mathcal{L}^{N+1}$ there. Furthermore, by induction, any word of $\mathcal{L}^{N+1}$ arises in this way; indeed, $w^{(N,i)}c = f^1(x)f^2(x)\cdots f^N(x)f^{N+1}(x)$ for $x \in [l_{2j}^{N+1}, l_{2j+2}^{N+1})$ where c is the unique extension of $w^{(N,i)}$, if that exists, or $c = a$ ($c = b$) if $w^{(N,i)}$ is right special and $x \in [l_{2j}^{N+1}, l_{2j+1}^{N+1})$ ($x \in [l_{2j+1}^{N+1}, l_{2j+2}^{N+1})$). □

We can take any odometer $(\mathcal{Z}, +1)$, but for concreteness, we take $\mathcal{Z} = \mathbb{Z}_2$. Consider a strictly increasing sequence $(t_n)_{n\in\mathbb{N}}$ in $\mathbb{Z}$ such that $t_n \to 0$ in $\mathbb{Z}_2$. For concreteness, we choose $t_n = 2^{l_0^n}$. Its expansion in $\mathbb{Z}_2$ has zeros everywhere except at position l_0^n where we have an entry 1. Note that the specifics of the following construction—and, in particular, the definition of f in equation (6.2)—depend to some extent on our particular choice of (t_n). For notational reasons, we keep this dependence implicit.

Define the clopen neighbourhood

$$U_n(z) := U_{\text{head}_{l_0^n}(z)} = \{z' \in \mathbb{Z}_2 : \text{head}_{l_0^n}(z) = \text{head}_{l_0^n}(z')\}.$$

It follows from condition (R1) and our choice of (t_n) that

$$U_n(t_n) \cap U_{n'}(t_{n'}) = \emptyset \quad \text{for all } n \neq n' \in \mathbb{N}. \tag{6.1}$$

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We set $D = \{t_n : n \in \mathbb{N}\} \cup \{0\}$, a closed set, and define $f : \mathcal{Z} \setminus D \to \{a, b\}$ with the help of the maps (f^n) from Lemma 6.1 by

$$f(z) = \begin{cases} f^n(L(z, t_n)) & \text{if } z \in U_n(t_n), \\ a & \text{otherwise,} \end{cases} \tag{6.2}$$

where, as before, $L(z, t_n)$ is the length of the common head between z and t_n. By equation (6.1), f is well defined. Since all functions f^n are continuous (trivially) and L is continuous away from the diagonal, f is continuous. We fix $\hat{z} \notin \mathbb{Z}$ (so that $\hat{z} + \mathbb{Z} \subseteq D^c$) and consider the restriction of f to $\hat{z} + \mathbb{Z}$ an $\{a, b\}$-valued semicocycle over $(\mathbb{Z}_2, +1, \hat{z})$. As D is not periodic, its stabiliser in $\mathbb{Z}_2$ is trivial and therefore f is separating. Altogether, f defines a semicocycle extension (X_f, σ) of $(\mathbb{Z}_2, +1)$.

6.2. *Tameness or otherwise.* The set of discontinuity points of the semicocyle is $D_f = D \subseteq \mathbb{Z}$, and consequently, all non-integer fibres are regular. The question of whether (X_f, σ) is tame or not is therefore a question about its fibre $\pi^{-1}(0)$. We now show that this fibre contains $X_{\mathcal{L}}$, the set of all unilateral sequences allowed by the language $\mathcal{L}$. More precisely, these unilateral sequences are precisely the subsequences of the sequences of $\pi^{-1}(0)$ along the times $(t_n)_n$.

PROPOSITION 6.2. *Let $(t_n)_{n\in\mathbb{N}}$ and f be as above and consider the dynamical system (X_f, σ). For any $y \in X_{\mathcal{L}}$, there exists $x \in \pi^{-1}(0)$ such that*

$$x_{t_n} = y_n \quad \textit{for all } n \in \mathbb{N}. \tag{6.3}$$

Conversely, for any $x \in \pi^{-1}(0)$, there is $y \in X_{\mathcal{L}}$ such that equation (6.3) holds true.

Proof. Fix $\hat{z} \in \mathbb{Z}_2 \backslash \mathbb{Z}$ and consider the sequence $\hat{f} \in X_f$ given by $\hat{f}_n = f(\hat{z} + n)$. Given $N \in \mathbb{N}$ and $w \in \mathcal{L}^N$, pick $i \in \mathbb{N}_0$ such that $w = w^{(N,i)}$ as in Lemma 6.1((2)). Choose $t_w \in \mathbb{Z}$ such that $L(t_w + \hat{z}, 0) = l_i^N$ (t_w has the same first l_i^N digits as $-\hat{z}$ and then disagrees). Then also $L(t_w + \hat{z} + t, t) = l_i^N$ for all $t \in \mathbb{Z}$. Hence, taking $t = t_n$, we deduce with equation (6.2) that

$$\hat{f}_{t_w+t_n} = f(t_w + \hat{z} + t_n) = f^n(L(t_w + \hat{z}, 0)) = f^n(l_i^N) = w_n$$

for all $n = 1, \ldots, N$.

In other words, the sequence $x = \sigma^{t_w}(\hat{f})$ verifies

$$x_{t_1} x_{t_2} \cdots x_{t_N} = f(t_w + \hat{z} + t_1) f(t_w + \hat{z} + t_2) \cdots f(t_w + \hat{z} + t_N) = w. \tag{6.4}$$

Now, given $y \in X_{\mathcal{L}}$, let x^y be a limit point of $\{\sigma^{t_{y_1 y_2 \cdots y_N}}(\hat{f}) : N \in \mathbb{N}\}$. As $L(t_{y_1 y_2 \cdots y_N} + \hat{z}, 0) \geq l_0^N \overset{N\to+\infty}{\longrightarrow} +\infty$, the sequence $t_{y_1 y_2 \cdots y_N} + \hat{z}$ tends to 0 and thus $\pi(x^y) = 0$. Due to equation (6.4), we further have $x^y_{t_n} = y_n$ for all $n = 1, \ldots, N$ and $N \in \mathbb{N}$, that is, equation (6.3). As $y \in X_{\mathcal{L}}$ was arbitrary, this shows the first statement. The second statement now follows with Lemma 6.1((1)). □

Remark 6.3. The above idea of embedding a subshift $X_{\mathcal{L}}$ in the almost automorphic shift X_f (as in the previous proposition) is similar in spirit to Williams' classical constructions in [**42**] despite the considerable differences on a technical level.

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THEOREM 6.4. *(X_f, σ) is backward tame. Further, (X_f, σ) is forward tame if and only if the $\mathbb{N}$-action $(X_{\mathcal{L}}, \sigma)$ is tame.*

Proof. To prove the first statement, let $z \in \mathbb{Z}_2$. Its backward orbit $\{z - n : n \in \mathbb{N}_0\}$ can intersect at most finitely many positive integers $\mathbb{Z}^+ \subseteq \mathbb{Z}_2$, hence only finitely many points of $D \subseteq \mathbb{Z}^+$. It therefore follows from Corollary 4.4 that if we restrict to the backward dynamics, then (X_f, σ) is tame.

Suppose the $\mathbb{N}$-action $(X_{\mathcal{L}}, \sigma)$ is non-tame. Then by Remark 1.6, there is a sequence of natural numbers (ν_n) such that for every choice function $\varphi \in \{a, b\}^{\mathbb{N}}$, there is $y^\varphi \in X_{\mathcal{L}}$ with ${y^\varphi}_{\nu_n} = \varphi(n)$ for each $n \in \mathbb{N}$. By Proposition 6.2, we find for every choice function φ, an element $x^\varphi \in X_f$ such that y^φ is the subsequence of x^φ corresponding to the times $(t_n)_n$. Hence, $(t_{\nu_n})_n$ is an independence set of (X_f, σ) for the pair of cylinder sets $[a]$ and $[b]$.

Conversely, suppose that (X_f, σ) is forward non-tame. As $D' = \{0\}$, the first part of Theorem 4.8 implies that there is an independence sequence (τ_n) for $\{a\}, \{b\}$ which converges to $z \in \mathcal{Z}$ and such that all choice functions are realized by elements of the fibre $\pi^{-1}(-z)$, that is, $\Sigma_{(\tau_n)}(\{0\}) = \{a, b\}^{\mathbb{N}}$. In particular, $\pi^{-1}(-z)$ must be infinite and hence $z \in \mathbb{Z}$.

Now, Lemma 4.2 gives that if $\{\tau_n - z : n \in \mathbb{N}\} \cap D$ was finite, then $\Sigma_{(\tau_n)}(\{z - z\})$ was finite, which is a contradiction. Therefore, $\{\tau_n - z : n \in \mathbb{N}\} \cap D$ is infinite. Thus, there is a subsequence $(t_{n_j})_j$ such that $\{(t_{n_j})_j\} = \{\tau_n - z : n \in \mathbb{N}\} \cap D$. Together with the second part of Proposition 6.2, we see that (n_j) is an independence set for $(X_{\mathcal{L}}, \sigma)$. □

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